The Mellin Transform

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1 Introduction

Robert Hjalmar Mellin was a Finnish mathematician who studied under Karl Weierstrass. He is accredited as the developer of the integral transform

$$\mathcal{M}[f(x); s] = f^*(s) = \int_0^\infty f(x)x^{s-1}dx$$

known as the Mellin transform.

We will first look at some examples and basic properties of the Mellin transform. Then we proceed to look at the correspondence between the asymptotic expansion of a function and singularities of the transformed function.

Finally we use the Mellin transform in asymptotic analysis for estimating asymptotically harmonic sums.

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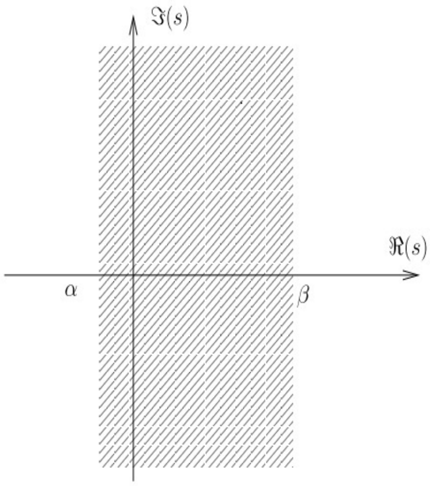
2 Basic properties of the Mellin transform

Definition 2.1 *Let $f(x)$ be absolutely integrable on any interval $0 \le x \le a$*

$$i.e. \int_0^a |f(x)|dx < \infty.$$

The largest open strip $\langle \alpha, \beta \rangle$ in which the integral converges is called the fundamental strip, where $\langle \alpha, \beta \rangle$ is the open strip of complex numbers $s = \sigma + it$ such that $\alpha < \sigma < \beta$.



Before we can say more about this fundamental strip, we turn our attention to the Laplace transform. Suppose for finite a that

$$\int_0^a |f(x)|dx < \infty$$

and $g(t) = O(e^{\alpha t})$ as $t \to \infty$ for some real constant α. Then the one-sided Laplace transform

$$\mathcal{L}^+[g(t); s] = \int_0^\infty g(t)e^{-st}dt$$

converges absolutely and is analytic in the right-half plane $\Re(s) > \alpha$. Similarly suppose

$$\int_0^a |g(-t)|dt < \infty$$

and $g(t) = O(e^{\beta t})$ as $t \to -\infty$ for some real constant β.

Then the one-sided Laplace transform

$$\mathcal{L}^{-}[g(t);s] = \int_{-\infty}^{0} g(t)e^{-st}dt$$
$$= \int_{0}^{\infty} g(-t)e^{st}dt$$

converges absolutely and is analytic in the left half plane $Re(s) < \beta$.
Under the same conditions on $g(t)$ and if $\beta > \alpha$ the two-sided Laplace transform

$$\mathcal{L}[g(t);s] = \int_{-\infty}^{\infty} g(t)e^{-st}dt$$

converges absolutely and is analytic in the vertical strip $\alpha < \Re(s) < \beta$.

Now if we let $t = -\log x$ and $g(-\log x) = f(x)$ then $e^{-st} = e^{s\log x} = x^{s}$.

Hence

$$\begin{aligned}\mathcal{L}[g(t);s] &= \int_{-\infty}^{\infty} g(t)e^{-st}dt\\ &= -\int_{\infty}^{0} x^{s-1}f(x)dx\\ &= \mathcal{M}[f(x);s].\end{aligned}$$

So the Mellin transform of $f(x)$ is the two-sided Laplace transform of $g(t)$ where $t = -\log x$ and it converges absolutely and is analytic in the vertical strip $\alpha < \Re(s) < \beta$.

Now since $g(t) = O(e^{\alpha t})$ as $t \to \infty$ we obtain $f(x) = O(x^{-\alpha})$ as $x \to 0^{+}$.

Also $g(t) = O(e^{\beta t})$ as $t \to -\infty$ implies $f(x) = O(x^{-\beta})$ as $x \to \infty$.
Summing up we have proved the following lemma:

Lemma 2.1 *The conditions $f(x) = O(x^{-\alpha})$ as $x \to 0^{+}$ and $f(x) = O(x^{-\beta})$ as $x \to \infty$ where $\alpha < \beta$ guarantee that $f^{*}(s)$ exists in the strip $\langle\alpha,\beta\rangle$.*

Hence monomials x^{c}, including constants do not have Mellin transforms.

Example 2.1

The function $f(x) = e^{-x}$ satisfies $e^{-x} = O(x^0)$ as $x \to 0^+$ and $e^{-x} = O(x^{-b})$ as $x \to \infty$ for any $b > 0$ so that its transform (the gamma function)

$$f^*(s) = \int_0^\infty e^{-x} x^{s-1} dx = \Gamma(s)$$

is defined and analytic on $\langle 0, \infty \rangle$.

Example 2.2

The function $f(x) = (e^x - 1)^{-1}$ satisfies $f(x) = O(x^{-1})$ as $x \to 0^+$ and $f(x) = O(x^{-b})$ for all $b > 0$ as $x \to \infty$. Hence $f(x)$ is analytic and defined on $\langle 1, \infty \rangle$. We find

$$\begin{aligned}
f^*(s) &= \int_0^\infty \frac{1}{1 - e^{-x}} x^{s-1} dx \\
&= \int_0^\infty \frac{e^{-x}}{(1 - e^{-x})} x^{s-1} dx \\
&= \int_0^\infty \sum_{n=1}^\infty e^{-nx} x^{s-1} dx \\
&= \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{s-1} dx \\
&= \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} \\
&= \Gamma(s)\zeta(s).
\end{aligned}$$

We require that $\Re(s) > 1$ for convergence of the Riemann-zeta function and we see that this validates the strip $\langle 1, \infty \rangle$ on which $f^*(s)$ is defined and analytic.

Example 2.3

The function $f(x) = (1 + x)^{-1}$ is $O(x^0)$ as $x \to 0^+$ and $O(x^{-1})$ as $x \to \infty$. Hence a guaranteed strip of existence for $f^*(s)$ is $\langle 0, 1 \rangle$. Set $x = \frac{t}{1-t}$. Then

$$\begin{aligned}
f^*(s) &= \int_0^1 \left(\frac{t}{1-t} \right)^{s-1} \frac{1}{1 + \frac{t}{1-t}} (1-t)^{-2} dt \\
&= \int_0^1 \left(\frac{t}{1-t} \right)^{s-1} (1-t)^{-1} dt \\
&= \int_0^1 t^{s-1} (1-t)^{-s} dt \\
&= \beta(s, 1-s) \\
&= \Gamma(s)\Gamma(1-s).
\end{aligned}$$

We can also evaluate the Mellin transform of $f(x)$ using complex analysis. Consider the branch of the function $\frac{z^{s-1}}{z+1}$ defined on the slit plane $\mathbb{C}\backslash[0,\infty)$ by

$$f(z) = \frac{r^{s-1}e^{i(s-1)\theta}}{z+1}; \quad z = re^{i\theta}, \quad 0 < \theta < 2\pi.$$

The values on the bottom edge are obtained from those on the top edge by multiplying by the phase factor $e^{2\pi i(s-1)}$.

