Sums of Products of Hypergeometric Bernoulli Polynomials

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Bernoulli Polynomials

Generating function:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

$$B_0(x) = 1, \qquad B_1(x) = x - \frac{1}{2},$$

$$B_2(x) = x^2 - x + \frac{1}{6}, \qquad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ldots$$

Bernoulli numbers:

$$B_n = B_n(0)$$

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0, \ldots$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$$

Some Properties of Bernoulli Numbers and Polynomials

1. $$\sum_{n=1}^{N} n^p = \frac{B_{p+1}(N+1) - B_{p+1}(0)}{p+1}$$ (Sums of Powers)

2. $$\zeta(2n) = \sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!} B_{2n}$$ (Riemann zeta)

3. $$B_n(1-x) = (-1)^n B_n(x)$$ (Functional identity)

4. $$\frac{d}{dx}[B_{n+1}(x)] = (n+1)B_n(x)$$ (Appell identity)

5. $$\sum_{k=0}^{n} \frac{B_k(x)}{k!(n+1-k)!} = \frac{x^n}{n!}$$ (Recursive relation)

Sums of Products

Theorem: (Nörlund, 1920) For $n \geq 2$

$$\sum_{i=0}^{n} \binom{n}{i} B_i(x) B_{n-i}(y) = -(n-1) B_n(x+y) + n(x+y-1) B_{n-1}(x+y)$$

Corollary: (Euler, 1755) For $n \geq 2$

$$\sum_{i=0}^{n} \binom{n}{i} B_i B_{n-i} = -(n-1) B_n - n B_{n-1}$$

Higher-Order Sums of Products

Theorem: (Dilcher, 1996) Suppose $x = x_1 + ... + x_p$. Then for $n \ge p$

$$\sum_{\substack{i_1 \ge 0, i_2 \ge 0, ..., i_p \ge 0 \\ i_1 + i_2 + ... + i_p = n}} \frac{n!}{i_1! \cdots i_p!} B_{i_1}(x_1) \cdots B_{i_p}(x_p) =$$

$$(-1)^{p-1} p \binom{n}{p} \sum_{k=0}^{p-1} \frac{(-1)^k}{n-k} \left(\sum_{m=0}^{k} \binom{p-1-k+m}{m} s(p, p-k+m) x^m \right) B_{n-k}(x)$$

where $s(p,m)$ are Stirling numbers of the first kind defined by

$$x(x-1)(x-2)\cdots(x-p+1) = \sum_{m=0}^{p} s(p,m) x^m$$

Proof:

$$\frac{te^{x_1 t}}{e^t - 1} \cdot \frac{te^{x_2 t}}{e^t - 1} \cdots \frac{te^{x_p t}}{e^t - 1} = \frac{t^p e^{xt}}{(e^t - 1)^p} \qquad (x = x_1 + \ldots + x_p)$$

LHS:

$$\frac{te^{x_1 t}}{e^t - 1} \cdot \frac{te^{x_2 t}}{e^t - 1} \cdots \frac{te^{x_p t}}{e^t - 1} = \left(\sum_{n=0}^{\infty} B_n(x_1) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{\infty} B_n(x_2) \frac{t^n}{n!} \right) \cdots \left(\sum_{n=0}^{\infty} B_n(x_p) \frac{t^n}{n!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{i_1 \geq 0, i_2 \geq 0, \ldots, i_p \geq 0 \\ i_1 + i_2 + \ldots + i_p = n}} \frac{n!}{i_1! \cdots i_p!} B_{i_1}(x_1) \cdots B_{i_p}(x_p) \right) \frac{t^n}{n!}$$

RHS:

$$\frac{t^p e^{xt}}{(e^t - 1)^p} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{t^n}{n!} \qquad \text{(Generalized BP of order } p\text{)}$$

Lemma:

$$\sum_{\substack{i_1 \geq 0, i_2 \geq 0, \ldots, i_p \geq 0 \\ i_1 + i_2 + \ldots + i_p = n}} \frac{n!}{i_1! \cdots i_p!} B_{i_1}(x_1) \cdots B_{i_p}(x_p) = B_n^{(p)}(x)$$

