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Asymptotics for Ménage polynomials and certain hypergeometric polynomials of type ${}_3F_1$

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Abstract

Using a uniform version of Laplace's method, strong asymptotics for suitably normalized Ménage polynomials and certain hypergeometric polynomials of type ${}_3F_1$ are established. Moreover, weak asymptotics and further properties of the zeros are derived.

Keywords: Strong asymptotics; Weak asymptotics; Asymptotic distribution of zeros; Ménage problem; Ménage polynomials; Hypergeometric polynomials

1. Introduction

The Ménage problem is one of the classical problems arising in combinatorics in the context of permutations with restricted positions. It deals with the number of ways seating n married couples at a circular table, men and women in alternating positions, so that no wife is next to her husband (see, e.g., [10, p. 163]). In 1934, Touchard [18] found the solution stating that these numbers are given by

$$2n! \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!, \quad n \geq 1,$$

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which today are called Ménage numbers. Neglecting the preceding factors yields the reduced Ménage numbers

$$U_n = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!, \quad n \geq 1.$$

In the context of rook problems (see, e.g., [10, p. 164]) these numbers give rise to considering the so-called Ménage polynomials

$$U_n(t) = \sum_{k=0}^{n} (t-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!, \quad t \in \mathbb{C}, \ n \geq 1, \tag{1.1}$$

which occasionally are termed as Ménage hit polynomials (see, e.g., [10, p. 197]). For the purpose of motivating our studies of these polynomials we will have a look at a related situation. It is based on a different combinatorial question, namely the rencontre problem, which asks for the number of permutations of $\{1, \dots, n\}$ without any fixed points. It is well known (see, e.g. [10, p. 57 ff.]) that the solution is given by the derangement numbers

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}, \quad n \geq 1.$$

Again motivated combinatorially, these numbers yield the polynomials

$$D_n(t) = n! \sum_{k=0}^{n} \frac{(t-1)^k}{k!} = (-1)^n \, n! \, L_n^{(-n-1)}(t-1), \quad t \in \mathbb{C}, \ n \geq 1,$$

where $L_n^{(\alpha)}(t)$ denote the generalized Laguerre polynomials (see, e.g. [13, p. 100]). The polynomials $S_n(t) = \frac{1}{n!} D_n(t+1)$ are very well known as they represent the partial sums of the exponential series. In 1924, the sequence of the partial sums of the exponential series was examined by Szegö (see [14]). He derived strong asymptotics for the polynomials $S_n(nt)$, that is, after choosing an adequate normalization, he obtained asymptotic results valid uniformly on compact subsets of certain domains in the complex t-plane, as $n \to \infty$. Using these results, he was able to study the behavior of the zeros of $S_n(nt)$ for large values of n. In doing so, he showed that the zeros accumulate on a curve in the complex plane, which can be defined by the equation $|z\, e^{1-z}| = 1, \ |z| \leq 1$, and today bears his name.

It is the main objective of this paper to establish strong and weak asymptotics for suitably normalized Ménage polynomials, and more general, for a class of hypergeometric polynomials of type ${}_3F_1$ containing the above mentioned polynomials. A further reason for these studies can be found in a work by Ismail and Askey [2], where the authors show the existence of integral representations for the solutions of certain combinatorial problems (e.g. the Ménage problem) and express their hope that these representations can be used to obtain asymptotic expansions.

The Ménage polynomials in (1.1) are connected with hypergeometric functions by the identity

$$U_n(t) = 2\,(t-1)^n \, {}_3F_1\left(\left. \begin{matrix} -n, & n, & 1 \\ & \frac{1}{2} & \end{matrix} \right| \frac{1}{4\,(1-t)} \right). \tag{1.2}$$

Therefore, we will be concerned with polynomials of the type

$$ {}_3F_1\left(\begin{matrix}-n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\,\middle|\, z\right) = \sum_{k=0}^{n} \frac{(-n)_k\,(n)_k\,(\alpha)_k}{\left(\frac{1}{2}\right)_k\,k!}\,z^k, \tag{1.3} $$

where the parameter α is supposed to be a positive integer. Although consideration of more general parameters is possible, we will restrict our attention to the integer case as this is most important for the study of the Ménage polynomials. For the polynomials in (1.3) we will use the notion of *generalized associated Ménage polynomials*. Beyond this context, polynomials of this kind also appear in the theory of rational approximations of solutions of ordinary differential equations (see, e.g. [5, p. 66 ff.]).

In the main results, a special curve plays an important role. We define a simple closed and continuously differentiable curve $\mathcal{C}$ in the complex plane by the equation

$$ \left|(z+\sqrt{z^2+1})\,e^{-\frac{1}{z}-\frac{\sqrt{z^2+1}}{z}}\right| = 1, \quad z \in \mathbb{C}, \tag{1.4} $$

where the multivalued expressions are defined below in the context of Lemma 2.1. Moreover, the exterior of $\mathcal{C}$ will be denoted by $\mathcal{E}(\mathcal{C})$ and the interior of $\mathcal{C}$ will be denoted by $\mathcal{I}(\mathcal{C})$.

In Section 2 we will state some preliminary results. For suitably normalized polynomials $F_n(z)$, where

$$ F_n(z) = {}_3F_1\left(\begin{matrix}-n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\,\middle|\, \frac{z}{2n}\right), \tag{1.5} $$

we will prove the following strong asymptotics in Theorem 3.1

$$ \begin{aligned} F_n(z) = {} & \frac{(-1)^n}{\Gamma(\alpha)}\, n^{\alpha-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \left(\frac{1}{z}+\frac{\sqrt{z^2+1}}{z}\right)^{\alpha-1} \left(\frac{\sqrt{z^2+1}}{z}\right)^{-\frac{1}{2}} \\ & \times \left\{(z+\sqrt{z^2+1})\,e^{-\frac{1}{z}-\frac{\sqrt{z^2+1}}{z}}\right\}^n \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned} $$

as $n \to \infty$, holding uniformly on compact subsets of $\mathcal{E}(\mathcal{C})$, and

$$ F_n(z) = \left(\frac{2}{n}\right)^{\alpha} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}} \left(\frac{1}{z}\right)^{\alpha} \cos(\alpha\,\pi) \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), $$

as $n \to \infty$, holding uniformly on compact subsets of $\mathcal{I}(\mathcal{C})$.

At the beginning of Section 4, in Theorem 4.4 and in Theorem 4.5, we will investigate the curve $\mathcal{C}$ and its closed interior $\overline{\mathcal{I}(\mathcal{C})}$ from a potential theoretic perspective, which serves as a preparation for our study of the asymptotic behavior of the zeros of the polynomials $F_n(z)$. After showing in Lemma 4.1 that the zeros of the polynomials $F_n(z)$ form a bounded set in the complex plane, we will prove in Lemma 4.2, that the zeros accumulate on the curve $\mathcal{C}$. Finally, in Theorem 4.6 we will describe the modality of this accumulation by showing that the asymptotic limit distribution of the zeros is given by the equilibrium measure μ for the set $\overline{\mathcal{I}(\mathcal{C})}$, that is, the zero counting measures μ_n associated to the polynomials $F_n(z)$ converge in the weak* topology to the measure μ. Moreover, the limit distribution μ is absolutely continuous with respect to the arc measure on the curve $\mathcal{C}$ and its Radon–Nikodym derivative is given explicitly by

$$\mathrm{d}\mu(z) = \frac{1}{2\pi} \, |D(z)| \, |\mathrm{d}z|, \quad z \in \mathcal{C},$$

where

$$D(z) = \frac{1}{\sqrt{z^2+1}-1}, \quad z \in \mathcal{C}.$$

All described results are obtained without making use of the notion of orthogonality. Throughout the following, all potential theoretic notions like logarithmic potential, logarithmic capacity, Green function, weak* topology, zero counting measure and equilibrium measure are used the way as defined in the standard monograph of Saff and Totik [11].

2. Auxiliary results

The first device we want to employ is a uniform version of Laplace's method for obtaining approximations for parameter integrals, which is taken from [6]. It is based on the well known classical procedure as it is presented in [8, pp. 121–127]. We may assume, that the path of integration $\mathcal{P}$ is a piecewise continuously differentiable non-closed simple curve such that the derivative of a given parametrization on an open "interval of smoothness" always admits a continuous extension to the closure of this interval, and that this extension does not vanish there.

Theorem 2.1. *Let $G \subset \mathbb{C}$ be a domain and $S \subset \mathbb{C}$ a nonempty compact set equipped with the subspace topology. Let $\mathcal{P}$ be a path, as defined above, with finite starting point $a \in G$ and finite or infinite endpoint b such that $[a, b)_{\mathcal{P}} \subset G$. Let*

$$p\,, q : G \times S \to \mathbb{C}$$

be continuous functions such that $p_z := p(\cdot, z)$ and $q_z := q(\cdot, z)$ are holomorphic on G for all $z \in S$. In the neighborhood of a, for all $z \in S$ the functions p_z and q_z can be expanded in convergent series. Let these series be given by

$$p_z(t) = p_z(a) + \sum_{\nu=0}^{\infty} b_\nu(z)\,(t-a)^{\mu+\nu}, \quad \mu \in \mathbb{N},\ z \in S,\ b_0(z) \neq 0 \text{ on } S,$$

$$q_z(t) = \sum_{\nu=0}^{\infty} c_\nu(z)\,(t-a)^{\nu+\lambda-1}, \quad \lambda \in \mathbb{N},\ z \in S,\ c_0(z) \neq 0 \text{ on } S.$$

Suppose, that for all $t \in (a, b)_{\mathcal{P}}$ the following condition holds

$$\inf_{z \in S} \Re(p_z(t) - p_z(a)) > 0, \tag{2.1}$$

and, that there is a positive lower bound for this expression as $t \to b$ on $\mathcal{P}$. In addition, suppose that there is a number $N \in \mathbb{N}$ such that the complex contour integrals

$$I(n,z) := \int_{\mathcal{P}} e^{-n\,p_z(t)}\, q_z(t)\,\mathrm{d}t$$

converge absolutely for all $n \geq N$ and all $z \in S$, and that the expression

$$\int_{\mathcal{P}} e^{-N\,\Re(p_z(t))}\, |q_z(t)|\,|\mathrm{d}t|$$

is bounded when considered as function of z on S.