For $\epsilon > 0$ small and $R > 0$ large, we consider the keyhole domain D (see Figure 1 in the appendix) consisting of z in the slit plane $\mathbb{C}\backslash[0,\infty)$ satisfying $\epsilon < |z| < R$. $f(z)$ has a pole in D, a simple pole at $z = -1$, with residue

$$\begin{aligned}
\operatorname{Res}\left[\frac{z^{s-1}}{z+1}, -1\right] &= \lim_{z\to -1}(z+1)\frac{z^{s-1}}{z+1} \\
&= \lim_{z\to -1}\frac{z^s}{z} \\
&= -e^{\pi i s}
\end{aligned}$$

Hence the residue theorem yields

$$\int_{\partial D} f(z)dz = -2\pi i e^{\pi i s}$$

The integral around δD breaks into the sum of 4 integrals.

$$\implies -2\pi i e^{\pi i s} = \int_{\epsilon}^{R}\frac{x^{s-1}}{1+x}dx + \int_{\Gamma_R}\frac{z^{s-1}}{z+1}dz + \int_{R}^{\epsilon}\frac{e^{2\pi i(s-1)}x^{s-1}}{1+x}dx + \int_{\Gamma_\epsilon}\frac{z^{s-1}}{z+1}dz$$

For the integrals over Γ_R and Γ_ϵ we have

$$\left|\int_{\Gamma_R}\frac{z^{s-1}}{z+1}dz\right| \leq \frac{R^{s-1}}{R-1}2\pi R = O\left(R^{s-1}\right)$$

$$\left|\int_{\Gamma_\epsilon}\frac{z^{s-1}}{1+z}dz\right| \leq \frac{\epsilon^{s-1}}{1-\epsilon}2\pi\epsilon = O\left(\epsilon^{s}\right)$$

Since $0 < s < 1$ both these integrals vanish as $R \to \infty$ and $\epsilon \to 0$.

$$\implies -2\pi i e^{\pi i s} = \left(1 - e^{2\pi i(s-1)}\right)\int_0^\infty \frac{x^{s-1}}{1+x}dx$$

$$\begin{aligned}
\implies \int_0^\infty \frac{x^{s-1}}{1+x}dx &= \frac{-2\pi i e^{\pi i s}}{1-e^{2\pi i(s-1)}} \\
&= \frac{2\pi i}{e^{\pi i s - 2\pi i} - e^{-\pi i s}} \\
&= \frac{\pi}{\sin(\pi s)}
\end{aligned}$$

Hence $\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$.

Example 2.4

Suppose $f(x) = (1 + x)^{-n}$. Setting $t = \frac{x}{1+x}$ we obtain

$$\begin{aligned} f^*(s) &= \int_0^1 t^{s-1}(1-t)^{n-s-1}dt \\ &= \beta(s, n-s) \\ &= \frac{\Gamma(s)\Gamma(n-s)}{\Gamma(n)}. \end{aligned}$$

Example 2.5

Suppose $f(t) = \sin t = \frac{1}{2i}(e^{it} - e^{-it})$. Consider the region D as in Figure 2 of the appendix.

Since $e^{-t}t^{s-1}$ is analytic in D we have by Cauchy's theorem:

$$0 = \oint_{\partial D} e^{-t}t^{s-1}dt$$

$$= \int_\epsilon^R e^{-u}u^{s-1}du + i\int_0^R e^{-(R+iu)}(R+iu)^{s-1}du - \int_0^R e^{-(u+iR)}(u+iR)^{s-1}du$$

$$-i\int_\epsilon^R e^{-iu}(iu)^{s-1}du - \int_0^{\frac{\pi}{2}} e^{-\epsilon e^{i\theta}}(\epsilon e^{i\theta})^{s-1}i\epsilon e^{i\theta}d\theta$$

Letting $\epsilon \to 0$ and $R \to \infty$ we have:

$$\int_0^{\frac{\pi}{2}} \left|e^{-\epsilon e^{i\theta}}(\epsilon e^{i\theta})^{s-1}i\epsilon e^{i\theta}\right| d\theta = O(\epsilon^s) \to 0 \quad if \quad \mathrm{Res}(s) > 0$$

$$\int_0^R \left|e^{-(R+iu)}(R+iu)^{s-1}\right| du = O\left(e^{-R}R^{Re(s-1)}R\right) \to 0$$

$$\int_0^R \left|e^{-(u+iR)}(u+iR)^{s-1}\right| du = O\left(R^{Re(s-1)}\int_0^R e^{-u}du\right) \to 0 \quad if \quad \mathrm{Res}(s) < 1$$

$$\implies \int_0^\infty e^{-u}u^{s-1}du = i^s\int_0^\infty e^{-iu}u^{s-1}du$$

$$\implies \int_0^\infty e^{-iu}u^{s-1}du = i^{-s}\int_0^\infty e^{-u}u^{s-1}du = e^{-s\log i}\Gamma(s) = e^{-\frac{i\pi s}{2}}\Gamma(s)$$

Likewise replacing i with $-i$ in the previous equation we obtain

$$\int_0^\infty e^{iu}u^{s-1}du = e^{\frac{i\pi s}{2}}\Gamma(s).$$

Combining the two we get

$$\int_0^\infty (\sin t)t^{s-1}dt = \frac{1}{2i}\left(e^{-\frac{i\pi s}{2}} - e^{\frac{i\pi s}{2}}\right) = \sin\left(\frac{\pi s}{2}\right)\Gamma(s).$$

We proceed to look at some functional properties of the Mellin transform.

