LAGUERRE-TYPE BELL POLYNOMIALS

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We develop an extension of the classical Bell polynomials introducing the Laguerre-type version of this well-known mathematical tool. The Laguerre-type Bell polynomials are useful in order to compute the nth Laguerre-type derivatives of a composite function. Incidentally, we generalize a result considered by L. Carlitz in order to obtain explicit relationships between Bessel functions and generalized hypergeometric functions.

1. Introduction

The Bell polynomials [1] appear in different frameworks. They are often used in combinatorial analysis [20], and even in statistics [14], although without explicit references. Moreover these polynomials have been applied even in many other contexts, such as the Blissard problem (see [20, page 46]), the representation of Lucas polynomials of the first and second kinds [4, 9], the representation formulas of Newton sum rules for polynomials' zeros [12, 13], the recurrence relations for a class of Freud-type polynomials [3], the representation of symmetric functions of a countable set of numbers, generalizing the classical algebraic Newton-Girard formulas [15]. Consequently they were also used [6] in order to find reduction formulas for the *orthogonal invariants* of a strictly positive compact operator, deriving in a simple way the so-called Robert formulas [21].

Some generalized forms of Bell polynomials already appeared in literature (see, e.g., [11, 17, 19]). A generalization of the Bell polynomials suitable for the differentiation of multivariable composite functions can also be found in [18]. Lastly, in [2], the so-called multidimensional Bell polynomials of higher order were introduced, which are suitable for representing the derivative of a composite function of several (say m) variables $f(\varphi^{(1)}(t),\varphi^{(2)}(t),\dots,\varphi^{(m)}(t))$, where $\varphi^{(i)}(t)=\phi^{(i,1)}(\phi^{(i,2)}(\cdots\phi^{(i,r_i)}(t)))$, $(i=1,2,\dots,m)$.

In this article we find explicit representation formulas for the nth Laguerre-type derivatives of a composite function. The case of the first Laguerre derivative DxD, $D:=d/dx$ is essentially related to an article by Carlitz [5], originated by a preceding paper by Lardner [16] in which the powers $(DxD)^n$ of this derivative appear.

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2. Recalling the Bell polynomials

We recall that the Bell polynomials are a classical mathematical tool for representing the nth derivative of a composite function. In fact by considering the composite function $\Phi(t) := f(g(t))$ of functions $x = g(t)$ and $y = f(x)$ defined in suitable intervals of the real axis and n times differentiable with respect to the relevant independent variables and by using the following notations:

$$\Phi_h := D_t^h \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_h := D_t^h g(t),$$
$$([f,g]_n) := (f_1,g_1;f_2,g_2;\dots;f_n,g_n), \tag{2.1}$$

they are defined as follows:

$$Y_n([f,g]_n) := \Phi_n. \tag{2.2}$$

For example one has

$$\begin{aligned} Y_1([f,g]_1) &= f_1g_1, \\ Y_2([f,g]_2) &= f_1g_2 + f_2g_1^2, \\ Y_3([f,g]_3) &= f_1g_3 + f_2(3g_2g_1) + f_3g_1^3. \end{aligned} \tag{2.3}$$

Further examples can be found in [20, page 49].

Inductively, we can write

$$Y_n([f,g]_n) = \sum_{k=1}^{n} A_{n,k}(g_1,g_2,\dots,g_n) f_k, \tag{2.4}$$

where the coefficient $A_{n,k}$, for any $k = 1,\dots,n$, is a polynomial in $g_1,g_2,\dots,g_n$, homogeneous of degree k and *isobaric* of weight n (i.e., it is a linear combination of monomials $g_1^{k_1}g_2^{k_2}\cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \cdots + nk_n = n$).

For them the following result holds true.