Then $I(n,z)$ possesses a complete asymptotic expansion of the form

$$I(n,z) = \int_{\mathcal{P}} e^{-n\,p_z(t)}\, q_z(t)\,\mathrm{d}t \sim e^{-n\,p_z(a)} \sum_{\nu=0}^{\infty} \Gamma\left(\frac{\nu+\lambda}{\mu}\right) \frac{a_\nu(z)}{n^{(\nu+\lambda)/\mu}}, \quad n \to \infty.$$

This expansion holds uniformly with respect to $z \in S$ and the first coefficients are given by

$$a_0(z) = \frac{c_0(z)}{\mu(b_0(z))^{\lambda/\mu}}, \qquad a_1(z) = \left(\frac{c_1(z)}{\mu} - \frac{(\lambda+1)b_1(z)c_0(z)}{\mu^2 b_0(z)}\right)\frac{1}{(b_0(z))^{(\lambda+1)/\mu}}.$$

Here, the powers of $b_0(z)$ are constructed by using the branch $\omega_0(z) := \arg b_0(z)$, which is uniquely determined by $|\omega_0(z) + \mu\omega| \leq \frac{\pi}{2}$, where the value $\omega := \lim_{\substack{t\to a\\ t\in\mathcal{P}}} \arg(t-a)$ is chosen arbitrary but fixed.

Next we turn to the special curve mentioned in (1.4). Let $[\,i,-i\,] := \{i-2it \mid t \in [0,1]\}$, $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ and $\overline{\mathbb{D}}^c := \{z \in \mathbb{C} \mid |z| > 1\}$. Furthermore, let the mapping

$$\mathbb{C} \setminus [\,i,-i\,] \to \overline{\mathbb{D}}^c, \qquad z \mapsto z + \sqrt{z^2+1} \tag{2.2}$$

be defined as the inverse mapping of

$$\overline{\mathbb{D}}^c \to \mathbb{C} \setminus [\,i,-i\,], \qquad w \mapsto \frac{1}{2}\left(w - \frac{1}{w}\right)$$

which maps the exterior of the unit disk conformally onto $\mathbb{C} \setminus [\,i,-i\,]$ preserving the point at infinity. We may define the values of the function in (2.2) on the cut $[\,i,-i\,]$ by stipulating that the line segment $[\,i,-i\,]$ is mapped onto the curve $\gamma(t) = e^{it}$, $\frac{\pi}{2} \leq t \leq \frac{3}{2}\pi$. By this setting, also the mapping $\mathbb{C} \to \mathbb{C}$, $z \mapsto \sqrt{z^2+1}$ is defined properly and clearly it is holomorphic on $\mathbb{C} \setminus [\,i,-i\,]$. Moreover, let the mapping $\varphi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\varphi(z) := (z + \sqrt{z^2+1}) \exp\left\{\frac{-2}{z + \sqrt{z^2+1} - 1} - 1\right\},$$

where an elementary calculation shows

$$\varphi(z) = (z + \sqrt{z^2+1})\, e^{-\frac{1}{z} - \frac{\sqrt{z^2+1}}{z}}, \quad z \in \mathbb{C}.$$

Let $\mathrm{Im}(\mathcal{C}) := \{z \in \mathbb{C} : |\varphi(z)| = 1\}$, as well as (note that for $z \in [\,i,-i\,]$ we have $|\varphi(z)| = 1$)

$$\mathcal{E}(\mathcal{C}) := \{z \in \mathbb{C} : |\varphi(z)| > 1\} = \{z \in \mathbb{C} \setminus [\,i,-i\,] : |\varphi(z)| > 1\}$$

and

$$\mathcal{I}(\mathcal{C}) := \{z \in \mathbb{C} : |\varphi(z)| < 1\} = \{z \in \mathbb{C} \setminus [\,i,-i\,] : |\varphi(z)| < 1\}.$$

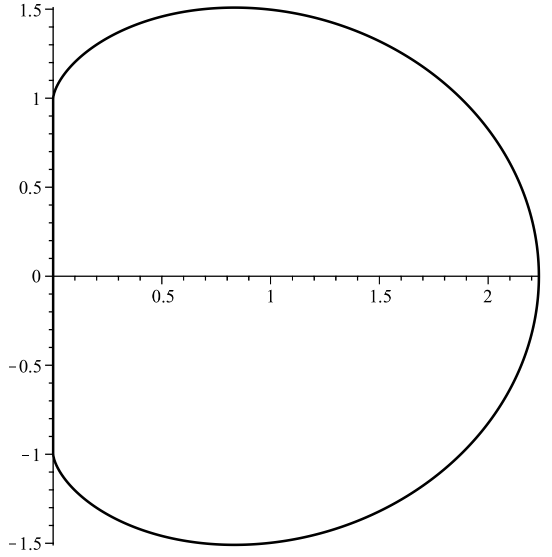

Fig. 1. The curve $\mathcal{C}$.

Lemma 2.1. *The set $\mathcal{E}(\mathcal{C})$ is a domain and $\varphi : \mathcal{E}(\mathcal{C}) \to \overline{\mathbb{D}}^c$ is a conformal mapping which possesses an homeomorphic extension $\varphi : \overline{\mathcal{E}(\mathcal{C})} \to \mathbb{C} \setminus \mathbb{D}$ such that $\partial\mathcal{E}(\mathcal{C}) = \mathrm{Im}(\mathcal{C})$ is mapped onto $\partial\mathbb{D}$. Moreover, $\mathrm{Im}(\mathcal{C})$ is the image of a simple closed and continuously differentiable curve $\mathcal{C}$, the exterior and interior of which are given by $\mathcal{E}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ respectively (see Fig. 1).*

The proof essentially is based on an application of Rouché's Theorem and a theorem of Caratheodory. It should be pointed out that the curve $\mathcal{C}$ is continuously differentiable at all points but it fails to be analytic in the points $z = i$ and $z = -i$. We will omit the proof of Lemma 2.1 here, details can be found in [7, pp. 64–68].

Now we will state several lemmata in order to prepare for the proof of our main results. In doing so, because of symmetry we can restrict our attention to the closed upper half-plane $\{z \in \mathbb{C} \mid \Im(z) \geq 0\}$. In the sequel we will always use the branch of argument given by $\arg(w) \in [0, 2\pi)$ for all $w \in \mathbb{C} \setminus \{0\}$ and α will be a positive integer. Moreover, let $a(z) := \arg(z + \sqrt{z^2+1})$ for $z \in \mathbb{C}$.

We begin by proving an integral representation for normalized generalized associated Ménage polynomials. Therefore, considering $z \in \mathbb{C} \setminus [\,i, -i\,]$ as a parameter, let the straight lines $J_1(z),\ J_2(z)$ be defined by

$$J_1(z) : w_1(t) := i\pi + it\,(a(z) - \pi), \quad 0 \leq t \leq 1,$$
$$J_2(z) : w_2(t) := t + i\,a(z), \quad t \geq 0.$$

Lemma 2.2. *Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus [\,i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$. Then we have*

$$ {}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\,\middle|\, \frac{z}{2n}\right) = \frac{(-1)^n}{\Gamma(\alpha)} \left(\frac{n}{z}\right)^{\alpha} e^{-\frac{n}{z}} \times \int_{K(z)} e^{-n\frac{\cosh(w)}{z}}\, (\cosh(w)+1)^{\alpha-1} \cosh(nw) \sinh(w)\, \mathrm{d}w,$$

where $K(z) := J_1(z) + J_2(z)$, that is, the path $K(z)$ is formed by connecting $J_1(z)$ and $J_2(z)$ consecutively.

Proof. Using a standard hypergeometric representation for Chebyshev polynomials of the first kind T_n (see, e.g. [3, p. 81], $\alpha = \beta = -\frac{1}{2}$ there), it is easy to see that (cf. the integral representation for Ménage numbers in [2, p. 858])

$$\begin{aligned}{}_3F_1\left(\begin{matrix}-n, & n, & \alpha\\ & \frac{1}{2} & \end{matrix}\,\middle|\,\frac{z}{2n}\right) &= \frac{(-1)^n}{\Gamma(\alpha)}\int_0^\infty e^{-t}\,t^{\alpha-1}\,T_n\left(\frac{zt}{n}-1\right)\mathrm{d}t\\ &= \frac{(-1)^n}{\Gamma(\alpha)}\,n^\alpha\int_0^\infty e^{-ns}\,s^{\alpha-1}\,T_n\left(zs-1\right)\mathrm{d}s\\ &= \frac{(-1)^n}{\Gamma(\alpha)}\left(\frac{n}{z}\right)^\alpha e^{-\frac{n}{z}}\int_{K_1(z)} e^{-\frac{ny}{z}}\,(y+1)^{\alpha-1}\,T_n(y)\mathrm{d}y,\end{aligned}$$

where the path $K_1(z)$ is given by

$$K_1(z) : y_1(t) := zt - 1, \quad 0 \le t < \infty.$$

Let the mapping

$$\mathbb{C}\setminus[-1,1] \to \overline{\mathbb{D}}^c, \qquad y \mapsto y + \sqrt{y^2-1}$$

be defined as the inverse mapping of

$$\overline{\mathbb{D}}^c \to \mathbb{C}\setminus[-1,1], \qquad t \mapsto \frac{1}{2}\left(t + \frac{1}{t}\right),$$