Theorem 2.1 *Let $f(x)$ be a function whose transform admits the fundamental strip $\langle\alpha,\beta\rangle$. Let ρ, μ and v be positive real numbers. Then the following relations hold:*

$$(i)\quad \mathcal{M}[f(\mu x);s] = \mu^{-s}f^*(s) \quad s\in\langle\alpha,\beta\rangle$$

$$(ii)\quad \mathcal{M}\left[\sum_{k\in A}\lambda_k f(\mu_k x);s\right] = \left(\sum_{k\in A}\lambda_k {\mu_k}^{-s}\right)f^*(s) \quad ,\mu_k>0$$

$$(iii)\quad \mathcal{M}[x^v f(x);s] = f^*(s+v) \quad s\in\langle\alpha-v,\beta-v\rangle$$

$$(iv)\quad \mathcal{M}[f(x^\rho);s] = \frac{1}{\rho}f^*\left(\frac{s}{\rho}\right) \quad s\in\langle\rho\alpha,\rho\beta\rangle$$

Proof:

(i) $\mathcal{M}[f(\mu x);s]$

$$= \int_0^\infty x^{s-1}f(\mu x)dx$$

$$= \int_0^\infty f(t)\left(\frac{t}{\mu}\right)^{s-1}\frac{dt}{\mu}$$

$$= \mu^{-s}f^*(s).$$

(ii) Since the Mellin transform is linear this follows immediately from (i).

(iii) $\mathcal{M}[x^v f(x);s]$

$$= \int_0^\infty f(x)x^{s+v-1}dx$$

$$= f^*(s+v).$$

(iv) $\mathcal{M}[f(x^\rho);s]$

$$= \int_0^\infty f(x^\rho)x^{s-1}dx$$

$$= \int_0^\infty f(t)t^{\frac{s}{\rho}-1}\frac{dt}{\rho}$$

$$= \frac{1}{\rho}f^*\left(\frac{s}{\rho}\right).$$

We have for example from Theorem 2.1(iv) $\mathcal{M}[e^{-x^2}; s] = \frac{1}{2}\Gamma\left(\frac{s}{2}\right)$ on $\langle 0, \infty\rangle$ with $f(x) = e^{-x}$ and $\rho = 2$. Wanting to expand our range of Mellin transforms we find by differentiation under the integral sign:

$$\frac{d}{ds} f^*(s) = \mathcal{M}[f(x)\log x; s]$$

For instance the transform of $e^{-x}\log x$ is $\Gamma'(s)$. If we want to transform the derivative of a function integration by parts yields:

$$\int_0^\infty f'(x)x^{s-1}dx = \left[f(x)x^{s-1}\right]_0^\infty - (s-1)\int_0^\infty f(x)x^{s-2}dx$$

The term $\left[f(x)x^{s-1}\right]_0^\infty$ equals 0 since we assume the reason why $f^*(s)$ exists for $s \in \langle\alpha, \beta\rangle$ is because $\lim_{x\to 0} x^\rho f(x) = 0$ for $\Re(s) > \alpha$ and $\lim_{x\to\infty} x^\rho f(x) = 0$ for $\Re(s) < \beta$. Thus $\mathcal{M}[f'(x); s] = -(s-1)f^*(s-1)$. For instance from Example 2.3 we can derive the Mellin transform of $f(x) = \log(1+x)$

$$\frac{\pi}{\sin(\pi s)} = \mathcal{M}[f'(x); s] = -(s-1)\mathcal{M}[\log(1+x); s-1]$$

$$\implies \mathcal{M}[f(x); s-1] = -\frac{\pi}{(s-1)\sin(\pi s)}$$

$$\implies \mathcal{M}[f(x); s] = -\frac{\pi}{s\sin(\pi(s+1))} = \frac{\pi}{s\sin(\pi s)}$$

for $s \in \langle -1, 0\rangle$.

<u>The inversion formula:</u>

The inversion formula for the two-sided Laplace transform of $g(t)$ is given by $g(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{L}[g(t); s]e^{st}ds$ for any c inside the region of convergence, provided that g is continuous at t. Setting $t = -\log x$ and $f(x) = g(-\log x)$ we obtain the inversion formula for the Mellin transform:

$$f(x) = g(-\log x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{M}[f(x); s]x^{-s}ds$$

We will prove the inversion formula for the Laplace transform, from which the inversion formula of the Mellin transform follows. Before we can do this we need to prove the following lemma (Riemann-Lebesgue):

Lemma 2.2

$I(\omega) = \int_a^b f(t)e^{i\omega t}dt \to 0$ as $\omega \to \infty$ if $f(t)$ is continuous on $[a, b]$.

Proof:

$$
\begin{aligned}
I(\omega) &= \int_a^{a+\frac{\pi}{\omega}} f(t)e^{i\omega t}dt + \int_{a+\frac{\pi}{\omega}}^{b} f(t)e^{i\omega t}dt \\
&= \int_a^{a+\frac{\pi}{\omega}} f(t)e^{i\omega t}dt - \int_a^{b-\frac{\pi}{\omega}} f\left(t+\frac{\pi}{\omega}\right)e^{i\omega t}dt
\end{aligned}
$$

and

$$
I(\omega) = \int_a^{b-\frac{\pi}{\omega}} f(t)e^{i\omega t}dt + \int_{b-\frac{\pi}{\omega}}^{b} f(t)e^{i\omega t}dt
$$

$$
\implies I(\omega) = \frac{1}{2}\int_a^{a+\frac{\pi}{\omega}} f(t)e^{i\omega t}dt + \frac{1}{2}\int_{b-\frac{\pi}{\omega}}^{b} f(t)e^{i\omega t}dt + \frac{1}{2}\int_a^{b-\frac{\pi}{\omega}} \left(f(t) - f\left(t+\frac{\pi}{\omega}\right)\right)e^{i\omega t}dt
$$

In the first two integrals $f(t)e^{i\omega t}$ is bounded, since it is a continuous function on a closed interval. Since the length of these integrals goes to 0 as $\omega \to \infty$ they both vanish. Similarly for the last integral we find $\left(f(t) - f\left(t+\frac{\pi}{\omega}\right)\right) \to 0$ as $\omega \to \infty$ and since the length of this integral is bounded it also vanishes so that $\lim_{\omega\to\infty} I(\omega) = 0$

Note that the proof also holds for an infinite interval if

$$
\int_0^{\infty} |f(t)|dt < \infty :
$$

Choose T large enough so that for all $\epsilon > 0$

$$
\int_T^{\infty} |f(t)|dt \leq \epsilon
$$

$$
\begin{aligned}
&\implies \int_0^{\infty} f(t)e^{i\omega t}dt = \int_0^{T} f(t)e^{i\omega t}dt + \int_T^{\infty} f(t)e^{i\omega t}dt \\
&\implies \lim_{\omega\to\infty}\int_0^{\infty} f(t)e^{i\omega t}dt = 0
\end{aligned}
$$

since ϵ is arbitrary.

