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The Li criterion and the Riemann hypothesis for the Selberg class II

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ABSTRACT

In this paper, we prove an explicit asymptotic formula for the arithmetic formula of the Li coefficients established in Omar and Mazhouda (2007) [10] and Omar and Mazhouda (2010) [11]. Actually, for any function $F(s)$ in the Selberg class $\mathcal{S}$, we have

$$RH \quad \Leftrightarrow \quad \lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

with

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log(\lambda Q_F^2), \quad \lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j},$$

where γ is the Euler constant.

1. Introduction

Let ρ be range over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$. The Li criterion asserts that the Riemann hypothesis is true if and only if all terms of the sequence

$$\lambda_n = \sum_{\rho} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right], \quad n \geqslant 1,$$

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are non-negative [7]. Bombieri and Lagarias [1], Coffey [2,3] and Li [8] obtained an arithmetic expression for the Li coefficients λ_n and gave an asymptotic formula as $n \to \infty$. More recently, Maslanka [9] computed λ_n for $1 \leqslant n \leqslant 3300$ and empirically studied the growth behavior of the Li coefficients. In [10] and [11], the authors give a generalization of the Li criterion for the Selberg class and established an explicit formula for the Li coefficients.

In this paper, we give an explicit asymptotic formula for the generalized Li coefficients which is equivalent to the Riemann hypothesis for the Selberg class.

The Selberg class $\mathcal{S}$ [14] consists of Dirichlet series

$$F(s) = \sum_{n=1}^{+\infty} \frac{a(n)}{n^s}, \quad \Re(s) > 1$$

satisfying the following hypothesis.

- **Analytic continuation:** there exists a non-negative integer m such $(s-1)^m F(s)$ is an entire function of finite order. We denote by m_F the smallest integer m which satisfies this condition;
- **Functional equation:** for $1 \leqslant j \leqslant r$, there are positive real numbers Q_F, λ_j and there are complex numbers μ_j, ω with $\Re(\mu_j) \geqslant 0$ and $|\omega| = 1$, such that

  $$\phi_F(s) = \omega \overline{\phi_F(1-\bar{s})}$$

  where

  $$\phi_F(s) = F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j);$$

- **Ramanujan hypothesis:** $a(n) = O(n^{\epsilon})$;
- **Euler product:** $F(s)$ satisfies

  $$F(s) = \prod_p \exp\left(\sum_{k=1}^{+\infty} \frac{b(p^k)}{p^{ks}}\right)$$

  with suitable coefficients $b(p^k)$ satisfying $b(p^k) = O(p^{k\theta})$ for some $\theta < \frac{1}{2}$.

It is expected that for every function in the Selberg class the analogue of the Riemann hypothesis holds, i.e., that all non-trivial (non-real) zeros lie on the critical line $\Re(s) = \frac{1}{2}$. The degree of $F \in \mathcal{S}$ is defined by

$$d_F = 2\sum_{j=1}^{r} \lambda_j.$$

The logarithmic derivative of $F(s)$ has also the Dirichlet series expression

$$-\frac{F'}{F}(s) = \sum_{n=1}^{+\infty} \Lambda_F(n) n^{-s}, \quad \Re(s) > 1,$$

where $\Lambda_F(n) = b(n)\log n$ is the generalized von Mangoldt function. If $N_F(T)$ counts the number of zeros of $F(s) \in \mathcal{S}$ in the rectangle $0 \leqslant \Re e(s) \leqslant 1$, $0 < \Im(s) \leqslant T$ (according to multiplicities) one can show by standard contour integration the formula

$$N_F(T) = \frac{d_F}{2\pi} T \log T + c_1 T + O(\log T)$$

in analogy to the Riemann–von Mangoldt formula for Riemann's zeta-function $\zeta(s)$, the prototype of an element in $\mathcal{S}$. For more details concerning the Selberg class we refer to the surveys of Kaczorowski [4], Kaczorowski and Perelli [5] and Perelli [12].