which maps the exterior of the unit disk conformally onto $\mathbb{C}\setminus[-1,1]$ preserving the point at infinity. Let $y \in \mathbb{C}\setminus[-1,\infty)$, then we define $\operatorname{arcosh}(y) := \log(y + \sqrt{y^2-1})$, where the branch of the logarithm is determined uniquely by the inequality $0 < \Im(\log(y + \sqrt{y^2-1})) < 2\pi$. Moreover, on the cut $[-1,\infty)$ we define the values of arcosh by the stipulation that the interval $[-1,1]$ is mapped onto the line segment connecting the points $i\pi$ and 0, and the interval $(1,\infty)$ is mapped onto $(0,\infty)$. By this setting, the mapping arcosh is defined properly on $\mathbb{C}$ and clearly it is holomorphic on $\mathbb{C}\setminus[-1,\infty)$. Let the path $K_2(z)$ be defined by

$$K_2(z) : w_2(t) := \operatorname{arcosh}(zt-1), \quad 0 \le t < \infty.$$

Using $y = \cosh(w)$ and the identity $T_n(\cosh(w)) = \cosh(nw)$, we find

$$\begin{aligned}{}_3F_1\left(\begin{matrix}-n, & n, & \alpha\\ & \frac{1}{2} & \end{matrix}\,\middle|\,\frac{z}{2n}\right) &= \frac{(-1)^n}{\Gamma(\alpha)}\left(\frac{n}{z}\right)^\alpha e^{-\frac{n}{z}}\int_{K_1(z)} e^{-\frac{ny}{z}}\,(y+1)^{\alpha-1}\,T_n(y)\mathrm{d}y\\ &= \frac{(-1)^n}{\Gamma(\alpha)}\left(\frac{n}{z}\right)^\alpha e^{-\frac{n}{z}}\int_{K_2(z)} e^{-n\frac{\cosh(w)}{z}}\,(\cosh(w)+1)^{\alpha-1}\\ &\quad\times\cosh(nw)\sinh(w)\mathrm{d}w.\end{aligned}$$

Now, by applying Cauchy's integral theorem, it is not difficult to see that the path $K_2(z)$ in the last integral can be replaced by the path $K(z)$ defined in the assumptions above (see Figs. 2 and 3, details can be found in [7, pp. 38–41]). □

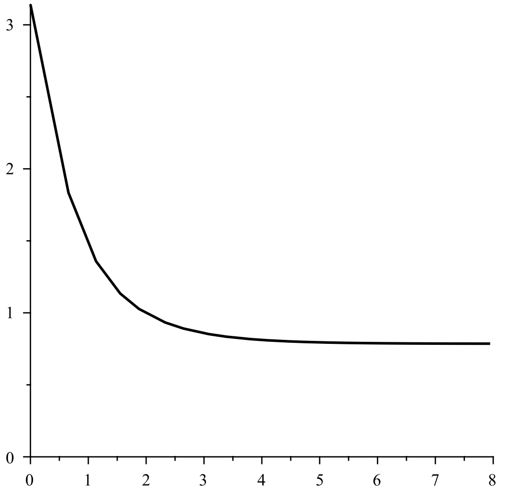

Fig. 2. The path $K_2(z)$ for $z = 1 + i$.

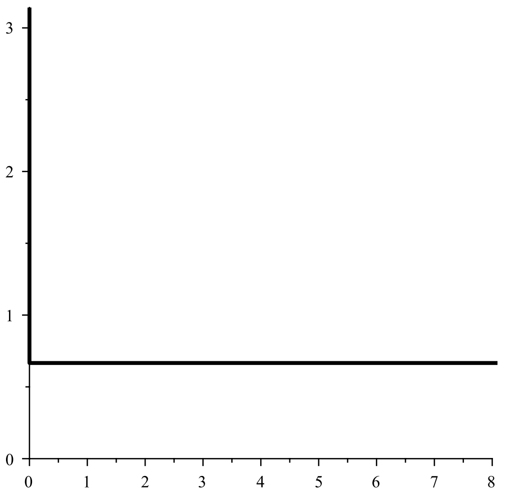

Fig. 3. The Path $K(z)$ for $z = 1 + i$.

In order to apply Theorem 2.1, we will give a slight modification of the representation in Lemma 2.2. Therefore, let the function $f : \mathbb{C} \times \mathbb{C} \setminus [\, i, -i\,] \times \mathbb{N} \to \mathbb{C}$ be defined by

$$f(w, z, n) := e^{-n\left(\frac{\cosh(w)}{z} - w\right)} (\cosh(w) + 1)^{\alpha - 1} \sinh(w).$$

Furthermore, for $z \in \mathbb{C} \setminus [\, i, -i\,]$ let the straight lines $\check{J}_1(z),\ \check{J}_2(z)$ be given by

$$\check{J}_1(z) : \check{w}_1(t) := -i\pi - it\,(a(z) - \pi), \quad 0 \le t \le 1,$$
$$\check{J}_2(z) : \check{w}_2(t) := -t - i\,a(z), \quad t \ge 0.$$

Lemma 2.3. *Let $n \in \mathbb{N}$ and $z \in \mathbb{C} \setminus [\,i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$. Then we have*

$$ {}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\middle| \frac{z}{2n}\right) = \frac{(-1)^n}{2\,\Gamma(\alpha)}\left(\frac{n}{z}\right)^{\alpha} e^{-\frac{n}{z}}\left\{\int_{J_1(z)} f(w,z,n)\,\mathrm{d}w \right. $$
$$ \left. + \int_{J_2(z)} f(w,z,n)\,\mathrm{d}w + \int_{\check{J}_1(z)} f(w,z,n)\,\mathrm{d}w + \int_{\check{J}_2(z)} f(w,z,n)\,\mathrm{d}w \right\}. $$

This representation follows without difficulty from Lemma 2.2. We now want to study the asymptotic behavior of the integrals appearing in Lemma 2.3, whereat Theorem 2.1 will emerge to be a crucial tool. In doing so, we will start our studies with the integral which turns out to provide the main contribution to the asymptotic behavior of F_n in the exterior $\mathcal{E}(\mathcal{C})$.

Lemma 2.4. *Let K be a nonempty compact subset of $\mathbb{C} \setminus [\,i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$. Then we have*

$$ \int_{J_2(z)} f(w,z,n)\,\mathrm{d}w = \sqrt{\frac{2\pi}{n}}\left(1+\sqrt{z^2+1}\right)^{\alpha-1} z \left(\frac{\sqrt{z^2+1}}{z}\right)^{-\frac{1}{2}} $$
$$ \times (z+\sqrt{z^2+1})^n\, e^{-n\frac{\sqrt{z^2+1}}{z}} \left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), $$

as $n \to \infty$, holding uniformly with respect to $z \in K$.

Proof. Let $\varphi_z(w) := \frac{\cosh(w)}{z} - w$ for $z \in \mathbb{C} \setminus [\,i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$ and $w \in \mathbb{C}$, then we have $\frac{\mathrm{d}}{\mathrm{d}w}\varphi_z(w) = \frac{\sinh(w)}{z} - 1$. Consequently, considering a fixed z, there is exactly one point w lying in the strip $\{w \in \mathbb{C} \mid \Re(w) > 0,\ 0 \leq \Im(w) < 2\pi\}$ which is a simple zero of $\frac{\mathrm{d}}{\mathrm{d}w}\varphi_z(w)$. This point is given by $w = \operatorname{arsinh}(z) := \log|z+\sqrt{z^2+1}| + i\,a(z)$, where $\log|z+\sqrt{z^2+1}|$ is defined to be real and positive. We now consider the function $\Re(\varphi_z(w))$ on the path $J_2(z)$. Using the appropriate parametrization, we find for $t \geq 0$

$$ \Re(\varphi_z(w_2(t))) = \Re(\varphi_z(t+ia(z))) = \Re\left\{\frac{\cosh(t+ia(z))}{z}\right\} - t $$
$$ = -t + \Re\left(\frac{1}{z}\right)\cos(a(z))\cosh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\sinh(t). $$

Hence, by differentiation

$$ \frac{\mathrm{d}}{\mathrm{d}t}\Re(\varphi_z(w_2(t))) = -1 + \Re\left(\frac{1}{z}\right)\cos(a(z))\sinh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\cosh(t) $$
$$ = -1 + \Re\left\{\frac{\sinh(t+ia(z))}{z}\right\}. $$

Therefore, if z is fixed, the function $\frac{\mathrm{d}}{\mathrm{d}t}\Re(\varphi_z(w_2(t)))$ is easily seen to be strictly increasing on $[0,\infty)$ and vanishing at $t = \Re(\operatorname{arsinh}(z))$. This implies, that the function $\Re(\varphi_z(w_2(t)))$ is strictly decreasing on $[0, \Re(\operatorname{arsinh}(z)))$, attaining its minimum at $t = \Re(\operatorname{arsinh}(z))$, and strictly increasing on $(\Re(\operatorname{arsinh}(z)), \infty)$. According to this, the path $J_2(z)$ passes through a simple saddle point of $\varphi_z(w)$, such that the function $\Re(\varphi_z(w))$ attains its minimum at this point, which actually gives reason for the choice of $J_2(z)$. By splitting $J_2(z)$ at $t = \Re(\operatorname{arsinh}(z))$ into two parts, we obtain

$$\begin{aligned}\int_{J_2(z)} f(w,z,n)\,\mathrm{d}w &= \int_0^1 f((1-t)\Re(\operatorname{arsinh}(z)) + ia(z), z, n)\,\Re(\operatorname{arsinh}(z))\,\mathrm{d}t \\ &\quad + \int_0^\infty f(t+\Re(\operatorname{arsinh}(z)) + ia(z), z, n)\,\mathrm{d}t \\ &=: I_1(n,z) + I_2(n,z). \end{aligned} \tag{2.3}$$