Dirichlet conditions and Dirichlet integrals:

$f(t)$ satisfies the Dirichlet conditions if it only has a finite number of discontinuities with finite jumps and a finite number of minima/maxima. (i.e. f is piecewise monotone)
The second mean value theorem for integrals states:
If $f(t)$ is monotone $[a, b]$, then for any continuous $g(t)$

$$
\int_a^b f(t)g(t)dt = f(a)\int_a^c g(t)dt + f(b)\int_c^b g(t)dt
$$

for some $c \in [a,b]$.
Assume $f(t)$ satisfies the Dirichlet conditions. We would like to determine

$$\lim_{\omega\to\infty}\int_0^b \frac{\sin\omega t}{t}dt = \lim_{\omega\to\infty} Im\int_0^b \frac{e^{i\omega t}}{t}dt$$

Choose c such that $f(t)$ is continuous and monotone on $[0,c]$.

$$\int_0^b f(t)\frac{\sin\omega t}{t}dt = \int_0^c f(t)\frac{\sin\omega t}{t}dt + \int_c^b f(t)\frac{\sin\omega t}{t}dt$$

$$\lim_{\omega\to\infty}\int_c^b \frac{\sin\omega t}{t}dt = 0$$

by the Riemann-Lebesgue lemma.
Apply the mean value theorem:

$$\begin{aligned}\int_0^c f(t)\frac{\sin\omega t}{t}dt &= f(0^+)\int_0^h \frac{\sin\omega t}{t}dt + f(c)\int_h^c \frac{\sin\omega t}{t}dt\\ &= f(0^+)\int_0^c \frac{\sin\omega t}{t}dt + (f(c)-f(0^+))\int_h^c \frac{\sin\omega t}{t}dt\end{aligned}$$

$$\begin{aligned}\int_0^c \frac{\sin\omega t}{t}dt &= \int_0^{\omega c} \frac{\omega\sin x}{x}\frac{dx}{\omega}\\ &= \int_0^{\omega c}\frac{\sin x}{x}dx\\ &= \lim_{\omega\to\infty}\int_0^{\omega}\frac{\sin x}{x}dx\end{aligned}$$

$\int_0^\infty \frac{\sin x}{x}dx$ can not be evaluated by elementary analytic methods. To evaluate it we use complex analysis and the following lemma (Jordan's lemma):

Lemma 2.3 *If Γ_R is the semicircular contour $z(\theta) = Re^{i\theta}$ where $0 \le \theta \le \pi$, in the upper half-plane, then $\int_{\Gamma_R} |e^{iz}||dz| < \pi$.*

Now we show that $\displaystyle\lim_{R\to\infty}\int_0^R \frac{\sin x}{x}dx = \frac{\pi}{2}$.

Since the integrand is an even function it is equivalent to $\displaystyle\lim_{R\to\infty}\int_{-R}^R \frac{\sin x}{x}dx = \pi$

To evaluate the limit we consider $f(z) = e^{iz}$ which only has one pole, a simple pole at $z = 0$.

$$Res\left[\frac{e^{iz}}{z}, 0\right] = 1$$

Consider Figure 3 in the appendix:

The integrand around ∂D breaks into 4:

$$\int_{\delta D} f(z)dz = \left(\int_{-R}^{-\epsilon} + \int_{C_\epsilon} + \int_{\epsilon}^{R} + \int_{\Gamma_R}\right) f(z)dz$$

Since f is analytic on D, Cauchy's theorem gives:

$$0 = \int_{\partial D} f(z)dz \quad (*)$$

The integrand C_ϵ is handled by the fractional residue theorem with angle $-\pi$:

$$\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{iz}}{z}dz = -\pi i Res\left[\frac{e^{iz}}{z}, 0\right] = -\pi i$$

Taking the limit as $\epsilon \to 0$ in $(*)$ we obtain:

$$0 = \int_{-R}^{R} \frac{e^{ix}}{x}dx - \pi i + \int_{\Gamma_R} \frac{e^{iz}}{z}dz$$

Taking imaginary parts gives:

$$\pi = \int_{-R}^{R} \frac{\sin x}{x}dx + Im\int_{\Gamma_R} \frac{e^{iz}}{z}dz$$

Now the integral over Γ_R is handled by Jordan's lemma:

$$\left|\int_{\Gamma_R} \frac{e^{iz}}{z}dz\right| \le \int_{\Gamma_R} \left|\frac{e^{iz}}{z}\right| |dz| < \frac{\pi}{R} \to 0$$

as $R \to \infty$. Thus we have $\lim_{R\to\infty}\int_{-R}^{R} \frac{\sin x}{x}dx = \pi$. Hence

$$\lim_{R\to\infty}\int_{0}^{R} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

Returning to where we were just before Jordan's lemma we have for the other part:

$$(f(c) - f(0^+))\int_{h}^{c} \frac{\sin \omega t}{t}dt \to 0$$

if $\omega \to \infty$, since $(f(c) - f(0^+))$ can be made arbitrarily close to 0 by the choice of c and since $\int_{h}^{c} \frac{\sin \omega t}{t}dt$ is bounded. It follows that $\lim_{\omega\to\infty}\int_{0}^{b} f(t)\frac{\sin \omega t}{t}dt = \frac{\pi}{2}f(0^+)$. Likewise $\lim_{\omega\to\infty}\int_{-b}^{0} f(t)\frac{\sin \omega t}{t}dt = \frac{\pi}{2}f(0^-)$. By the Riemann-Lebesgue lemma we have

$$\lim_{\omega\to\infty}\int_{b}^{\infty} f(t)\frac{\sin \omega t}{t}dt = 0.$$

Hence:

$$\lim_{\omega\to\infty}\int_{0}^{\infty} f(t)\frac{\sin \omega t}{t}dt = \frac{\pi}{2}f(0^+).$$

$$\lim_{\omega\to\infty}\int_{-\infty}^{0} f(t)\frac{\sin\omega t}{t}dt = \frac{\pi}{2}f(0^-)$$

So that

$$\lim_{\omega\to\infty}\int_{-\infty}^{\infty} f(t)\frac{\sin\omega t}{t}dt = \frac{\pi}{2}\left(f(0^+) + f(0^-)\right)$$

Now we can finally prove the inversion formula of the one-sided Laplace transform. Assume f is continuous at t. Since the integral of the Laplace transform is uniformly convergent we can switch integrals in the following expression:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-iR}^{c+iR} F(s)e^{st}ds &= \frac{1}{2\pi i}\int_{c-iR}^{c+iR}\left(\int_{0}^{\infty} f(u)e^{-su}du\right)e^{st}ds \\
&= \frac{1}{2\pi i}\int_{0}^{\infty} f(u)\int_{c-iR}^{c+iR} e^{s(t-u)}dsdu \\
&= \frac{1}{2\pi i}\int_{0}^{\infty} f(u)\int_{-R}^{R} e^{(c+ir)(t-u)}idrdu \quad , s = c + ir \\
&= \frac{1}{2\pi}\int_{0}^{\infty} f(u)e^{c(t-u)}\int_{-R}^{R} e^{ir(t-u)}drdu
\end{aligned}$$

$$\begin{aligned}
\int_{-R}^{R} e^{ir(t-u)}dr &= \frac{e^{(t-u)iR} - e^{-(t-u)iR}}{i(t-u)} \\
&= \frac{2\sin\left((t-u)R\right)}{t-u}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{2\pi i}\int_{c-iR}^{c+iR} F(s)e^{st}ds &= \frac{1}{\pi}\int_{0}^{\infty} f(u)e^{c(t-u)}\frac{\sin\left((u-t)R\right)}{u-t}du \\
&= \frac{1}{\pi}\int_{-t}^{\infty} f(v+t)e^{-cv}\frac{\sin(vR)}{v}dv
\end{aligned}$$