2. The Li criterion

Let F be a function in the Selberg class non-vanishing at $s = 1$ and let us define the xi-function $\xi_F(s)$ by

$$\xi_F(s) = s^{m_F}(s-1)^{m_F}\phi_F(s).$$

The function $\xi_F(s)$ satisfies the functional equation

$$\xi_F(s) = \omega\overline{\xi_F(1-\bar{s})}.$$

The function ξ_F is an entire function of order 1. Therefore, by the Hadamard product, it can be written as

$$\xi_F(s) = \xi_F(0)\prod_\rho\left(1 - \frac{s}{\rho}\right),$$

where the product is over all zeros of $\xi_F(s)$ in the order given by $|\Im(\rho)| < T$ for $T \to \infty$. Let $\lambda_F(n)$, $n \in \mathbb{Z}$, be a sequence of numbers defined by a sum over the non-trivial zeros of $F(s)$ as

$$\lambda_F(n) = \sum_\rho\left[1 - \left(1 - \frac{1}{\rho}\right)^n\right],$$

where the sum over ρ is

$$\sum_\rho = \lim_{T\mapsto\infty}\sum_{|\Im\rho|\leqslant T}.$$

These coefficients are expressible in terms of power-series coefficients of functions constructed from the ξ_F-function. For $n \leqslant -1$, the Li coefficients $\lambda_F(n)$ correspond to the following Taylor expansion at the point $s = 1$

$$\frac{d}{dz}\log\xi_F\left(\frac{1}{1-z}\right) = \sum_{n=0}^{+\infty}\lambda_F(-n-1)z^n$$

and for $n \geqslant 1$, they correspond to the Taylor expansion at $s = 0$

$$\frac{d}{dz}\log\xi_F\left(\frac{-z}{1-z}\right) = \sum_{n=0}^{+\infty}\lambda_F(n+1)z^n.$$

Let $\mathcal{Z}$ be the multi-set of zeros of $\xi_F(s)$ (counted with multiplicity). The multi-set $\mathcal{Z}$ is invariant under the map $\rho \mapsto 1-\overline{\rho}$. We have

$$1-\left(1-\frac{1}{\rho}\right)^{-n}=1-\left(\frac{\rho-1}{\rho}\right)^{-n}=1-\left(\frac{-\rho}{1-\rho}\right)^{n}=1-\overline{\left(1-\frac{1}{1-\overline{\rho}}\right)^{n}}$$

and this gives the symmetry $\lambda_F(-n)=\overline{\lambda_F(n)}$. Using the corollary in [1, Theorem 1], we get the following generalization of the Li criterion for the Riemann hypothesis.

Theorem 2.1. *Let $F(s)$ be a function in the Selberg class $\mathcal{S}$ non-vanishing at $s=1$. Then, all non-trivial zeros of $F(s)$ lie on the line $\Re e(s)=1/2$ if and only if $\Re e(\lambda_F(n))>0$ for $n=1,2,\ldots.$*

Under the same hypothesis of Theorem 2.1, the Riemann hypothesis is also equivalent to each of the two following conditions (a) or (b):

- (a) For each $\epsilon>0$, there is a positive constant $c(\epsilon)$ such that

$$\Re e\big(\lambda_F(n)\big)\geqslant -c(\epsilon)e^{\epsilon n}\quad \text{for all } n\geqslant 1.$$

- (b) The Li coefficients $\lambda_F(n)$ satisfy

$$\lim_{n\to\infty}\big|\lambda_F(n)\big|^{1/n}\leqslant 1.$$

The proof is the same as in [6, Theorem 2.2]. Next, we recall the following explicit formula for the coefficients $\lambda_F(n)$. Let consider the following hypothesis:

$\mathcal{H}$: *there exists a constant $c>0$ such that $F(s)$ is non-vanishing in the region*:

$$\left\{s=\sigma+it;\ \sigma\geqslant 1-\frac{c}{\log(Q_F+1+|t|)}\right\}.$$

Theorem 2.2. *Let $F(s)$ be a function in the Selberg class $\mathcal{S}$ satisfying $\mathcal{H}$. Then, we have*

$$\begin{aligned}\lambda_F(-n)={}&m_F+n\left(\log Q_F-\frac{d_F}{2}\gamma\right)\\&-\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{l-1}}{(l-1)!}\lim_{X\to+\infty}\left\{\sum_{k\leqslant X}\frac{\Lambda_F(k)}{k}(\log k)^{l-1}-\frac{m_F}{l}(\log X)^l\right\}\\&+n\sum_{j=1}^{r}\lambda_j\left(-\frac{1}{\lambda_j+\mu_j}+\sum_{l=1}^{+\infty}\frac{\lambda_j+\mu_j}{l(l+\lambda_j+\mu_j)}\right)\\&+\sum_{j=1}^{r}\sum_{k=2}^{n}\binom{n}{k}(-\lambda_j)^k\sum_{l=0}^{+\infty}\left(\frac{1}{l+\lambda_j+\mu_j}\right)^k,\end{aligned}\tag{1}$$

where γ is the Euler constant.