Recursive Formula for Generalized BP

RHS Again:

$$\frac{t^p e^{xt}}{(e^t-1)^p}=\frac{t^p e^{xt}[e^t-(e^t-1)]}{(e^t-1)^p}$$

$$=t^p e^{xt}\left[\frac{e^t}{(e^t-1)^p}-\frac{1}{(e^t-1)^{p-1}}\right]$$

$$=\frac{t}{1-p}\left[t^{p-1}e^{xt}\cdot\frac{d}{dt}\left(\frac{1}{(e^t-1)^{p-1}}\right)\right]-t\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}$$

$$=\frac{t}{1-p}\left[\frac{d}{dt}\left(\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}\right)-\frac{d}{dt}(t^{p-1}e^{xt})\cdot\left(\frac{1}{(e^t-1)^{p-1}}\right)\right]-t\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}$$

$$=-\frac{t}{p-1}\frac{d}{dt}\left(\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}\right)+\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}+\frac{xt}{p-1}\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}-t\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}$$

$$=-\frac{t}{p-1}\frac{d}{dt}\left(\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}\right)+\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}+t\left(\frac{x}{p-1}-1\right)\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}$$

$$\frac{t^p e^{xt}}{(e^t-1)^p} = -\frac{t}{p-1}\frac{d}{dt}\left(\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}\right) + \frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}} + t\left(\frac{x}{p-1}-1\right)\frac{t^{p-1}e^{xt}}{(e^t-1)^{p-1}}$$

Substitute and equate series coefficients:

$$\sum_{n=0}^{\infty} B_n^{(p)}(x)\frac{x^n}{n!}$$

$$= -\frac{t}{p-1}\frac{d}{dt}\left(\sum_{n=0}^{\infty} B_n^{(p-1)}(x)\frac{t^n}{n!}\right) + \sum_{n=0}^{\infty} B_n^{(p-1)}(x)\frac{t^n}{n!} + \left(\frac{x}{p-1}-1\right)t\sum_{n=0}^{\infty} B_n^{(p-1)}(x)\frac{t^n}{n!}$$

$$= B_0^{(p-1)}(x) + \sum_{n=1}^{\infty}\left[\left(1-\frac{n}{p-1}\right)B_n^{(p-1)}(x) + n\left(\frac{x}{p-1}-1\right)B_{n-1}^{(p-1)}(x)\right]\frac{t^n}{n!}$$

Lemma: For $n \ge 1, p > 1$

$$B_n^{(p)}(x) = \left(1-\frac{n}{p-1}\right)B_n^{(p-1)}(x) + n\left(\frac{x}{p-1}-1\right)B_{n-1}^{(p-1)}(x)$$

Reduction to Classical Bernoulli Polynomials

Lemma: For $n \geq p$

$$B_n^{(p)}(x) = (-1)^{p-1} p \binom{n}{p} \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} B_k^{(p)}(x) \frac{B_{n-k}(x)}{n-k} \qquad (n \geq p)$$

Lemma: For $k < p$

$$B_k^{(p)}(x) = \binom{p-1}{k}^{-1} \sum_{m=0}^{k} \binom{p-1-k+m}{m} s(p, p-k+m) x^m$$

Proof: Substitute $n = p - 1$ into recursive relation:

$$B_n^{(p)}(x) = \left(1 - \frac{n}{p-1}\right) B_n^{(p-1)}(x) + n\left(\frac{x}{p-1} - 1\right) B_{n-1}^{(p-1)}(x)$$

$$B_{p-1}^{(p)}(x) = \quad x - p + 1 \;\; B_{p-2}^{(p-1)}(x) = (x-1)(x-2)\cdots(x-p+1)$$

Integrate using Appell identity:

$$B_k^{(p)}(x) = \frac{k!}{(p-1)!}\frac{d^{\,p-1-k}}{dx^{p-1-k}}[(x-1)(x-2)\cdots(x-p+1)] \qquad (k < p)$$

$$= \frac{k!}{(p-1)!}\frac{d^{\,p-1-k}}{dx^{p-1-k}}[\sum_{m=1}^{p} s(p,m)x^{m-1}]$$

$$= \binom{p-1}{k}^{-1}\sum_{m=0}^{k}\binom{p-1-k+m}{m}s(p,p-k+m)x^m$$

Thus for $n \geq p$ we obtain Dilcher’s result:

$$\sum_{\substack{i_1\geq 0, i_2\geq 0,\ldots,.i_p\geq 0\\ i_1+i_2+\ldots+i_p=n}} \frac{n!}{i_1!\cdots i_p!} B_{i_1}(x_1)\cdots B_{i_p}(x_p) = B_n^{(p)}(x) =$$

$$(-1)^{p-1}p\binom{n}{p}\sum_{k=0}^{p-1}\frac{(-1)^k}{n-k}\left(\sum_{m=0}^{k}\binom{p-1-k+m}{m}s(p,p-k+m)x^m\right)B_{n-k}(x)$$

Hypergeometric Bernoulli Polynomials

	Classical	Hypergeometric
Bernoulli polynomials:	$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$	$\frac{(t^N / N!)e^{xt}}{e^t - \sum_{n=0}^{N-1} \frac{t^n}{n!}} = \sum_{n=0}^{\infty} B_{N,n}(x) \frac{t^n}{n!}$
Bernoulli numbers:	$B_n = B_n(0)$ $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \cdot x^{n-k}$	$B_{N,n} = B_{N,n}(0)$ $B_{N,n}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{N,k} \cdot x^{n-k}$
Generalized BP:	$\frac{t^p e^{xt}}{(e^t - 1)^p} = \sum_{n=0}^{\infty} B_n^{(p)}(x) \frac{x^n}{n!}$	$\frac{(t^N / N!)^p e^{xt}}{\left(e^t - \sum_{n=0}^{N-1} \frac{t^n}{n!}\right)^p} = \sum_{n=0}^{\infty} B_{N,n}^{(p)}(x) \frac{x^n}{n!}$