Since f is continuous if we let $R \to \infty$ the integral converges to

$$\frac{\pi}{2}\left(f(t^+) + f(t^-)\right) = \pi f(t)$$

So $f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}ds$ if f satisfies the Dirichlet conditions and is continuous at t. By the change of variable previously mentioned we obtain the inversion formula for the Mellin transform from the two-sided Laplace transform which is a sum of two one-sided Laplace transforms.

Theorem 2.2 *(Inversion formula)*
Let $f(x)$ be absolutely integrable with fundamental strip $\langle\alpha,\beta\rangle$. If f satisfies the Dirichlet conditions and is continuous at x then for any c in the interval (α,β)

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds\ .$$

3 The fundamental correspondence

There is a correspondence between the asymptotic expansion of a function at 0 and ∞ and poles of its Mellin transform in a left and respectively right half-plane. The correspondence fares both ways and forms the basis of an asymptotic process:
For estimating asymptotically a function $f(x)$, determine its Mellin transform and translate back its singularities into asymptotic terms in the expansion of $f(x)$.

3.1 Singularities

If $\phi(s)$ is meromorphic at $s = s_0$ it admits a Laurent expansion near s_0:

$$\phi(s) = \sum_{k \geq -r} c_k (s - s_0)^k.$$

The function $\phi(s)$ has a pole of order r if $r > 0$ and $c_{-r} \neq 0$.

Definition 3.1 *A singular element of $\phi(s)$ at s_0 is an initial sum of the Laurent expansion truncated at terms of order $O(1)$ or smaller.*

Example 3.1

Consider the function $\phi(s) = \dfrac{1}{s^2(s+1)}$, which has two poles in the complex plane, a double pole at $s_0 = 0$ and a simple pole at $s_0 = -1$.

$$\begin{aligned}
\frac{1}{s^2(s+1)} &= \frac{1}{s+1}\frac{1}{(1-(s+1))^2} \\
&= \frac{1}{s+1}\sum_{n=0}^{\infty}\binom{-2}{n}[-(s+1)]^n \\
&= \frac{1}{s+1} + 2 + 3(s+1) + ...
\end{aligned}$$

$$\begin{aligned}
\frac{1}{s^2(s+1)} &= \frac{1}{s^2}\frac{1}{1-(-s)} \\
&= \frac{1}{s^2} - \frac{1}{s} + 1 - s + ...
\end{aligned}$$

Singular elements at $s_0 = 0$ are:

$$\left[\frac{1}{s^2} - \frac{1}{s}\right], \left[\frac{1}{s^2} - \frac{1}{s} + 1\right], \left[\frac{1}{s^2} - \frac{1}{s} + 1 - s\right], ...$$

We can truncate the Laurent expansion wherever we want as long as we include all the terms with negative degree of s. Similarly we have the following singular elements for $s_0 = -1$:

$$\left[\frac{1}{s+1}\right], \left[\frac{1}{s+1} + 2\right], ...$$

Definition 3.2 *Let $\phi(s)$ be meromorphic in Ω with P including all the poles of $\phi(s)$ in Ω. A singular expansion of $\phi(s)$ in Ω is a formal sum of singular elements of $\phi(s)$ at all points of P. When E is a singular expansion of $\phi(s)$ in Ω, we write $\phi(s) \asymp E$ ($s \in \Omega$).*

Example 3.2

For instance

$$\frac{1}{s^2(s+1)} \asymp \left[\frac{1}{s+1}\right]_{s=-1} + \left[\frac{1}{s^2} - \frac{1}{s}\right]_{s=0} + \left[\frac{1}{2}\right]_{s=1}$$

This expansion is a concise way of combining information contained in the Laurent expansions of the function $\phi(s) \equiv s^{-2}(s+1)^{-1}$ at the three points of $P = \{-1, 0, 1\}$. Note that there is no pole at $s_0 = 1$, but we can include an extra singular element if we want. P is to include all poles of the function but may also include other points. Singular expansion is a formal sum which we do not want to evaluate. It only shows us which poles a function have.

Example 3.3

Recall the gamma function

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

for $s \in \langle 0, \infty \rangle$. The well-known functional equation $\Gamma(s+1) = s\Gamma(s)$ follows if we integrate by parts. This allows us to extend $\Gamma(s)$ to a meromorphic function in the whole of $\mathbb{C}$. We have from the functional equation:

$$\Gamma(s) = \frac{\Gamma(s+m+1)}{s(s+1)(s+2)...(s+m)}$$

Thus $\Gamma(s)$ has poles at the points $s = -m$, $m \in \mathbb{N} \cup \{0\}$ and $\Gamma(s) \sim \frac{(-1)^m}{m!}\frac{1}{s+m}$ as $s \to -m$ .This means the right side is a singular element of $\Gamma(s)$ at $s = -m$. Hence we obtain the following singular expansion in the whole of $\mathbb{C}$

$$\Gamma(s) \asymp \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}.$$

3.2 Direct mapping

e^{-x} has a Taylor expansion at $x = 0$:

$$e^{-x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$$

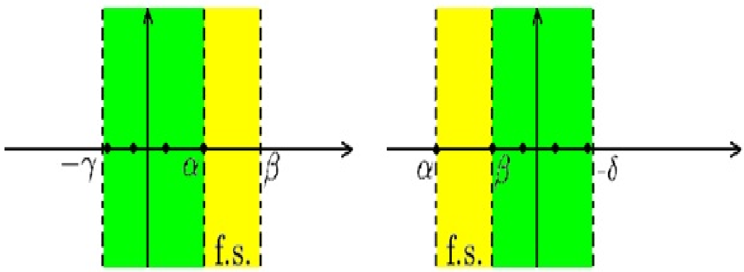
Comparing this with the singular expansion of $\Gamma(s)$ we see there is a striking coincidence of coefficients expressed by the rule $x^k \mapsto \frac{1}{s+k}$. We will see that this is a general phenomenon.
Some information about the connection between the asymptotic expansion of our initial function and properties of its Mellin transform are summed up in the following table:

$f(x)$	$f^*(s)$
Order at 0: $O(x^{-\alpha})$	Leftmost boundary of f.s. at $Re(s) = \alpha$
Order at ∞: $O(x^{-\beta})$	Rightmost boundary of f.s. at $\Re(s) = \beta$
Expansion till $O(x^{\gamma})$ at 0	Meromorphic continuation till $\Re(s) = -\gamma$
Expansion till $O(x^{\delta})$ at ∞	Meromorphic continuation till $\Re(s) = -\delta$

The following table will follow from the Direct mapping theorem:

$f(x)$	$f^*(s)$
Term $x^a(\log x)^k$ at 0	Pole with singular element $\dfrac{(-1)^k k!}{(s+a)^{k+1}}$
Term $x^a(\log x)^k$ at ∞	Pole with singular element $-\dfrac{(-1)^k k!}{(s+a)^{k+1}}$

Visualising these two tables:



Theorem 3.1

Let $f(x)$ have Mellin transform $f^*(s)$ with non-empty fundamental strip $\langle\alpha,\beta\rangle$.
(i) Assume that $f(x)$ admits as $x \to 0^+$ a finite asymptotic expansion of the form

$$f(x) = \sum_{(a,k)\in A} c_{a,k} x^a (\log x)^k + O(x^{\gamma})$$

where $-\gamma < -a \leq \alpha$ and the k are nonnegative. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle -\gamma,\beta\rangle$ where it admits the singular expansion

$$f^*(s) \asymp \sum_{(a,k)\in A} c_{a,k} \frac{(-1)^k k!}{(s+a)^{k+1}}, \; s \in \langle -\gamma,\beta\rangle.$$

(ii) Similarly assume that $f(x)$ admits as $x \to \infty$ a finite asymptotic expansion of the form

$$f(x) = \sum_{(a,k)\in A} c_{a,k} x^a (\log x)^k + O(x^\gamma)$$

where now $\beta \leq -a \leq -\gamma$. Then $f^*(s)$ is continuable to a meromorphic function in the strip $\langle \alpha, -\gamma \rangle$ where

$$f^*(s) \asymp -\sum_{(a,k)\in A} c_{a,k} \frac{(-1)^k k!}{(s+a)^{k+1}}, \; s \in \langle \alpha, -\gamma \rangle.$$

Thus terms in the asymptotic expansion of $f(x)$ at 0 induce poles of $f^*(s)$ in a strip to the left of the fundamental strip. Terms in the expansion at ∞ induce poles in a strip to the right.

Proof:
Since $\mathcal{M}\left[f\left(\frac{1}{x}\right); s\right] = \mathcal{M}\left[(f(x)); -s\right]$ we only need to treat case (i) corresponding to $x \to 0^+$.
By assumption $g(x) = f(x) - \sum_{(a,k)\in A} c_{a,k} x^a (logx)^k$ satisfies $g(x) = O(x^\gamma)$. For $s \in \langle \alpha, \beta \rangle$ we have:

$$\begin{aligned} f^*(s) &= \int_0^1 f(x) x^{s-1} dx + \int_1^\infty f(x) x^{s-1} dx \\ &= \int_0^1 g(x) x^{s-1} dx + \int_0^1 \left(\sum_{(a,k)\in A} c_{a,k} x^a (logx)^k \right) x^{s-1} dx + \int_1^\infty f(x) x^{s-1} dx \end{aligned}$$

In the last line the first integral defines an analytic function of s in the strip $\langle -\gamma, \beta \rangle$ since $g(x) = O(x^\gamma)$ as $x \to 0$, the third integral is analytic in $\langle -\infty, \beta \rangle$ so that the sum of these two are analytic in $\langle -\gamma, \beta \rangle$. Finally

$$\int_0^1 \left(\sum_{(a,k)\in A} c_{a,k} x^a (logx)^k \right) x^{s-1} dx = \sum_{(a,k)\in A} c_{a,k} \frac{(-1)^k k!}{(s+a)^{k+1}}$$

This result follows since

$$\int_0^1 (logx)^k x^{s+a-1} dx = -\frac{k}{s+a} \int_0^1 (logx)^{k-1} x^{s+a-1} dx$$

from integration by parts. Integrating by parts k times yields the above result. Now $\sum_{(a,k)\in A} c_{a,k} \frac{(-1)^k k!}{(s+a)^{k+1}}$ is meromorphic in all of $\mathbb{C}$ and provides the singular expansion of $f^*(s)$ in the extended strip. This completes the proof of the Direct mapping theorem.

Note that in the case where there exists a complete expansion of $f(x)$ at 0 (or ∞), the transform $f^*(s)$ becomes meromorphic in a complete left (or right) half-plane. This situation always occurs for functions that are analytic at 0 (or ∞).

Example 3.4

The Riemann zeta function plays a pivotal role in analytic number theory and has applications in physics, probability theory and applied statistics. Using singular expansion and the Direct mapping theorem we can easily derive some of its properties.

Recall Example 2.2 where we showed if $f(x) = \frac{e^{-x}}{1-e^{-x}}$ then $f^*(s) = \Gamma(s)\zeta(s)$ with fundamental strip $\langle 1, \infty \rangle$. $f(x)$ is exponentially small at infinity and it admits a complete expansion near $x = 0$:

$$\frac{e^{-x}}{1-e^{-x}} = \sum_{k=-1}^{\infty} B_{k+1} \frac{x^k}{(k+1)!}$$

which defines the Bernoulli numbers B_k: $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, ...$

Thus $f^*(s) = \Gamma(s)\zeta(s)$ is meromorphic on the whole of $\mathbb{C}$ with singular expansion $\Gamma(s)\zeta(s) \asymp \displaystyle\sum_{k=-1}^{\infty} \frac{B_{k+1}}{(k+1)!} \frac{1}{s+k}$. If we compare it with the singular expansion of the gamma function $\Gamma(s) \asymp \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{s+k}$ we can extract the singular expansion of the zeta function

$$\zeta(s) \asymp \left[\frac{1}{s-1}\right]_{s=1} + \sum_{j=0}^{\infty} \left[(-1)^j \frac{B_{j+1}}{j+1}\right]_{s=-j}$$

Hence the zeta function is meromorphic in the whole of $\mathbb{C}$ with only a simple pole at $s = 1$. It follows from the singular expansion that

$$\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}, m \in \mathbb{N} \cup \{ 0 \}.$$

Hence $\zeta(0) = -\frac{1}{2}$. Since $B_{2k+1} = 0$ for $k \in \mathbb{N}$ it follows that $\zeta(-2m) = 0$ for all $m \in \mathbb{N}$. These zeros are called the trivial zeros of the zeta function. The nontrivial zeros of the zeta function lie in the strip $\{ 0 \leq \Re(s) \leq 1 \}$ which is called the critical strip.