3. An asymptotic formula

A natural question is to know the asymptotic behavior of the numbers $\lambda_F(n)$. To do so, we use the arithmetic formula (1). Furthermore, we prove that it is equivalent to the Riemann hypothesis.

Theorem 3.1. *Let $F \in \mathcal{S}$, then*

$$RH \quad \Leftrightarrow \quad \lambda_F(n) = \frac{d_F}{2} n \log n + c_F n + O(\sqrt{n} \log n),$$

where

$$c_F = \frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log\bigl(\lambda Q_F^2\bigr), \quad \lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$$

and γ is the Euler constant.

Proof. The proof is an analogous of the argument used by Lagarias in [6].

($\Rightarrow$) First, recall that for $n \geqslant 1$,

$$\frac{d}{dz} \log \xi_F\left(\frac{z}{z-1}\right) = \sum_{n=0}^{+\infty} \lambda_F(n+1) z^n.$$

Writing,

$$\frac{\xi_F'}{\xi_F}(s+1) = \frac{F'}{F}(s+1) + \frac{m_F}{s} + \frac{m_F}{s+1} + \left(\log Q_F + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s + \lambda_j + \mu_j)\right).$$

Let define the coefficients $\{\eta_F(k),\ k \in \mathbb{N}\}$ by

$$-\frac{F'}{F}(s+1) - \frac{m_F}{s} = \sum_{k=0}^{+\infty} \eta_F(k) s^k$$

and the coefficients $\{\tau_F(k),\ k \in \mathbb{N}\}$ by

$$\log Q_F + \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s + \lambda_j + \mu_j) = \sum_{k=0}^{+\infty} \tau_F(k) s^k.$$

Then, we have

$$\lambda_F(-n) = m_F - \sum_{k=1}^{n} \binom{n}{k} \eta_F(k-1) + \sum_{k=1}^{n} \binom{n}{k} \tau_F(k-1), \quad \forall n \geqslant 1.$$

Now, we write

$$H_F(n) = \sum_{k=1}^{n} \binom{n}{k} \eta_F(k-1)$$

and

$$K_F(n)=\sum_{k=1}^{n}\binom{n}{k}\tau_F(k-1).$$

Furthermore, one has

$$\tau_F(0)=\log Q_F+\sum_{j=1}^{r}\lambda_j\psi(\lambda_j+\mu_j)=\log Q_F-\frac{d_F}{2}\gamma+\sum_{j=1}^{r}\lambda_j\left(-\frac{1}{\lambda_j+\mu_j}+\sum_{l=1}^{+\infty}\frac{\lambda_j+\mu_j}{l(l+\lambda_j+\mu_j)}\right),$$

where γ is the Euler constant, $\psi=\frac{\Gamma'}{\Gamma}$ is the digamma function and for all $k\geqslant 1$

$$\tau_F(k)=\sum_{j=1}^{r}(-\lambda_j)^{k+1}\sum_{l=0}^{+\infty}\left(\frac{1}{l+\lambda_j+\mu_j}\right)^{k+1}$$

using

$$\psi(z)=-\gamma-\frac{1}{z}+\sum_{l=1}^{+\infty}\frac{z}{l(l+z)}.$$

Therefore, we obtain

$$\begin{aligned}K_F(n)&=\sum_{j=1}^{r}\sum_{k=2}^{n}\binom{n}{k}(-\lambda_j)^k\sum_{l=0}^{+\infty}\frac{1}{(l+\lambda_j+\mu_j)^k}+\binom{n}{1}\left(\log Q_F+\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j+\mu_j)\right)\\&=\sum_{j=1}^{r}\sum_{k=2}^{n}\binom{n}{k}(-\lambda_j)^k\sum_{l=1}^{+\infty}\frac{1}{(l+\lambda_j+\mu_j-1)^k}+\binom{n}{1}\left(\log Q_F+\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j+\mu_j)\right).\end{aligned}\tag{2}$$