It is not difficult to see (details can be found in [7, pp. 45–46]) that all assumptions of Theorem 2.1 are satisfied for both integrals $I_1(n,z)$ and $I_2(n,z)$, and we obtain approximations of the form ($\omega = 0$, $\mu = 2$ and $\lambda = 1$ there)

$$\begin{aligned} I_1(n,z) &= e^{-n\left(\frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z)\right)} \left(\sqrt{\pi}\,\frac{a_0(z)}{\sqrt{n}} - \frac{a_1(z)}{n} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right), \\ I_2(n,z) &= e^{-n\left(\frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z)\right)} \left(\sqrt{\pi}\,\frac{a_0(z)}{\sqrt{n}} + \frac{a_1(z)}{n} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right), \end{aligned}$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. Note, that the leading coefficients coincide, whereas the second coefficients differ by their signs. Hence, we have

$$I_1(n,z) + I_2(n,z) = e^{-n\left(\frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z)\right)} \left(2\sqrt{\pi}\,\frac{a_0(z)}{\sqrt{n}} + \mathcal{O}\left(\frac{1}{n^{3/2}}\right)\right), \tag{2.4}$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. In order to calculate the coefficient $a_0(z)$, we consider the integral $I_2(n,z)$. Thus

$$I_2(n,z) = \int_0^\infty e^{-np_z(t)}\, q_z(t)\,\mathrm{d}t,$$

where, using some standard trigonometric identities, we have

$$\begin{aligned} p_z(t) &= \frac{\sqrt{z^2+1}}{z}\cosh(t) + \sinh(t) - t - \operatorname{arsinh}(z), \\ q_z(t) &= \left(\sqrt{z^2+1}\cosh(t) + z\sinh(t) + 1\right)^{\alpha-1} \left(\sqrt{z^2+1}\sinh(t) + z\cosh(t)\right). \end{aligned}$$

Consequently, in the notations of Theorem 2.1 we find $p_z(0) = \frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z)$, $b_0(z) = \frac{\sqrt{z^2+1}}{2z}$ and $c_0(z) = z\left(1+\sqrt{z^2+1}\right)^{\alpha-1}$. Hence, we obtain

$$a_0(z) = \frac{c_0(z)}{\mu(b_0(z))^{\lambda/\mu}} = \frac{z}{\sqrt{2}}\left(1+\sqrt{z^2+1}\right)^{\alpha-1}\left(\frac{\sqrt{z^2+1}}{z}\right)^{-\frac{1}{2}},$$

from which, by virtue of (2.3) and (2.4), the statement follows. □

Next we will treat those summands appearing in Lemma 2.3 which turn out to provide the main contributions to the asymptotic behavior of F_n in the interior $\mathcal{I}(\mathcal{C})$.

Lemma 2.5. *Let K be a nonempty compact subset of $\left\{z \in \mathbb{C} \setminus [\,i, -i\,] \mid \arg(z) \in \left[0, \frac{\pi}{2}\right)\right\}$. Then we have*

$$\begin{aligned}\int_{J_1(z)} f(w, z, n)\,\mathrm{d}w &+ \int_{\check{J}_1(z)} f(w, z, n)\,\mathrm{d}w \\ &= \frac{e^{\frac{n}{z}}(-1)^n\Gamma(2\alpha)}{n^{2\alpha}}\left(\frac{1}{2}\right)^{\alpha-1} 2\cos(\alpha\pi)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right),\end{aligned}$$

as $n \to \infty$, holding uniformly with respect to $z \in K$.

Proof. By using the parametrization $w_1(t) = i\pi + it(a(z) - \pi),\ 0 \le t \le 1$, we obtain

$$\begin{aligned}\int_{J_1(z)} f(w, z, n)\,\mathrm{d}w &= i(a(z) - \pi)\int_0^1 f(w_1(t), z, n)\,\mathrm{d}t \\ &= (a(z) - \pi)\int_0^1 e^{-np_z(t)}\,q_z(t)\,\mathrm{d}t,\end{aligned}$$

where, applying some standard trigonometric identities, we have

$$\begin{aligned}p_z(t) &= -\frac{1}{z}\cos(t(a(z) - \pi)) - i\pi - it(a(z) - \pi), \\ q_z(t) &= (1 - \cos(t(a(z) - \pi)))^{\alpha-1}\sin(t(a(z) - \pi)).\end{aligned}$$

Taking the real part of $p_z(t)$ yields $\Re(p_z(t)) = -\Re\left(\frac{1}{z}\right)\cos(t(a(z) - \pi))$, which implies that for a fixed z with $\arg(z) \in \left[0, \frac{\pi}{2}\right)$ the function $\Re(p_z(t))$ is strictly increasing on $[0, 1]$. Furthermore, using the notations of Theorem 2.1, we have ($\omega = 0$, $\mu = 1$ and $\lambda = 2\alpha$ there) $p_z(0) = -\frac{1}{z} - i\pi$, $b_0(z) = -i(a(z) - \pi)$ and $c_0(z) = \left(\frac{1}{2}\right)^{\alpha-1}(a(z) - \pi)^{2\alpha-1}$, which implies

$$a_0(z) = \frac{c_0(z)}{\mu(b_0(z))^{\lambda/\mu}} = \left(\frac{1}{2}\right)^{\alpha-1}\frac{(-1)^\alpha}{a(z) - \pi}.$$

It is easy to see that all assumptions of Theorem 2.1 are satisfied, and we readily obtain

$$\int_{J_1(z)} f(w, z, n)\,\mathrm{d}w = e^{-n\left(-\frac{1}{z} - i\pi\right)}\left(\Gamma(2\alpha)\left(\frac{1}{2}\right)^{\alpha-1}\frac{(-1)^\alpha}{n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{2\alpha+1}}\right)\right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. It can be shown in a similar manner that

$$\int_{\check{J}_1(z)} f(w, z, n)\,\mathrm{d}w = e^{-n\left(-\frac{1}{z} + i\pi\right)}\left(\Gamma(2\alpha)\left(\frac{1}{2}\right)^{\alpha-1}\frac{(-1)^\alpha}{n^{2\alpha}} + \mathcal{O}\left(\frac{1}{n^{2\alpha+1}}\right)\right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$, from which statement follows. □

Finally we state some lemmata which serve the purpose of estimating the remainders in the proof of the strong asymptotics for F_n. For this we have to distinguish between the right and the left half-plane.

Lemma 2.6. *Let K be a nonempty compact subset of $\left\{z \in \mathbb{C} \setminus [\, i, -i\,] \mid \arg(z) \in \left[0, \frac{\pi}{2}\right]\right\}$. Then there is a constant $M > 0$ such that for all $n \in \mathbb{N}$ and $z \in K$ we have*

$$\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le M \quad \textit{and} \quad \left| e^{-\frac{n}{z}} \int_{\check{J}_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le M.$$

Proof. We have for $n \in \mathbb{N}$ and $z \in K$

$$\begin{aligned}\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| &\le (\pi - a(z)) \int_0^1 e^{-n\Re\left(\frac{1}{z}\right)(1-\cos\{t(\pi - a(z))\})} \\ &\quad \times |1 - \cos\{t(\pi - a(z))\}|^{\alpha-1}\ |\sin\{t(\pi - a(z))\}|\, \mathrm{d}t,\end{aligned}$$

and by using the estimate $\Re\left(\frac{1}{z}\right)(1 - \cos\{t(\pi - a(z))\}) \ge 0$, we obtain

$$\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le (\pi - a(z)) 2^{\alpha-1} \le \pi\, 2^{\alpha-1}.$$

The second statement can be established in a similar manner. □

Lemma 2.7. *Let K be a nonempty compact subset of $\left\{z \in \mathbb{C} \setminus [\, i, -i\,] \mid \arg(z) \in \left[\frac{\pi}{2}, \pi\right]\right\}$. Then there is a constant $M > 0$ such that for all $n \in \mathbb{N}$ and $z \in K$ we have*

$$\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le M \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n,$$

$$\left| e^{-\frac{n}{z}} \int_{\check{J}_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le M \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n.$$

Proof. By using the parametrization $w(t) = ia(z) + it(\pi - a(z)),\ 0 \le t \le 1$, we have for all $n \in \mathbb{N}$ and $z \in K$

$$\begin{aligned}&\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le \left| e^{-\frac{n}{z}} (\pi - a(z)) \right| e^{-n\Re\left(\frac{1}{z}\right)\cos(a(z))} \\ &\quad \times \int_0^1 \exp\left\{ -n \left(\Re\left(\frac{1}{z}\right) \cos(a(z) + t(\pi - a(z))) - \Re\left(\frac{1}{z}\right) \cos(a(z)) \right) \right\} \\ &\quad \times |1 + \cos(a(z) + t(\pi - a(z)))|^{\alpha-1}\, |\sin(a(z) + t(\pi - a(z)))|\, \mathrm{d}t.\end{aligned}$$

Applying the estimate $\Re\left(\frac{1}{z}\right)\cos(a(z) + t(\pi - a(z))) - \Re\left(\frac{1}{z}\right)\cos(a(z)) \ge 0$, we obtain

$$\left| e^{-\frac{n}{z}} \int_{J_1(z)} f(w, z, n)\, \mathrm{d}w \right| \le \pi\, 2^{\alpha-1} \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n.$$

Again, the second statement can be established in a similar manner. □

Lemma 2.8. *Let K be a compact subset of $\mathbb{C} \setminus [\, i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \ge 0\}$. Then there is a constant $M > 0$ such that for all $n \in \mathbb{N}$ and $z \in K$ we have*

$$\left| e^{-\frac{n}{z}} \int_{\check{J}_2(z)} f(w, z, n)\, \mathrm{d}w \right| \le M \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n.$$