Sums of Products of Hypergeometric Bernoulli Polynomials

Lemma:

$$\sum_{\substack{i_1 \ge 0, i_2 \ge 0, \ldots, i_p \ge 0 \\ i_1 + i_2 + \ldots + i_p = n}} \frac{n!}{i_1! \cdots i_p!} B_{N,i_1}(x_1) \cdots B_{N,i_p}(x_p) = B_{N,n}^{(p)}(x)$$

Lemma:

$$B_{N,n}^{(p)}(x) = \underbrace{\left(1 - \frac{n}{N(p-1)}\right)}_{a_n(p)} B_{N,n}^{(p-1)}(x) + \underbrace{\frac{n}{N}\left(\frac{x}{p-1} - 1\right)}_{b_n(p)} B_{N,n-1}^{(p-1)}(x)$$

Iterate:

$$B_{N,n}^{(2)}(x) = \left(1 - \frac{n}{N}\right) B_{N,n}(x) + \frac{n}{N} \;\; x - 1 \;\; B_{N,n-1}(x)$$

$$= a_n(2) B_{N,n}(x) + b_n(2) B_{N,n-1}(x)$$

$$B_{N,n}^{(3)}(x) = a_n(3)a_n(2)B_{N,n}(x)$$
$$+[a_n(3)b_n(2)+b_n(3)a_{n-1}(2)]B_{N,n-1}(x)$$
$$+b_n(3)b_{n-1}(2)B_{N,n-2}(x)$$

$$B_{N,n}^{(4)}(x) = a_n(3)a_n(2)a_n(1)B_n(x)$$
$$+[a_n(3)a_n(2)b_n(1)+a_n(3)b_n(2)a_{n-1}(1)+b_n(3)a_{n-1}(2)a_{n-1}(1)]B_{n-1}(x)$$
$$+[a_n(3)b_n(2)b_{n-1}(1)+b_n(3)a_{n-1}(2)b_{n-1}(1)+b_n(3)b_{n-1}(2)a_{n-2}(1)]B_{n-2}(x)$$
$$+b_n(3)b_{n-1}(2)b_{n-2}(1)B_{n-3}(x)$$

$$B_{N,n}^{(5)}(x) = ?$$

$$B_{N,n}^{(p)}(x) = ?$$

Main Theorem: Suppose $x = x_1 + \ldots + x_p$. Then for $n \geq p$

$$\sum_{\substack{i_1 \geq 0, i_2 \geq 0, \ldots, i_p \geq 0 \\ i_1 + i_2 + \ldots + i_p = n}} \frac{n!}{i_1! \cdots i_p!} B_{N,i_1}(x_1) \cdots B_{N,i_p}(x_p) = \sum_{k=0}^{p-1} \left(\sum_{\sigma \in A_{p-1}(p-k-1)} c_{N,n}(\sigma) \right) B_{N,n-k}(x)$$

where

$$A_{p-1}(p-k-1) = \{(p-1)\text{-element subsets of } \{1, 2, \ldots, p-k-1\}\},$$

$$c_{N,n}(\sigma) = \prod_{r=1}^{k} a_{N,n-f_\sigma(i_r)}(i_r) \prod_{s=1}^{p-k} b_{N,n-g_\sigma(j_s)}(j_s),$$

and

$$f_\sigma(i_r) = \#\{j_s : i_r < j_s\}$$
$$g_\sigma(j_s) = \#\{j_w : j_s < j_w\}$$

Sums of Products of Mixed Hypergeometric Bernoulli Polynomials

Lemma:

$$\sum_{m=0}^{n}\binom{n}{m}B_{1,m}(x)B_{2,n-m}(x)=-\frac{n}{2}B_{1,n-1}(x+y)+B_{2,n}(x+y)$$

Theorem: Suppose $x = x_1 + x_2$. Then for $M < N$

$$\sum_{p=0}^{N-M-1}\sum_{m=0}^{n-p}\frac{M!B_{M,m}(x/2)B_{N,n-p-m}(x/2)}{(p+M)!m!(n-p-m)!}=-\frac{M!B_{M,m-N+M}(x)}{N!(n-N+M)!}+\frac{B_{N,n}(x)}{n!}$$

References

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[3] K. Kamano, *Sums of products of hypergeometric Bernoulli numbers*, preprint (2010).