The most famous unsolved problem in complex analysis is the Riemann hypothesis which states that the nontrivial zeros of the zeta function lie on the line $\{\Re(s) = \frac{1}{2} \}$. It is also one of the Clay Mathematics Institute Millennium Prize Problems.

Computer calculations have shown that the first 10 trillion zeros lie on the critical line. Whether the Riemann hypothesis is true remains a famous unresolved problem.

Example 3.5

Recall from Example 2.3 the Mellin pair $f(x) = (1+x)^{-1}$, $f^*(s) = \frac{\pi}{\sin \pi s}$ with fundamental strip $\langle 0, 1\rangle$. As $x \to 0^+$:

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$$

and as $x \to \infty$:

$$\frac{1}{1+x} = \frac{x^{-1}}{1+x^{-1}} = \sum_{n=1}^{\infty}(-1)^{n-1}x^{-n}$$

These expansions translate into:

$$f^*(s) \asymp \sum_{n=0}^{\infty} \frac{(-1)^n}{s+n} \quad s \in \langle -\infty, 1\rangle$$

which is the continuation of the transformed function to the left of the fundamental strip and

$$f^*(s) \asymp -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{s-n} \quad s \in \langle 0, \infty\rangle$$

which is the continuation of the transformed function to the right of the fundamental strip. We can combine the two singular expansions into one, giving us a singular expansion in the whole of $\mathbb{C}$:

$$f^*(s) = \frac{\pi}{\sin \pi s} \asymp \sum_{n \in \mathbb{Z}} \frac{(-1)^n}{s+n}\ , s \in \mathbb{C}.$$

3.3 Converse mapping:

Under a set of mild conditions, a converse to the Direct mapping theorem also holds: The singularities of a Mellin transform which is small enough towards $\pm i\infty$ encode the asymptotic properties of the original function.

Theorem 3.2

Let $f(x)$ be continuous with Mellin transform $f^*(s)$ having a non-empty fundamental strip $\langle \alpha, \beta\rangle$.
(i) Assume that $f^*(s)$ admits a meromorphic continuation to the strip $\langle \gamma, \beta\rangle$ for some $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\Re(s) = \gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that $f^*(s) = O(|s|^{-r})$ with $r > 1$ when $|s| \to \infty$ in $\gamma \leq \Re(s) \leq \eta$. If $f^*(s)$ admits the singular expansion for $s \in \langle \gamma, \alpha\rangle$,

$$f^*(s) \asymp \sum_{(a,k) \in A} d_{a,k} \frac{1}{(s-a)^k}$$

then an asymptotic expansion of $f(x)$ at 0 is

$$f(x) = \sum_{(a,k)\in A} d_{a,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-a} (\log x)^{k-1} \right) + O(x^{-\gamma}).$$

(ii) Similarly assume that $f^*(s)$ admits a meromorphic continuation to $\langle \alpha, \gamma \rangle$ for some $\gamma > \beta$ and is analytic on $\Re(s) = \gamma$. Assume again that $f^*(s) = O(|s|^{-r})$ with $r > 1$ when $|s| \to \infty$ in $\langle \eta, \gamma \rangle$ for some $\eta \in (\alpha, \beta)$. If

$$f^*(s) \asymp \sum_{(a,k)\in A} d_{a,k} \frac{1}{(s-a)^k}$$

for $s \in (\eta, \gamma)$, then the asymptotic expansion of $f(x)$ at ∞ is

$$f(x) = -\sum_{(a,k)\in A} d_{a,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-a} (\log x)^{k-1} \right) + O(x^{-\gamma}).$$

Proof:
As in the direct mapping theorem, it suffices to consider case (i) corresponding to continuation to the left. Consider the integral

$$I(T) = \frac{1}{2\pi i} \int_{\partial D} f^*(s) x^{-s} ds$$

where D denotes the rectangular contour defined by the segments

$$[\eta - iT, \eta + iT] \, , [\eta + iT, \gamma + iT] \, , [\gamma + iT, \gamma - iT] \, , [\gamma - iT, \eta - iT].$$

By the residue theorem $I(T)$ is equal to the sum of residues, which are:

$$\begin{aligned} R &= \sum_{(a,k)\in A} d_{a,k} \operatorname{Res} \left(\frac{x^{-s}}{(s-a)^k} \right)_{s=a} \\ &= \sum_{(a,k)\in A} d_{a,k} \left(\frac{(-1)^{k-1}}{(k-1)!} x^{-a} (\log x)^{k-1} \right) \end{aligned}$$

Let $T \to \infty$. The integrals along the two horizontal segments vanish, since they are $O(T^{-r})$ and tend to 0 as $T \to \infty$. The integral along the vertical line $\Re(s) = \eta$ lies inside the fundamental strip and tends to the inverse Mellin integral $f(x)$ by the inversion theorem since $f(x)$ is continuous. For the integral along the vertical line $\Re(s) = \gamma$ we have:

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} f^*(s) x^{-s} ds \right| &\leq \frac{1}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} |f^*(s)| x^{-\Re(s)} |ds| \\ &= O(1) \int_0^\infty \frac{x^{-\gamma}}{(1+t)^r} dt \\ &= O(x^{-\gamma}) \end{aligned}$$

Hence for $T \to \infty$ we have $R = f(x) + O(x^{-\gamma})$. So that $f(x) = R + O(x^{-\gamma})$. This completes the proof of the Converse mapping theorem.

The following table sums up aspects of the Converse mapping theorem:

$f^*(s)$	$f(x)$
Pole at a	Term in asymptotic expansion $\approx x^{-a}$
left of fundamental strip	expansion at 0
right of fundamental strip	expansion at ∞
Simple pole	
left: $\frac{1}{s-a}$	x^{-a} at 0
right: $\frac{1}{s-a}$	$-x^{-a}$ at ∞
Multiple pole	Logarithmic factor
left: $\frac{1}{(s-a)^{k+1}}$	$\frac{(-1)^k}{k!}x^{-a}(\log x)^k$ at 0
right: $\frac{1}{(s-a)^{k+1}}$	$-\frac{(-1)^k}{k!}x^{-a}(\log x)^k$ at ∞

Example 3.6

In example 2.4 we showed that $(1+x)^{-\nu}$ has Mellin transform $\frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)}$. Now consider $\phi(s) = \frac{\Gamma(s)\Gamma(\nu-s)}{\Gamma(\nu)}$ that is analytic in the strip $\langle 0, \nu\rangle$. We use the singular expansion of the gamma function to obtain the singular expansion to the left of $\Re(s) = 0$ (i.e. for $s \in \langle -\infty, \nu\rangle$) :

$$\phi(s) \asymp \frac{(-1)^k}{k}\frac{\Gamma(\nu+k)}{\Gamma(\nu)}\frac{1}{s+k}.$$