Proof. By using the parametrization $w(t) = -ia(z) - t,\ t \geq 0$, we have for all $n \in \mathbb{N}$ and $z \in K$

$$\begin{aligned}
&\left| e^{-\frac{n}{z}} \int_{\check{J}_2(z)} f(w,z,n)\,\mathrm{d}w \right| \\
&\quad = \left| e^{-\frac{n}{z}} \int_0^\infty e^{-n\left(\frac{\cosh(w(t))}{z} - w(t)\right)} (\cosh(w(t)) + 1)^{\alpha-1} \sinh(w(t))\,\mathrm{d}t \right| \\
&\quad \leq \left| e^{-\frac{n}{z}} \right| \int_0^\infty e^{-np_z(t)} \, |\cosh(w(t)) + 1|^{\alpha-1} \, |\sinh(w(t))|\,\mathrm{d}t,
\end{aligned}$$

where $p_z(t) = t + \Re\left(\frac{1}{z}\right)\cos(a(z))\cosh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\sinh(t)$. By differentiation we obtain $\frac{\mathrm{d}}{\mathrm{d}t} p_z(t) = 1 + \Re\left(\frac{1}{z}\right)\cos(a(z))\sinh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\cosh(t) \geq 0$, showing that for a fixed z the function $p_z(t)$ is strictly increasing on the interval $[0, \infty)$. By using the estimate ($t \geq 0$)

$$\begin{aligned}
0 \leq p_z(t) - p_z(0) = t + \Re\left(\frac{1}{z}\right)\cos(a(z))\cosh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\sinh(t) \\
- \Re\left(\frac{1}{z}\right)\cos(a(z)),
\end{aligned}$$

we find

$$\begin{aligned}
&\left| e^{-\frac{n}{z}} \int_{\check{J}_2(z)} f(w,z,n)\,\mathrm{d}w \right| \leq \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n e^{\Re\left(\frac{1}{z}\right)\cos(a(z))} \\
&\quad \times \int_0^\infty \exp\left\{ -\left(t + \Re\left(\frac{1}{z}\right)\cos(a(z))\cosh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\sinh(t) \right) \right\} \\
&\quad \times |\cosh(w(t)) + 1|^{\alpha-1} \, |\sinh(w(t))|\,\mathrm{d}t. \qquad (2.5)
\end{aligned}$$

Furthermore, for $t \geq 0$ we have the estimates $|\cosh(-t - ia(z)) + 1| \leq 2e^t$ and $|\sinh(-t - ia(z))| \leq e^t$. Let the constants m_1 and m_2 be defined by

$$\begin{aligned}
m_1 &:= \min_{z\in K} \left\{ \frac{1}{2}\Re\left(\frac{1}{z}\right)\cos(a(z)) - \frac{1}{2}\Im\left(\frac{1}{z}\right)\sin(a(z)) \right\} > 0 \\
m_2 &:= \min_{z\in K} \left\{ \frac{1}{2}\Re\left(\frac{1}{z}\right)\cos(a(z)) + \frac{1}{2}\Im\left(\frac{1}{z}\right)\sin(a(z)) \right\},
\end{aligned}$$

then for $t \geq 0$ we obtain

$$t + \Re\left(\frac{1}{z}\right)\cos(a(z))\cosh(t) - \Im\left(\frac{1}{z}\right)\sin(a(z))\sinh(t) \geq t + m_1 e^t + m_2 e^{-t}.$$

Applying this to (2.5) yields

$$\begin{aligned}
\left| e^{-\frac{n}{z}} \int_{\check{J}_2(z)} f(w,z,n)\,\mathrm{d}w \right| &\leq \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right|^n e^{\Re\left(\frac{1}{z}\right)\cos(a(z))} \\
&\quad \times 2^{\alpha-1} \int_0^\infty \exp\left\{-t - m_1 e^t - m_2 e^{-t}\right\} e^{\alpha t}\,\mathrm{d}t,
\end{aligned}$$

where the expression $e^{\Re\left(\frac{1}{z}\right)\cos(a(z))}$ is bounded on K. This completes the proof. $\square$

Now we turn to the last lemma which we need for the proof of the strong asymptotics.

Lemma 2.9. *Let K be a nonempty compact subset of $\mathbb{C} \setminus [\, i, -i\,] \cap \{z \in \mathbb{C} \mid \Im(z) \geq 0\}$. Then we have*

$$\sup_{z\in K} \left| \frac{e^{-\frac{1}{z}-\frac{1}{z}\cos(a(z))}}{(z+\sqrt{z^2+1})\, e^{-\frac{1}{z}-\frac{\sqrt{z^2+1}}{z}}} \right| < 1.$$

Proof. Exploiting the continuity of all functions involved, it suffices to prove for all $z \in K$

$$\left| e^{\frac{\sqrt{z^2+1}}{z}} \, (z+\sqrt{z^2+1})^{-1} \right| < \left| e^{\frac{1}{z}\cos(a(z))} \right|.$$

However, this is equivalent to

$$\Re\left\{ \frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z) \right\} < \Re\left\{ \frac{1}{z}\cos(a(z)) \right\}, \qquad z \in K.$$

In order to prove this inequality we define $p_z(t) := \Re\left(\frac{\cosh(w(t))}{z} - w(t) \right)$, $t \geq 0$, where we use $w(t) = t + ia(z)$. Then we have $p_z(\Re(\operatorname{arsinh}(z))) = \Re\left\{ \frac{\sqrt{z^2+1}}{z} - \operatorname{arsinh}(z) \right\}$ and $p_z(0) = \Re\left\{ \frac{1}{z}\cos(a(z)) \right\}$. From the proof of Lemma 2.4 we know that for fixed z the function $p_z(t)$ is strictly decreasing on the interval $[0, \Re(\operatorname{arsinh}(z))]$, which furnishes the proof. $\square$

3. Strong asymptotics

By gathering the results of the previous section, we now can prove the following strong asymptotics for generalized associated Ménage polynomials. Note, that by symmetry or by analogous arguments, all the statements of the Lemmatas 2.2–2.9 can be established in appropriate forms with respect to the lower half-plane. The notions $\mathcal{E}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ are defined in the context of Lemma 2.1 above.

Theorem 3.1. *Let α be a positive integer, then we have the following statements:*

(i) *Let K be a nonempty compact subset of $\mathcal{E}(\mathcal{C})$, then we have*

$$ {}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix} \,\middle|\, \frac{z}{2n} \right) = \frac{(-1)^n}{\Gamma(\alpha)}\, n^{\alpha-\frac{1}{2}} \sqrt{\frac{\pi}{2}} \left(\frac{1}{z} + \frac{\sqrt{z^2+1}}{z} \right)^{\alpha-1} \left(\frac{\sqrt{z^2+1}}{z} \right)^{-\frac{1}{2}} \times \left\{ (z+\sqrt{z^2+1})\, e^{-\frac{1}{z}-\frac{\sqrt{z^2+1}}{z}} \right\}^n \left(1 + \mathcal{O}\left(\frac{1}{n} \right) \right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$.

(ii) *Let K be a nonempty compact subset of $\mathcal{I}(\mathcal{C})$, then we have*

$$ {}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix} \,\middle|\, \frac{z}{2n} \right) = \left(\frac{2}{n} \right)^{\alpha} \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}} \left(\frac{1}{z} \right)^{\alpha} \cos(\alpha\pi) \left(1 + \mathcal{O}\left(\frac{1}{n} \right) \right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$.

Proof. We may restrict our attention to compact subsets of $\{z \in \mathbb{C} \setminus [i, -i] \mid \Im(z) \geq 0\}$. Let $c_n(z) = \frac{(-1)^n}{2\Gamma(\alpha)} \left(\frac{n}{z}\right)^\alpha$, then by virtue of Lemma 2.3, we have

$$
\begin{aligned}
{}_3F_1\left(\left.\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\right| \frac{z}{2n}\right) &= c_n(z)e^{-\frac{n}{z}} \int_{J_1(z)} f(w,z,n)\,\mathrm{d}w + c_n(z)e^{-\frac{n}{z}} \\
&\quad \times \int_{J_2(z)} f(w,z,n)\,\mathrm{d}w \\
&\quad + c_n(z)e^{-\frac{n}{z}} \int_{\check{J}_1(z)} f(w,z,n)\,\mathrm{d}w + c_n(z)e^{-\frac{n}{z}} \\
&\quad \times \int_{\check{J}_2(z)} f(w,z,n)\,\mathrm{d}w \\
&=: I_1(n,z) + I_2(n,z) + I_3(n,z) + I_4(n,z). \qquad (3.1)
\end{aligned}
$$

Using Lemma 2.4, we obtain $I_2(n,z) = H_n(z)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$, as $n \to \infty$, holding uniformly on compact subsets of $\{z \in \mathbb{C} \setminus [i, -i] \mid \Im(z) \geq 0\}$, where

$$
\begin{aligned}
H_n(z) &:= \frac{(-1)^n}{\Gamma(\alpha)} n^{\alpha - \frac{1}{2}} \sqrt{\frac{\pi}{2}} \left(\frac{1}{z} + \frac{\sqrt{1+z^2}}{z}\right)^{\alpha-1} \left(\frac{\sqrt{1+z^2}}{z}\right)^{-\frac{1}{2}} \\
&\quad \times \left\{(z + \sqrt{z^2+1})\, e^{-\frac{1}{z} - \frac{\sqrt{z^2+1}}{z}}\right\}^n.
\end{aligned}
$$

(i) (1) First, let K be a nonempty compact subset of

$$\mathcal{E}(\mathcal{C}) \cap \left\{z \in \mathbb{C} \setminus [i, -i] \mid \arg(z) \in \left[0, \frac{\pi}{2}\right]\right\},$$

then, using Lemma 2.6, we find

$$\frac{I_1(n,z)}{H_n(z)} = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{I_3(n,z)}{H_n(z)} = \mathcal{O}\left(\frac{1}{n}\right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. Furthermore, combining Lemmatas 2.8 and 2.9, we obtain

$$\frac{I_4(n,z)}{H_n(z)} = \mathcal{O}\left(\frac{1}{n}\right), \qquad (3.2)$$

as $n \to \infty$, holding uniformly with respect to $z \in K$.