Noting the first term of the right-hand side of (2) by $T_1(n)$. We have

$$\begin{aligned}T_1(n)&=\sum_{j=1}^{r}\left\{\sum_{l=1}^{+\infty}\sum_{k=2}^{n}\binom{n}{k}\frac{(-\lambda_j)^k}{(l+\lambda_j+\mu_j-1)^k}\right\}\\&=\sum_{j=1}^{r}\left\{\sum_{l=1}^{+\infty}\left(\left(1-\frac{\lambda_j}{l+\lambda_j+\mu_j-1}\right)^n-1+\frac{\lambda_j n}{l+\lambda_j+\mu_j-1}\right)\right\}.\end{aligned}$$

Writing,

$$\begin{aligned}\sum_{l=1}^{+\infty}\left(\left(1-\frac{\lambda_j}{l+\lambda_j+\mu_j-1}\right)^n-1+\frac{\lambda_j n}{l+\lambda_j+\mu_j-1}\right)&=\sum_{l=1}^{n}\cdots+\sum_{l=n+1}^{+\infty}\cdots\\&=T_1^{(1)}(n)+T_1^{(2)}(n).\end{aligned}$$

The estimates of $T_1^{(1)}(n)$ and $T_1^{(2)}(n)$ are given by the following lemma [6, Lemma 5.1 and Lemma 5.2].

Lemma 3.2.

(i) *For all* $n \geqslant |\lambda_j + \mu_j - 1|^2$, *we have*

$$T_1^{(1)}(n) = \lambda_j(n\log n) + \left(-\lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) + \lambda_j I + e^{-\lambda_j} - 1\right)n + O\big(|\lambda_j + \mu_j - 1|^2 + 1\big), \quad (3)$$

where the integral I *is defined by*

$$I = \int_1^{+\infty} e^{-\lambda_j t}\frac{dt}{t}.$$

(ii) *For all* $n \geqslant |\lambda_j + \mu_j - 1| + 2$, *we have*

$$T_1^{(2)}(n) = \big(\lambda_j - e^{-\lambda_j} + \lambda_j I'\big)n + O\big(|\lambda_j + \mu_j - 1| + 1\big), \quad (4)$$

where the integral I' *is defined by*

$$I' = \int_0^1 \big(1 - e^{-\lambda_j t}\big)\frac{dt}{t}.$$

Therefore, we have

$$T_1(n) = \sum_{j=1}^{r}\left(\lambda_j(n\log n) + \lambda_j\left(I' - I - 1 - \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j)\right)n + O(K_j + 1)\right),$$

where $K_j = \max\{|\lambda_j + \mu_j - 1|^2\colon\ 1 \leqslant j \leqslant r\}$. Then

$$T_1(n) = \frac{d_F}{2} n\log n + \frac{d_F}{2}\big(I' - I - 1\big)n - \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) + O\big(r(K_j + 1)\big).$$

Using the formula

$$\int_0^w \big(1 - e^{-t}\big)\frac{dt}{t} - \int_w^{+\infty} e^{-t}\frac{dt}{t} = \gamma + \log w,$$

where γ is the Euler constant, we get

$$\begin{aligned} I' - I &= \int_0^1 \big(1 - e^{-\lambda_j t}\big)\frac{dt}{t} - \int_1^{+\infty} e^{-\lambda_j t}\frac{dt}{t} \\ &= \lambda_j\left[\int_0^{\lambda_j} \big(1 - e^{-t}\big)\frac{dt}{t} - \int_{\lambda_j}^{+\infty} e^{-t}\frac{dt}{t}\right] \\ &= \lambda_j[\gamma + \log\lambda_j]. \end{aligned}$$

Hence,

$$T_1(n) = \frac{d_F}{2} n \log n + \left[\frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log\lambda\right] n - \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) + O\big(r(K_j + 1)\big),$$

with $\lambda = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}$. Assembling all the results above, we find

$$\lambda_F(-n) = \frac{d_F}{2} n \log n + \left\{\frac{d_F}{2}(\gamma - 1) + \frac{1}{2}\log\big(\lambda Q_F^2\big)\right\} n + H_F(n) + O\big(r(K_j + 1)\big). \tag{5}$$

Now, a bound for $H_F(n)$ is stated in the following lemma.