Proposition 2.1. *The Bell polynomials satisfy the recurrence relation*

$$\begin{aligned} Y_0([f,g]_0) &:= f_1, \\ Y_{n+1}([f,g]_{n+1}) &= \sum_{k=0}^{n} \binom{n}{k} Y_{n-k}([f_1,g]_{n-k}) g_{k+1}, \end{aligned} \tag{2.5}$$

where

$$([f_1,g]_{n-k}) := (f_2,g_1;f_3,g_2;\ldots;f_{n-k+1},g_{n-k}). \tag{2.6}$$

An explicit expression for the Bell polynomials is also given by the Faà di Bruno formula [10]:

$$\Phi_n = Y_n([f,g]_n) = \sum_{\pi(n)} \frac{n!}{j_1!j_2!\cdots j_n!} f_j \left[\frac{g_1}{1!}\right]^{j_1} \left[\frac{g_2}{2!}\right]^{j_2} \cdots \left[\frac{g_n}{n!}\right]^{j_n}, \tag{2.7}$$

where the sum runs over all partitions $\pi(n)$ of the integer n, that is, $n = j_1 + 2j_2 + \cdots + nj_n$. In (2.7) j_h denotes the number of parts of size h, and $j = j_1 + j_2 + \cdots + j_n$ denotes the number of parts of the considered partition. A proof of the Faà di Bruno formula can be found in [20]. In [22] the proof is based on the *umbral calculus* (see [23] and the references therein).

3. Laguerre-type derivatives

The Laguerre-type derivatives were introduced in [7, 8] in connection with a differential isomorphism denoted by the symbol $\mathcal{T} := \mathcal{T}_x$, acting onto the space $\mathcal{A} := \mathcal{A}_x$ of analytic functions of the x variable by means of the correspondence

$$D := \frac{d}{dx} \longrightarrow \hat{D}_L := DxD; \qquad x\cdot \longrightarrow \hat{D}_x^{-1}, \tag{3.1}$$

where

$$\begin{aligned} \hat{D}_x^{-1} f(x) &:= \int_0^x f(\xi)\,d\xi, \\ \hat{D}_x^{-n} f(x) &:= \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1} f(\xi)\,d\xi, \end{aligned} \tag{3.2}$$

so that

$$\mathcal{T}_x(x^n) = \hat{D}_x^{-n}(1) := \frac{1}{(n-1)!} \int_0^x (x-\xi)^{n-1}\,d\xi = \frac{x^n}{n!}. \tag{3.3}$$

According to this isomorphism, the exponential operator e^x is transformed into the first Laguerre-type exponential $e_1(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$ which is an eigenfunction of the *Laguerre derivative operator* $D_L := DxD$. We have, in fact,

$$\begin{aligned} \mathcal{T}_x(e^x) &= \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2} = e_1(x), \\ \hat{D}_L e_1(ax) &= a e_1(ax), \quad \forall a \in \mathbb{C}. \end{aligned} \tag{3.4}$$

This result can be generalized by considering the rLaguerre-type exponential $e_r(x) := \sum_{k=0}^{\infty} x^k/(k!)^{r+1}$, the rth Laguerre-type derivative operator $D_{rL} := DxDxD\cdots DxD$ (containing $r+1$ ordinary derivatives), and the iterated isomorphism $\mathcal{T}^r$, since

$$\mathcal{T}_x^r(e^x) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_x(x^k)}{(k!)^r} = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{r+1}} = e_r(x),$$
$$\hat{D}_{rL} e_r(ax) = a e_r(ax), \quad \forall a \in \mathbb{C}. \tag{3.5}$$

Remark 3.1. The above results show that, for every positive integer r, we can define a Laguerre-type exponential function $e_r(x)$, satisfying an eigenfunction property, which is an analog of the elementary property of the exponential. This function reduces to the exponential function when $r = 0$, so that we can put by definition

$$e_0(x) := e^x, \qquad \hat{D}_{0L} := D. \tag{3.6}$$

Obviously, $\hat{D}_{1L} := \hat{D}_L$.