(2) Now let K be a nonempty compact subset of

$$\mathcal{E}(\mathcal{C}) \cap \left\{z \in \mathbb{C} \setminus [i, -i] \mid \arg(z) \in \left[\frac{\pi}{2}, \pi\right]\right\},$$

then the combination of Lemmas 2.7 and 2.9 yields

$$\frac{I_1(n,z)}{H_n(z)} = \mathcal{O}\left(\frac{1}{n}\right) \quad \text{and} \quad \frac{I_3(n,z)}{H_n(z)} = \mathcal{O}\left(\frac{1}{n}\right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. Moreover, the assertion (3.2) remains valid. By virtue of (3.1), this completes the proof of the statement (i).

(ii) Let K be a nonempty compact subset of $\mathcal{I}(\mathcal{C}) \cap \{z \in \mathbb{C} \setminus [\,i, -i\,] \mid \Im(z) \geq 0\}$. It is easy to see, that K therefore is a subset of $\left\{z \in \mathbb{C} \setminus [\,i, -i\,] \mid \ \arg(z) \in \left[0, \frac{\pi}{2}\right)\right\}$. Using Lemma 2.5 and the duplication formula for the gamma function (see, e.g., [3, p. 8]), we obtain

$$\begin{aligned} I_1(n,z) + I_3(n,z) &= \frac{1}{n^\alpha} \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \frac{1}{z^\alpha} \left(\frac{1}{2}\right)^{\alpha-1} \cos(\alpha\pi) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \\ &= G_n(z) \left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right), \end{aligned}$$

as $n \to \infty$, holding uniformly with respect to $z \in K$, where

$$G_n(z) := \left(\frac{2}{n}\right)^\alpha \frac{\Gamma\left(\alpha + \frac{1}{2}\right)}{\sqrt{\pi}} \left(\frac{1}{z}\right)^\alpha \cos(\alpha\,\pi).$$

Moreover, we have

$$\sup_{z \in K} \left| (z + \sqrt{z^2+1})\, e^{-\frac{1}{z} - \frac{\sqrt{z^2+1}}{z}} \right| < 1,$$

hence, by Lemma 2.9, we find

$$\sup_{z \in K} \left| e^{-\frac{1}{z} - \frac{1}{z}\cos(a(z))} \right| < 1.$$

This implies

$$\frac{I_2(n,z)}{G_n(z)} = \mathcal{O}\left(\frac{1}{n}\right),$$

and by Lemma 2.8 we obtain

$$\frac{I_4(n,z)}{G_n(z)} = \mathcal{O}\left(\frac{1}{n}\right),$$

as $n \to \infty$, holding uniformly with respect to $z \in K$. By virtue of (3.1), this finishes the proof. $\square$

4. Weak asymptotics and further properties of the zeros

We begin by stating some additional results which are needed for the study of the behavior of the zeros in this section.

Theorem 4.1. *Let P and Q be polynomials of degree $n \in \mathbb{N}$ of the form*

$$P(z) = \sum_{k=0}^{n} \binom{n}{k} a_k\, z^k, \qquad Q(z) = \sum_{k=0}^{n} \binom{n}{k} b_k\, z^k,$$

such that the moduli of all roots of P are not greater than $r > 0$ and the moduli of all roots of Q are not greater than $s > 0$. Then the moduli of all roots of the polynomial

$$R(z) = \sum_{k=0}^{n} \binom{n}{k} a_k\, b_k\, z^k$$

are not greater than rs.

For a proof of Theorem 4.1, see [15]. In order to state the following theorem, Szegö introduced the German notion *Abbildungskonstante*. However, it is easy to see that it can be replaced by use of the more contemporary notion of logarithmic capacity.

Theorem 4.2. *Let $\mathcal{K}$ be a closed simple curve in the complex plane consisting of finitely many analytic arcs. By ρ we denote the logarithmic capacity of the closed interior of $\mathcal{K}$. Let (n_k) be a subsequence of the positive integers and for every index n_k let P_{n_k} be a polynomial of the form*

$$P_{n_k}(x) = a_0^{(n_k)} + a_1^{(n_k)} x + \cdots + a_{n_k}^{(n_k)} x^{n_k} \quad (a_{n_k}^{(n_k)} \neq 0).$$

Furthermore, suppose that the sequence (P_{n_k}) is uniformly convergent on every compact subset of $\mathcal{I}(\mathcal{K})$ such that its limit function does not vanish identically. Then we have

(i)
$$\limsup_{k\to\infty} |a_{n_k}^{(n_k)}|^{1/n_k} \leq \frac{1}{\rho}.$$

(ii) *If the inequality in* (i) *is an equality, then every point on the curve $\mathcal{K}$ is an accumulation point of zeros of the polynomials P_{n_k}.*

For the proof of Theorem 4.2 see [16]. For the purpose of determining the Radon–Nikodym derivative of an equilibrium measure, we will need the following result.

Theorem 4.3. *Let $K \subset \mathbb{C}$ be a compact set and let μ_K be the equilibrium measure for K such that the intersection of its support S with a domain is a simple $C^{1+\delta}$-curve γ for some $\delta > 0$. Then the restriction of μ_K to γ is absolutely continuous with respect to the arc measure and we have*

$$\mathrm{d}\mu_K(z) = -\frac{1}{2\pi}\left(\frac{\partial \mathcal{U}^{\mu_K}}{\partial n_+}(z) + \frac{\partial \mathcal{U}^{\mu_K}}{\partial n_-}(z)\right)|\mathrm{d}z|, \quad z \in \gamma,$$

where $\frac{\partial \mathcal{U}^{\mu_K}}{\partial n_+}$ and $\frac{\partial \mathcal{U}^{\mu_K}}{\partial n_-}$ denote differentiation of the equilibrium potential in the direction of the two normals to γ.

Theorem 4.3 is a special case of a theorem in [11, p. 211]. For the definition of the above mentioned $C^{1+\delta}$-curves, see [11, p. 89]. Now we will state some potential theoretic results on the curve $\mathcal{C}$ defined in Lemma 2.1.

Theorem 4.4. (i) *The Green function $g(z) = g_{\mathcal{E}(\mathcal{C})}(z, \infty)$ of the exterior of $\mathcal{C}$ with pole at infinity is given by*

$$g(z) = \log|\varphi(z)|, \quad z \in \mathcal{E}(\mathcal{C}).$$

(ii) *The logarithmic capacity of $\overline{\mathcal{I}(\mathcal{C})}$ is given by*

$$\operatorname{cap}(\overline{\mathcal{I}(\mathcal{C})}) = \frac{e}{2}.$$

(iii) *The logarithmic potential of the equilibrium measure μ for the set $\overline{\mathcal{I}(\mathcal{C})}$ is given by*

$$\begin{aligned}\mathcal{U}^{\mu}(z) &= \log\left(\frac{2}{e}\right) - \log|\varphi(z)| \\ &= \Re\left(\frac{2}{z+\sqrt{z^2+1}-1}\right) - \log|z+\sqrt{z^2+1}| + \log\left(\frac{2}{e}\right) + 1,\end{aligned}$$

for $z \in \mathcal{E}(\mathcal{C})$,

and

$$\mathcal{U}^{\mu}(z) = \log\left(\frac{2}{e}\right), \quad \textit{for } z \in \overline{\mathcal{I}(\mathcal{C})}.$$

Proof. (i) This statement is an immediate consequence of Lemma 2.1 (see, e.g., [11, p. 109]).
(ii) By virtue of the definition of Green functions (see, e.g., [11, p. 108]), we obtain

$$-\log\left(\mathrm{cap}(\overline{\mathcal{I}(\mathcal{C})})\right) = \lim_{|z|\to\infty} (\log|\varphi(z)| - \log|z|).$$

Exploiting the explicit form of the mapping φ, we find

$$\lim_{|z|\to\infty} (\log|\varphi(z)| - \log|z|) = \log\left(\frac{2}{e}\right),$$

which yields $\mathrm{cap}(\overline{\mathcal{I}(\mathcal{C})}) = \frac{e}{2}$.
(iii) Using the previous two results, this statement follows easily from the general theory (see [11, pp. 52–54 and p. 108]). □

Now we will describe the equilibrium measure for the closed interior of the curve $\mathcal{C}$. Therefore, let $\psi : \partial\mathbb{D} \to \mathrm{Im}(\mathcal{C})$ denote the restriction of the inverse mapping of $\varphi : \overline{\mathcal{E}(\mathcal{C})} \to \mathbb{C} \setminus \mathbb{D}$ to $\partial\mathbb{D}$.

Theorem 4.5. *The equilibrium measure μ for the set $\overline{\mathcal{I}(\mathcal{C})}$ is given by the image of the arc measure on $\partial\mathbb{D}$ induced by the mapping ψ. Moreover, μ is absolutely continuous with respect to the arc measure on $\mathcal{C}$ and its Radon–Nikodym derivative is given by*

$$\mathrm{d}\mu(z) = \frac{1}{2\pi}\,|D(z)|\,|\mathrm{d}z|, \quad z \in \mathcal{C},$$

where

$$D(z) = \frac{2\,(z+\sqrt{z^2+1})}{(z+\sqrt{z^2+1}-1)^2} = \frac{1}{\sqrt{z^2+1}-1}, \quad z \in \mathcal{C}.$$

The expressions $z+\sqrt{z^2+1}$ and $\sqrt{z^2+1}$ are defined in the context of Lemma 2.1.