Lemma 3.3. *If the Riemann hypothesis holds for $F \in \mathcal{S}$, then*

$$H_F(n) = O(\sqrt{n}\log n).$$

Proof. We use a contour integral argument and we introduce the kernel function

$$k_n := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{l=1}^{n} \binom{n}{l} \left(\frac{1}{s}\right)^l.$$

If C is a contour enclosing the point $s = 0$ counterclockwise on a circle of small enough positive radius, the residue theorem gives

$$I(n) = \frac{1}{2i\pi} \int_C k_n(s)\left(-\frac{F'}{F}(s + 1)\right) ds = \sum_{l=1}^{n} \binom{n}{l} \eta_{l-1} = H_F(n).$$

Let $-3 < \sigma_0 < -2$, $\sigma_1 = 2\sqrt{n}$ and $T = \sqrt{n} + \epsilon_n$, for some $0 < \epsilon_n < 1$. Changing the contour C by the contour C' consisting of vertical lines with real part $\Re e(s) = \sigma_0$, $\Re e(s) = \sigma_1$ and the horizontal lines $\Im(s) = \pm T$. Then, using the residue theorem, we obtain

$$\begin{aligned} I'(n) &= \frac{1}{2i\pi} \int_{C'} k_n(s)\left(-\frac{F'}{F}(s + 1)\right) ds \\ &= H_F(n) + \sum_{\rho;\, |\Im\rho| \leqslant T} \left(1 + \frac{1}{\rho - 1}\right)^n - 1 + O(1). \end{aligned}$$

The term $O(1)$ evaluates the residues coming from the trivial zeros of $F(s)$. Using the symmetry $\rho \mapsto 1 - \overline{\rho}$, we can write

$$\left(\frac{1 - \overline{\rho}}{-\overline{\rho}}\right)^n - 1 = \left(\frac{\overline{\rho} - 1}{\overline{\rho}}\right)^n - 1.$$

Then

$$I'(n) = H_F(n) - \lambda_F(-n, T) + O(1),$$

where

$$\lambda_F(n,T)=\sum_{\rho;\,|\Im\rho|\leqslant T}1-\left(1-\frac{1}{\rho}\right)^n$$

with a parameter T. Observing that $|T-\sqrt{n}|<1$ and that there are $O(\log n)$ zeros in an interval of length one at this height. Furthermore, for each zero $\rho=\beta+i\gamma$ with $\sqrt{n}\leqslant|\Im(\rho)|<\sqrt{n}+1$, we have

$$\left|\left(\frac{\rho-1}{\rho}\right)\right|\leqslant\left|1+\frac{1}{n}\right|^{n/2}\leqslant 2.$$

Hence,

$$\left|\lambda_F(-n,\sqrt{n})-\lambda_F(-n,T)\right|=O(\log n).$$

We now choose the parameters σ_0 and T appropriately to avoid poles of the integrand. We may choose σ_0 so that the contour avoids any trivial zero and $T=\sqrt{n}+\epsilon_n$ with $0\leqslant\epsilon_n\leqslant 1$ so that the horizontal lines do not approach closer than $O(\log n)$ to any zero of $F(s)$. Recall that [15], for $-2<\Re e(s)<2$ there holds

$$\frac{F'}{F}(s)=\sum_{\{\rho;\,|\Im(\rho-s)|<1\}}\frac{1}{s-\rho}+O\big(\log\big(Q_F\big(1+|s|\big)\big)\big).$$

Then on the horizontal line in the interval $-2\leqslant\Re e(s)\leqslant 2$, we have

$$\left|\frac{F'}{F}(s+1)\right|=O\big(\log^2 T\big).$$

The Euler product for $F(s)$ converges absolutely for $\Re e(s)>1$. Hence the Dirichlet series for $\frac{F'}{F}(s)$ converges absolutely for $\Re e(s)>1$. More precisely for $\sigma=\Re e(s)>1$

$$\left|\frac{F'}{F}\right|(\sigma)<\infty.$$

For $\sigma=\Re e(s)>2$, we obtain the bound

$$\left|\frac{F'}{F}(s)\right|\leqslant\left|\frac{F'}{F}\right|(\sigma)\leqslant 2^{-(\sigma-2)}.$$

Consider the integral $I'(n)$ on the vertical segment (L_1) having $\sigma_1=2\sqrt{n}$. We have

$$\left|\left(1-\frac{1}{s}\right)^n-1\right|\leqslant\left(1+\frac{1}{\sigma_1}\right)^n+1\leqslant\left(1+\frac{1}{2\sqrt{n}}\right)^n\leqslant\exp(\sqrt{n}/2)<2^{\sqrt{n}}.$$