For this reason we will refer to such functions as L-exponential functions, or shortly L-exponentials.

4. Laguerre-type Bell polynomials

The problem of constructing Bell polynomials can be extended in the natural way to the case of the Laguerre-type derivatives.

To this aim, by using notations in (2.1), we introduce the following definition.

Definition 4.1. The nth Laguerre-type Bell polynomial, denoted by ${}_{rL}Y_n(x;[f,g]_n)$, represents the nth rLaguerre-type derivative of the composite function $f(g(t))$.

We will show that ${}_{rL}Y_n$ can be expressed as a polynomial in the independent variable x, depending on $f_1,g_1;f_2,g_2;\ldots;f_n,g_n$ in terms of the classical Bell polynomials.

We start noting that, according to a general result due to Viskov [24], the Laguerre derivative satisfy

$$(D_L)^n = (DxD)^n = D^n x^n D^n, \tag{4.1}$$

and furthermore, for any order r, it turns out that

$$(D_{rL})^n = (DxDx\cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n. \tag{4.2}$$

According to the above equations, the proof of Carlitz [5] can be reduced to a simple application of the Leibnitz rule, since

$$(DxD)^n = D^n(x^n D^n) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k} x^n D^{n+k}$$
$$= \sum_{k=0}^{n} \left[\binom{n}{k}\right]^2 (n-k)! x^k D^{n+k} = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n}{k} x^k D^{n+k}. \tag{4.3}$$

Therefore, the following representation formula for the Laguerre-type Bell polynomials, denoted by ${}_LY_n$, holds true.

Theorem 4.2. *The ${}_LY_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation*

$$ {}_LY_n(x;[f,g]_n) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} x^k Y_{n+k}([f,g]_{n+k}). \tag{4.4}$$

The above results can be easily generalized, since

$$\begin{aligned}(D_{2L})^n &= (DxDxD)^n = D^n x^n (D^n x^n D^n)\\ &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\binom{n}{k_1}\binom{n}{k_2} x^{k_1+k_2} D^{n+k_1+k_2}.\end{aligned} \tag{4.5}$$

5. The general case

The following result follows by induction.

Theorem 5.1. *The powers of the rth Laguerre-type derivative operator $D_{rL} := DxDxD\cdots DxD$ (containing $r+1$ ordinary derivatives) can be expanded in the form*

$$\begin{aligned}(D_{rL})^n &= (DxDx\cdots DxD)^n = D^n x^n D^n x^n \cdots D^n x^n D^n\\ &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\cdots\sum_{k_r=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\cdots\frac{(n+k_1+k_2+\cdots+k_{r-1})!}{(k_1+k_2+\cdots+k_r)!}\\ &\quad\times\binom{n}{k_1}\binom{n}{k_2}\cdots\binom{n}{k_r} x^{k_1+k_2+\cdots+k_r} D^{n+k_1+k_2+\cdots+k_r}.\end{aligned} \tag{5.1}$$

Therefore, for the rth Laguerre-type Bell polynomials denoted by ${}_{rL}Y_n$, the following result holds true.

Theorem 5.2. *The ${}_{rL}Y_n$ polynomials are expressed in terms of the ordinary Bell polynomials according to the equation*

$$\begin{aligned}{}_{rL}Y_n(x;[f,g]_n) &= \sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\cdots\sum_{k_r=0}^{n} \frac{n!}{k_1!}\frac{(n+k_1)!}{(k_1+k_2)!}\cdots\frac{(n+k_1+k_2+\cdots+k_{r-1})!}{(k_1+k_2+\cdots+k_r)!}\\ &\quad\times\binom{n}{k_1}\binom{n}{k_2}\cdots\binom{n}{k_r} x^{k_1+k_2+\cdots+k_r} Y_{n+k_1+k_2+\cdots+k_r}([f,g]_{n+k_1+k_2+\cdots+k_r}).\end{aligned} \tag{5.2}$$

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