Proof. The characterization of μ as the image of the arc measure on $\partial\mathbb{D}$ is an immediate consequence of Lemma 2.1 (see, e.g. [1, p. 21]). From this we can deduce that the support of μ coincides with the image of $\mathcal{C}$. Moreover, it is easy to see that $\mathcal{C}$ is analytic at every point $z \neq \pm i$, so that we can apply Theorem 4.3 in order to calculate the Radon–Nikodym derivative in these points. As we know the equilibrium potential $\mathcal{U}^{\mu}$ explicitly from Theorem 4.4, it is clear that the derivative of $\mathcal{U}^{\mu}$ in the direction of the inner normal vanishes. Hence, we find

$$\mathrm{d}\mu(z) = -\frac{1}{2\pi}\frac{\partial\,\mathcal{U}^{\mu}}{\partial n}(z)\,|\mathrm{d}z|,$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the outer normal. An elementary calculation shows

$$-\frac{\partial\,\mathcal{U}^{\mu}}{\partial n}(z) = \frac{\partial\, g_{\mathcal{A}(\mathcal{C})}(z,\infty)}{\partial n} = |\varphi'(z)| = \left|\frac{2\,(z+\sqrt{z^2+1})}{(z+\sqrt{z^2+1}-1)^2}\right| = \left|\frac{1}{\sqrt{z^2+1}-1}\right|.$$

Furthermore, $\{i, -i\}$ is a set of measure zero with respect to μ as well as with respect to the arc measure on $\mathcal{C}$, which furnishes the proof. □

Next we turn to properties of the zeros of generalized associated Ménage polynomials. Here and in the sequel let α denote a fixed positive integer and let F_n be as defined in (1.3). First we show the boundedness of the zeros.

Lemma 4.1. *The set* $\mathcal{N} := \{z \in \mathbb{C} \mid \exists n \in \mathbb{N} \text{ with } F_n(z) = 0\}$ *is bounded.*

Proof. It is easy to see that we have

$$F_n(z) = \sum_{k=0}^{n} \binom{n}{k} A_k\, B_k\, (-z)^k, \quad n \in \mathbb{N},\ z \in \mathbb{C},$$

where

$$A_k := \frac{(n)_k}{(2n)^k}, \qquad B_k := \frac{(\alpha)_k}{\left(\frac{1}{2}\right)_k}, \quad 0 \le k \le n.$$

Furthermore, we have

$$P_n(z) := \sum_{k=0}^{n} \binom{n}{k} A_k\, (-z)^k = \frac{(n)_n\, (-n)_n}{(-2n+1)_n\, (2n)^n}\, z^n\, L_n^{(-2n)}\left(-\frac{2n}{z}\right),$$

where $L_n^{(-2n)}(w)$ denotes the n-th Laguerre polynomial with parameter $-2n$. From [4] we know that the zeros of the sequence $L_n^{(-2n)}(nw)$ tend uniformly to a curve in the complex plane with a positive distance from the origin. This implies the boundedness of the set

$$\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} \text{ with } P_n(z) = 0\}.$$

Moreover, we have

$$Q_n(z) := \sum_{k=0}^{n} \binom{n}{k} B_k\, (-z)^k = {}_2F_1\left(-n, \alpha; \frac{1}{2}; z\right).$$

Using arguments similar to those applied in [12] in order to study the behavior of the zeros of a more general family of hypergeometric polynomials, a classical linear transformation formula for hypergeometric functions (see, e.g. [9, p. 390, 15.8.4]) yields

$$Q_n(z) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+n-\alpha\right)}{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-\alpha\right)}\, {}_2F_1\left(-n, \alpha; \frac{1}{2}-n+\alpha; 1-z\right).$$

Hence, in order to show the boundedness of the set

$$\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} \text{ with } Q_n(z) = 0\} \tag{4.1}$$

it is sufficient to prove the boundedness of the set

$$\left\{z \in \mathbb{C} \mid \exists n \in \mathbb{N} \text{ with } {}_2F_1\left(-n, \alpha; \frac{1}{2}-n+\alpha; z\right) = 0\right\}.$$

To achieve this we show that for every $0 < \epsilon < 1$ there is an positive integer n_0 such that

$$\tilde{Q}_n(z) := z^n\, {}_2F_1\left(-n, \alpha; \frac{1}{2}-n+\alpha; \frac{1}{z}\right) \neq 0 \tag{4.2}$$

is valid for z with $|z| \le \epsilon$ and $n \ge n_0$. Therefore we observe the identity

$$\tilde{Q}_n(z) = \frac{(-1)^n\,(\alpha)_n}{\left(\alpha + \frac{1}{2} - n\right)_n} \sum_{k=0}^{n} \frac{\left(\frac{1}{2} - \alpha\right)_k (-n)_k}{k!(1 - \alpha - n)_k} z^k,$$

which results from reverse summation and standard manipulations for the Pochhammer symbol (see, e.g. [3, p. 9]). Using the estimate $\left|\frac{(-n)_k}{(1-\alpha-n)_k} - 1\right| \le \frac{\alpha-1}{\alpha} k$ for $0 \le k \le n$, it is not difficult to see (e.g. by applying Lebesgue's dominated convergence theorem) that we have

$$\frac{(-1)^n \left(\alpha + \frac{1}{2} - n\right)_n}{(\alpha)_n} \tilde{Q}_n(z) \longrightarrow \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2} - \alpha\right)_k}{k!} z^k = (1 - z)^{\alpha - \frac{1}{2}},$$

as $n \to \infty$, holding uniformly with respect to $z \in \{w \in \mathbb{C} \mid |w| \le \epsilon\}$. Taking into account that the limit function has no zeros in the unit disc, this establishes the property in (4.2) and hence it shows the boundedness of the set (4.1). Now an application of Theorem 4.1 yields the claimed boundedness of the set $\mathcal{N}$. □

The following result gives a more significant description of the set $\mathcal{N}$ defined in the previous lemma.

Lemma 4.2. *The set of accumulation points of $\mathcal{N}$ coincides with the image of $\mathcal{C}$.*

Proof. (i) Let

$$P_n(z) := n^\alpha\,{}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix}\middle|\, \frac{z}{2n}\right), \qquad n \in \mathbb{N},\ z \in \mathbb{C}.$$

Using the second part of Theorem 3.1, we obtain the uniform convergence of the sequence (P_n) on every compact subset of $\mathcal{I}(\mathcal{C})$ to a function which does not vanish identically. If $a_n^{(n)}$ denotes the coefficient of z^n in the polynomial P_n, then we have

$$a_n^{(n)} = n^\alpha \frac{(-n)_n\,(n)_n\,(\alpha)_n}{n!\,(1/2)_n\,(2n)^n}.$$

An application of Stirling's formula yields

$$\lim_{n\to\infty} |a_n^{(n)}|^{1/n} = \frac{2}{e} = \frac{1}{\rho},$$

where ρ denotes the logarithmic capacity of $\overline{\mathcal{I}(\mathcal{C})}$ (see Theorem 4.4). Hence, using Theorem 4.2, we obtain that every point on the curve $\mathcal{C}$ is an accumulation point of the set $\mathcal{N}$.

(ii) Combining Lemma 4.1 and the first part of Theorem 3.1, we find that there is no accumulation point of zeros lying in the exterior $\mathcal{E}(\mathcal{C})$. On the other hand, from the second part of Theorem 3.1 it is clear, that there is no accumulation point of zeros lying in the interior $\mathcal{I}(\mathcal{C})$, which completes the proof. □

As a final remark of his paper [17], Szegö states (without proof) that in the situation of Theorem 4.2, the mode of accumulation of the zeros on the curve can be described to proceed in an "equidistributed" way. Using the strong asymptotics in Theorem 3.1, we now will render this more precisely for the generalized associated Ménage polynomials F_n by identifying the limit distribution of the zeros with the equilibrium measure μ.

Theorem 4.6. *Let μ_n denote the zero counting measure associated to the polynomial*

$$F_n(z) = {}_3F_1\left(\begin{matrix} -n, & n, & \alpha \\ & \frac{1}{2} & \end{matrix} \middle| \frac{z}{2n}\right), \quad n \in \mathbb{N}.$$

Then the sequence (μ_n) converges in the weak* *topology to the equilibrium measure μ for the set $\overline{\mathcal{I}(\mathcal{C})}$.*

Proof. It suffices to show that every subsequence of (μ_n) possesses a subsequence which converges in the weak* topology to the measure μ. Therefore, let (μ_{n_k}) be an arbitrary subsequence of (μ_n). By virtue of Lemma 4.1, we can choose a compact set containing the supports of all measures μ_n. Using Helly's selection theorem (see, e.g. [11, p. 3]), we can find a subsequence $n(l) := n_{k_l}$ of (n_k) and a unit measure ν such that the sequence $(\mu_{n(l)})$ converges in the weak* topology to the measure ν. By virtue of the arguments used in the proof of Lemma 4.1, applying Theorem 3.1 yields that the support of ν is contained in the image of the curve $\mathcal{C}$. Let $a_n^{(n)}$ denote the coefficient of z^n in the polynomial F_n and let the zeros of F_n be denoted by $z_1^{(n)}, \ldots, z_n^{(n)}$. If $z \in \mathcal{E}(\mathcal{C})$ is fixed, again by Theorem 3.1, we can choose a compact superset of $\overline{\mathcal{I}(\mathcal{C})}$ such that it does not contain the point z, but all supports of measures μ_n for large n. Then we have for the logarithmic potential of ν

$$\begin{aligned}
\int \log|z-t|^{-1}\,\mathrm{d}\nu(t) &= \lim_{l\to\infty}\int \log|z-t|^{-1}\,\mathrm{d}\mu_{n(l)}(t) \\
&= \lim_{l\to\infty}\frac{1}{n(l)}\sum_{k=1}^{n(l)}\log|z-z_k^{(n(l))}|^{-1} \\
&= \lim_{l\to\infty}\frac{1}{n(l)}\log\left|\frac{a_{n(l)}^{(n(l))}}{F_{n(l)}(z)}\right| \\
&= \lim_{l\to\infty}\log\left|a_{n(l)}^{(n(l))}\right|^{1/n(l)} - \lim_{l\to\infty}\log\left|F_{n(l)}(z)\right|^{1/n(l)}.
\end{aligned}$$

Now, an application of Stirling's formula yields

$$\lim_{l\to\infty}\log\left|a_{n(l)}^{(n(l))}\right|^{1/n(l)} = \log\left(\frac{2}{e}\right).$$

Furthermore, using the first part of Theorem 3.1 we obtain

$$\lim_{l\to\infty}\log\left|F_{n(l)}(z)\right|^{1/n(l)} = \log|\varphi(z)|,$$

where we remind that the mapping φ is defined in the context of Lemma 2.1. Consequently, we have

$$\int \log|z-t|^{-1}\,\mathrm{d}\nu(t) = \log\left(\frac{2}{e}\right) - \log|\varphi(z)|, \quad z \in \mathcal{E}(\mathcal{C}).$$

Hence, by virtue of Theorem 4.4, we obtain that the potential of ν coincides with the potential of the equilibrium measure μ on the exterior $\mathcal{E}(\mathcal{C})$. An application of Carleson's unicity theorem (see, e.g. [11, p. 123]) yields $\nu = \mu$, which finishes the proof. □

Finally we will have a look at a sequence of plots of the zeros of the polynomials F_n in the most important case $\alpha = 1$ (see Figs. 4–11).