Then

$$\left|\frac{F'}{F}(s)\right|\leqslant C_0 2^{-2(\sqrt{n}+2)}.$$

Furthermore, the length of the contour is $O(\frac{n}{\log n})$, we obtain $|I'_{L_1}| = O(1)$. Let $s = \sigma + it$ be a point on one of the tow horizontal segment. We have $T \geqslant \sqrt{n}$, so that

$$\left|1 + \frac{1}{s}\right| \leqslant 1 + \frac{\sigma + 1}{\sigma^2 + T^2}.$$

By hypothesis $T^2 \geqslant n$, so for $-2 \leqslant \sigma \leqslant 2$, we have

$$\left|k_n(s)\right| \leqslant \left(1 + \frac{3}{4 + n}\right)^n + 1 = O(1)$$

and

$$\left|\frac{F'}{F}(s)\right| = O\left(\log^2 T\right) = O\left(\log^2 n\right)$$

since we have chosen the ordinate T to stay away from zeros of $F(s)$. We step across the interval (L_2) toward the right, in segments of length 1, starting from $\sigma = 2$. Furthermore

$$\left|\frac{k_n(s+1)+1}{k_n(s)+1}\right| \leqslant \left(1 + \frac{1}{T^2}\right)^n \leqslant e,$$

we obtain an upper bound for $|k_n(s)\frac{F'}{F}(s)|$ that decreases geometrically at each step, and after $O(\log n)$ steps it becomes $O(1)$ the upper bound is

$$\left|I'_{L_2,L_4}(n)\right| = O\left(\log^2 n + \sqrt{n}\right) = O(\sqrt{n}).$$

For the vertical segment (L_3) with $\Re e(s) = \sigma_0$, we have $|k_n(s)| = O(1)$ and $|\frac{F'}{F}(s)| = O(Q_F(\log(|s| + 1)))$. Since the segment (L_3) has length $O(\sqrt{n})$, we obtain

$$\left|I'_{L_3}\right| = O(\sqrt{n}\log n).$$

Totaling all these bounds above gives

$$H_F(n) = \lambda_F(-n, T) + O(\sqrt{n}\log n),$$

with $T = \sqrt{n} + \epsilon_n$.

If the Riemann hypothesis holds for $F(s)$, then we have

$$\left|1 - \frac{1}{\rho}\right| = 1.$$

Since each zero contributes a term of absolute value at most 2 to $\lambda_F(n, T)$, we obtain using the zero density estimate

$$\lambda_F(n, T) = O(T \log T + 1).$$

Therefore

$$\lambda_F(-n, \sqrt{n}) = \overline{\lambda_F(n, \sqrt{n})} = O(\sqrt{n}\log n)$$

and Lemma 3.3 follows. □

Using Lemma 3.3, the expression (5) of $\lambda_F(-n)$ and $\lambda_F(-n) = \overline{\lambda_F(n)}$, we obtain

$$RH \quad \Rightarrow \quad \lambda_F(n) = \frac{d_F}{2} n \log n + \left\{ \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log\bigl(\lambda Q_F^2\bigr) \right\} n + O(\sqrt{n} \log n). \qquad \square$$

Conversely, if

$$\lambda_F(n) = \frac{d_F}{2} n \log n + \left\{ \frac{d_F}{2}(\gamma - 1) + \frac{1}{2} \log\bigl(\lambda Q_F^2\bigr) \right\} n + O(\sqrt{n} \log n),$$

then, $\lambda_F(n)$ grows polynomially in n. Therefore, if *RH* is false then from conditions (a) or (b), some Li coefficients become exponentially large in n and negative and the asymptotic formula of $\lambda_F(n)$ rules out.

Examples. 1. In the case of the Riemann zeta-function, we have $d_\zeta = 1$, $Q_\zeta = \pi^{-1/2}$ and $\lambda = 1/2$. This reproves the asymptotic formula established by A. Voros in [16, Eq. (17), p. 59].

2. Lagarias established a similar asymptotic formula for $\lambda_n(\pi)$ [6, Eqs. (1.12) and (1.13)] in the case of the principal L-function $L(s, \pi)$ attached to an irreducible cuspidal unitary automorphic representation of $GL(N)$, as in Rudnick and Sarnak [13, §2]. Lagarias' result is about a subclass of automorphic L-functions and such functions belong to the Selberg class only if we assume the Ramanujan hypothesis.

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