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Letter Section

Mellin transforms of a generalization of Legendre polynomials *

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Abstract

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We show that the zeros and poles of the Mellin transforms of polynomials on $[-1, 1]$ orthogonal with respect to the weight $|x|^{2r}$, with $r > -\frac{1}{2}$, are real and simple.

Keywords: Mellin transforms; Legendre polynomials.

Several authors have considered Mellin transforms of orthogonal polynomials. The zeros of the Mellin transforms of the Laguerre polynomials L_n^0 have real part $\frac{1}{2}$ [4]. In [1], it is shown that the zeros of Mellin transforms of the Hermite polynomials also have real part $\frac{1}{2}$. Essentially the same transforms arise in [5]. The generalization to Mellin transforms of the Laguerre polynomials L_n^α is made in [3], where "exceptional" real zeros can occur in some cases. In a more general setting, [6] considers other unitary transforms, while [7] surveys some of the underlying group theory.

In this paper, we consider polynomials which are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)|x|^{2r}\,\mathrm{d}x, \quad r > -\tfrac{1}{2}.$$

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Let $\{p_n(x)\}_{n \geqslant 0}$ be the set of such polynomials where $p_n(x)$ is monic of degree n. Each $p_n(x)$ is also dependent on the parameter r found in the inner product. For the purposes of this paper, we take as our definition of the Mellin transform

$$M_n(s) = 2\int_0^1 p_n(x) x^{r+s} \frac{\mathrm{d}x}{x}, \quad \operatorname{Re}(r+s) > 0. \tag{1}$$

Our main result is the following theorem.

Theorem 1. *The zeros and poles of $M_n(s)$ are real and simple.*

Proof. Any system of monic orthogonal polynomials $\{p_n\}$ satisfies a recurrence relation of the form

$$p_n(x) = (x - c_n) p_{n-1}(x) - \lambda_n p_{n-2}(x),$$

where $c_n \in \mathbb{R}$ and $\lambda_n > 0$ [2, p.8], [8, p.41]. Since the weight function $|x|^{2r}$ is symmetric, $c_n = 0$ for all n and each $p_n(x)$ is either odd or even. Therefore, the formula reduces to

$$p_n(x) = x p_{n-1}(x) - \lambda_n p_{n-2}(x). \tag{2}$$

If we substitute (2) into (1), we find that

$$\begin{aligned} M_n(s) &= 2\int_0^1 (x p_{n-1}(x) - \lambda_n p_{n-2}(x)) x^{r+s} \frac{\mathrm{d}x}{x} \\ &= 2\int_0^1 p_{n-1}(x) x^{r+s+1} \frac{\mathrm{d}x}{x} - \lambda_n 2\int_0^1 p_{n-2}(x) x^{r+s} \frac{\mathrm{d}x}{x} \\ &= M_{n-1}(s+1) - \lambda_n M_{n-2}(s). \end{aligned} \tag{3}$$

We can now recursively generate the system of Mellin transforms, once we have the values $\{\lambda_n\}$ from (2). Using orthogonal projection and the fact that p_n is orthogonal to all polynomials of smaller degree, we arrive at [2, p.19]

$$\lambda_{n+1} = \frac{\langle p_n(x), x^n \rangle}{\langle p_{n-1}(x), x^{n-1} \rangle} = \frac{M_n(r+n+1)}{M_{n-1}(r+n)}. \tag{4}$$

We are now ready to complete the proof of the theorem. We use the notation

$$(a)_j = a(a+1) \cdots (a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}.$$

We claim that

$$M_n(s) = -\frac{\lfloor \frac{1}{2}n \rfloor !\, \left(\frac{1}{2}(s-r-n+1)\right)_{\lfloor n/2 \rfloor}}{\left(\frac{1}{2}(-2r-2n+1)\right)_{\lfloor n/2 \rfloor} \left(\frac{1}{2}(-s-r-n)\right)_{\lfloor n/2 \rfloor + 1}}. \tag{5}$$

Note that this formula makes sense for all r except negative half-integers.

The inductive proof begins by verifying (5) for $n = 0$ and $n = 1$. Now assume the claim for M_k, $0 \leqslant k \leqslant n$. From (3) we have

$$M_{n+1}(s) = M_{n-1}(s) \left(\frac{M_n(s+1)}{M_{n-1}(s)} - \lambda_{n+1} \right);$$

hence

$$M_{n+1}(s)=M_{n-1}(s)\left(\frac{M_n(s+1)}{M_{n-1}(s)}-\frac{M_n(r+n+1)}{M_{n-1}(r+n)}\right)$$

by (4). Grouping similar terms in the ratios, first for n even, we obtain

$$\begin{aligned}M_{n+1}(s)&=M_{n-1}(s)\left(\frac{n(s-r)}{(2r+2n-1)(s+r+n+1)}-\frac{n^2}{(2r+2n-1)(2r+2n+1)}\right)\\&=M_{n-1}(s)\left(\frac{\left(\frac{1}{2}n\right)\cdot\left(\frac{1}{2}(s-r-n)\right)}{\left(\frac{1}{2}(-2r-2n-1)\cdot\frac{1}{2}(-2r-2n+1)/\left(\frac{1}{2}(-2r-n-1)\right)\right)\cdot\left(\frac{1}{2}(-s-r-n-1)\right)}\right),\end{aligned}$$

and (5) follows for $n+1$. Similarly for n odd, we obtain

$$\begin{aligned}M_{n+1}(s)&=M_{n-1}(s)\left(\frac{(2r+n)(s+r)}{(2r+2n-1)(s+r+n+1)}-\frac{(2r+n)^2}{(2r+2n-1)(2r+2n+1)}\right)\\&=M_{n-1}(s)\left(\frac{\left(\frac{1}{2}(n+1)\right)\cdot\left(\frac{1}{2}(s-r-n)\right)}{\left(\frac{1}{2}(-2r-2n-1)\cdot\frac{1}{2}(-2r-2n+1)/\left(\frac{1}{2}(-2r-n-1)\right)\right)\cdot\left(\frac{1}{2}(-s-r-n-1)\right)}\right),\end{aligned}$$

and (5) again follows.

The induction is complete. We immediately conclude the zeros and poles of $M_n(s)$ are real and simple. □

We remark that the zeros and poles of the Mellin transforms of several other families of orthogonal polynomials, such as Gegenbauer polynomials, do not have such properties.

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