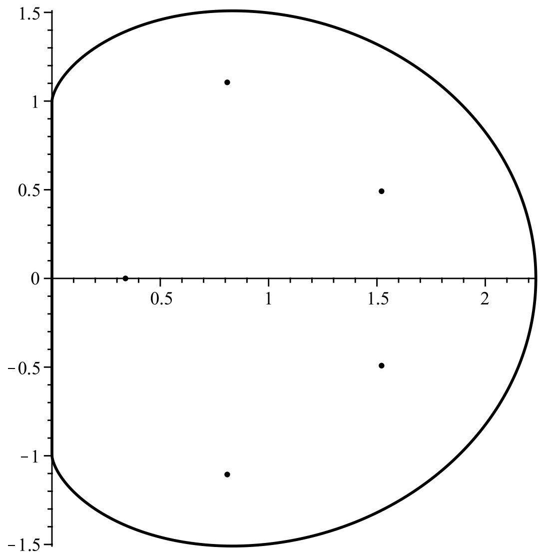

Fig. 4. Zeros of $F_5, \alpha = 1$.

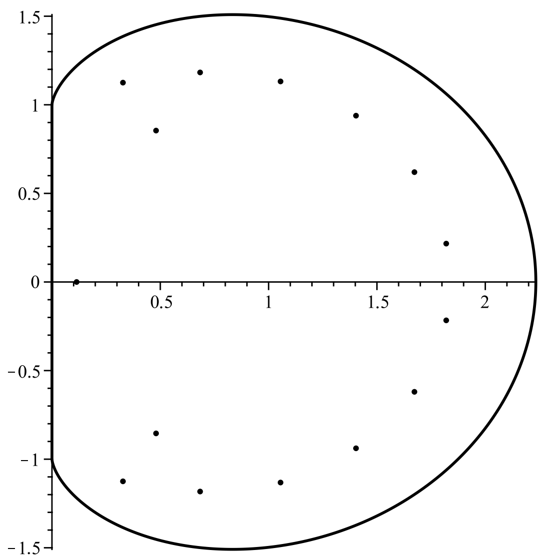

Fig. 5. Zeros of $F_{15}, \alpha = 1$.

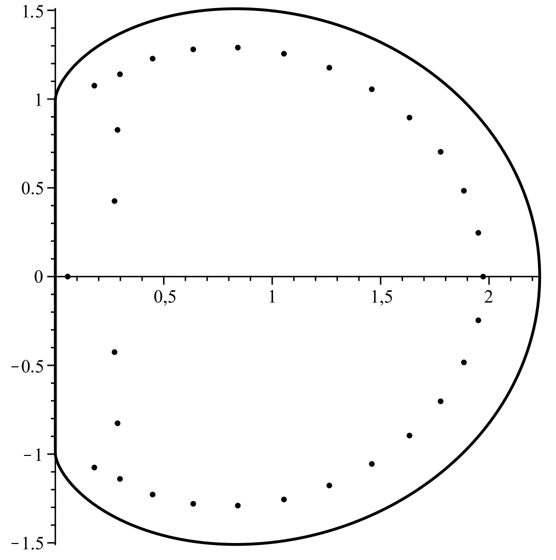

Fig. 6. Zeros of $F_{30}, \alpha = 1$.

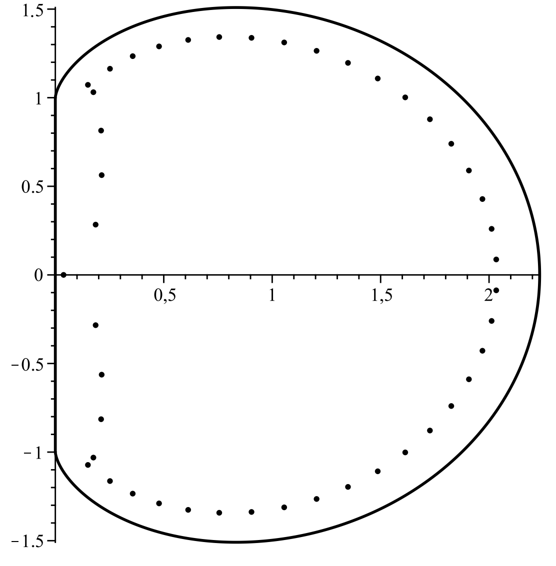

Fig. 7. Zeros of $F_{45}, \alpha = 1$.

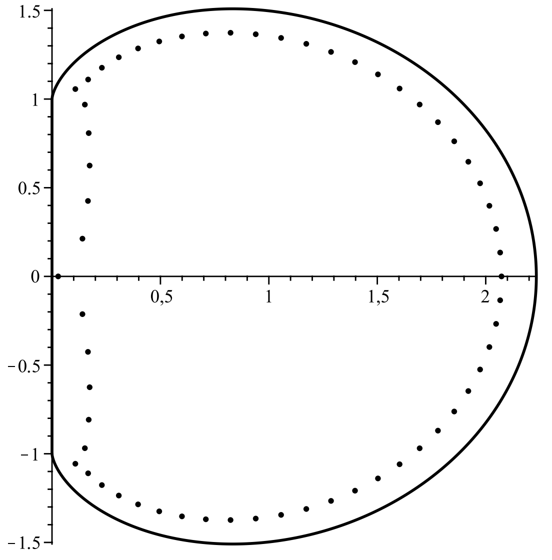

Fig. 8. Zeros of $F_{60}, \alpha = 1$.

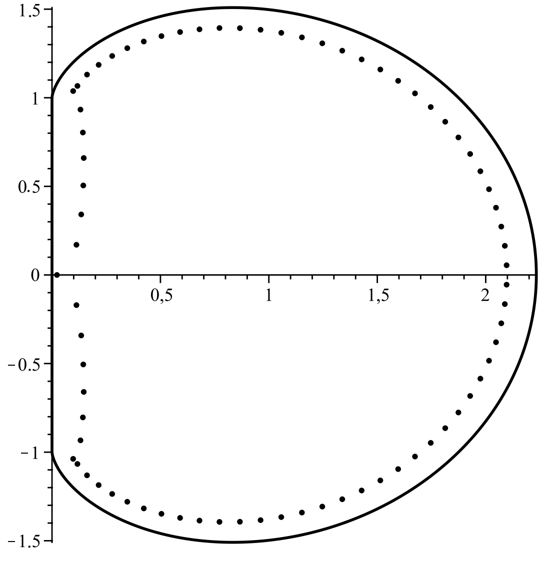

Fig. 9. Zeros of $F_{75}, \alpha = 1$.

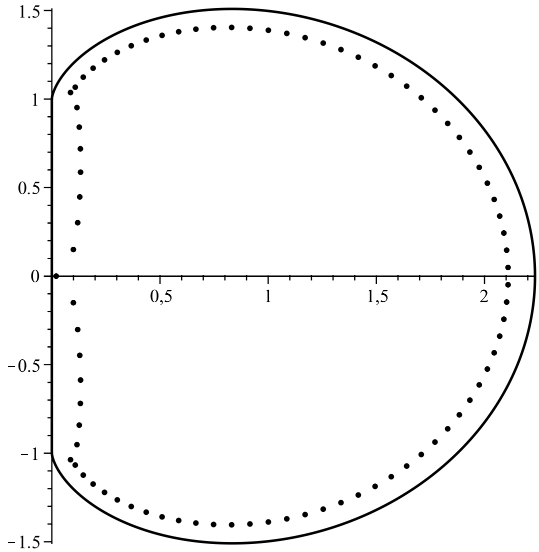

Fig. 10. Zeros of $F_{85}, \alpha = 1$.

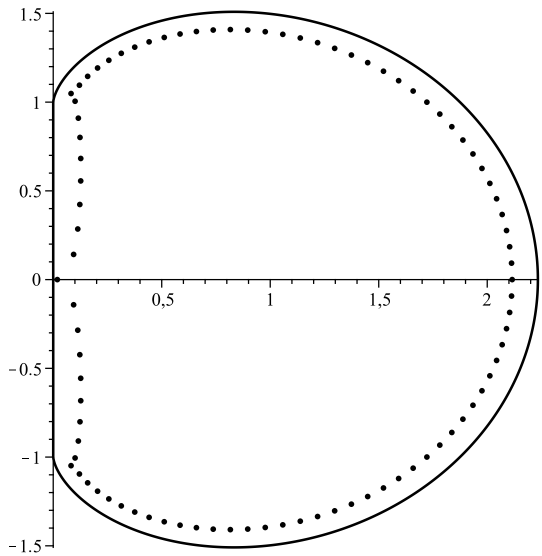

Fig. 11. Zeros of $F_{90}, \alpha = 1$.

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