Since the gamma function decreases along vertical lines it follows that $\phi(s)$ decreases along vertical lines, so that the conditions of the converse mapping theorem is satisfied.
$\phi(s)$ is the Mellin transform of some function $f(x)$, which can be obtained by the inversion formula:

$$f(x) = \frac{1}{2\pi i}\int_{\frac{\nu}{2}-i\infty}^{\frac{\nu}{2}+i\infty}\phi(s)x^{-s}ds$$

Singularities of $\phi(s)$ in $\langle -\infty, \nu\rangle$ encode asymptotics for $f(x)$ at 0:

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\nu+k)}{\Gamma(\nu)}x^k$$

$$\begin{aligned}\text{Furthermore } (1+x)^{-\nu} &= \sum_{k=0}^{\infty}\binom{-\nu}{k}x^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\nu(\nu+1)(\nu+2)...(\nu+k-1)x^k \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\nu+k)}{\Gamma(\nu)}x^k\end{aligned}$$

Hence we again found the Mellin pair $f(x) = \dfrac{1}{(1+x)^v}$, $f^*(s) = \dfrac{\Gamma(s)\Gamma(v-s)}{\Gamma(v)}$, this time using the Inversion formula and the Converse mapping theorem.

4 Application to Harmonic Sums

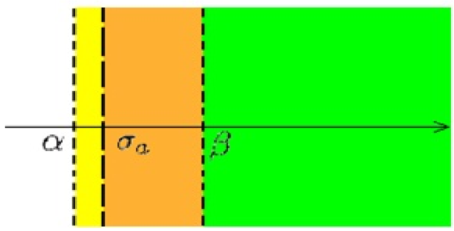
In this section we look at an application of the Mellin transform to harmonic sums using the functional properties of the Mellin transform and the fundamental correspondence.

Definition 4.1 *A sum of the form* $G(x) = \sum_k \lambda_k g(\mu_k x)$ *is called a harmonic sum with base function* $g(x)$ *, frequencies* μ_k *and amplitudes* λ_k*.*

The series $\Lambda(s) = \sum_k \lambda_k (\mu_k)^{-s}$ is called a Dirichlet series. Recall the following useful property of the Mellin transform: $\mathcal{M}[g(\mu x), s] = \mu^{-s} g^*(s)$.
Hence if $G(x) = \sum_k \lambda_k g(\mu_k x)$ then $G^*(s) = \Lambda(s) g^*(s)$.
So we have the following proposition:

Proposition 4.1 *The Mellin transform of the harmonic sum* $G(x) = \sum_k \lambda_k g(\mu_k x)$ *is defined in the intersection of the fundamental strip of* $g(x)$ *and the domain of absolute convergence of the Dirichlet series* $\Lambda(s) = \sum_k \lambda_k (\mu_k)^{-s}$ *(which is of the form* $Re(s) > \sigma_a$ *for some real* σ_a *) and is given by* $G^*(s) = \Lambda(s) g^*(s)$*.*

This is illustrated in the following picture:



Example 4.1

Consider the function

$$h(x) = \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+x} \right]$$
$$= \sum_{k=1}^{\infty} \frac{1}{k} \frac{\frac{x}{k}}{1 + \frac{x}{k}}$$

This is a harmonic sum with $\mu_k = \frac{1}{k}$, $\lambda_k = \frac{1}{k}$ and base function $g(x) = \frac{x}{1+x}$. Note that $h(n) = 1 + \frac{1}{2} + ... + \frac{1}{n} = H_n$, the harmonic number.

$$\begin{aligned}\Lambda(s) &= \sum_k \frac{\lambda_k}{{\mu_k}^s} \\ &= \sum_{k=1}^{\infty} k^{s-1} \\ &= \zeta(1-s)\end{aligned}$$

Transform of the base function is $g^*(s) = \dfrac{\pi}{\sin \pi(s+1)} = -\dfrac{\pi}{\sin \pi s}$ with fundamental strip $\langle -1, 0\rangle = \langle 0-1, 1-1\rangle$, which follows when applying theorem 2.1(iii) to example 2.3. Now

$$h^*(s) = \Lambda(s) g^*(s) = -\zeta(1-s)\frac{\pi}{\sin \pi s} \quad , s \in \langle -1, 0\rangle.$$

We showed that $\zeta(s) \sim \frac{1}{s-1}$ is the beginning of the Laurent expansion of the zeta function. It is well known that the coefficient of zero degree in the Laurent expansion is $\gamma \approx 0.57721566$, the Euler-Mascheroni constant. This can be found in [5]. Hence $\zeta(s) = \frac{1}{s-1} + \gamma + ...$ as $s \to 1$. Hence we have the following singular expansion of $h^*(s)$:

$$h^*(s) \asymp \left[\frac{1}{s^2} - \frac{\gamma}{s}\right] - \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(1-k)}{s-k} \quad , s \in \langle -1, \infty\rangle.$$

Remembering that $\zeta(-m) = (-1)^m \frac{B_{m+1}}{m+1}$ for $m \in \mathbb{N} \cup \{0\}$, we use the converse mapping theorem and obtain:

$$H_n \sim \log n + \gamma + \frac{1}{2n} - \sum_{k \geq 2} \frac{(-1)^k B_k}{k} \frac{1}{n^k} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - ...$$

Example 4.2

It follows from the product decomposition of the Gamma function that

$$l(x) = \log \Gamma(x+1) + \gamma x = \sum_{n=1}^{\infty} \left[\frac{x}{n} - \log\left(1 + \frac{x}{n}\right)\right].$$

This is a harmonic sum with amplitudes $\lambda_n = 1$, frequencies $\mu_n = \frac{1}{n}$ and base function $g(x) = x - \log(1+x)$. The Dirichlet series is

$$\begin{aligned}\Lambda(s) &= \sum_{n=1}^{\infty} \lambda_n (\mu_n)^{-s} \\ &= \sum_{n=1}^{\infty} n^s \\ &= \zeta(-s).\end{aligned}$$

Remember that the Mellin transform of $f(x) = -\log(1+x)$ is $f^*(s) = -\frac{\pi}{s\sin\pi s}$ for $s \in \langle -1, 0\rangle$. Notice that $g(x) = x[1 - x^{-1}\log(1+x)]$. $1 - x^{-1}\log(1+x)$ is $O(x)$ as $x \to 0^+$ and $O\left(x^0\right)$ as $x \to \infty$. Hence $1 - x^{-1}\log(1+x)$ is $f^*(s-1)$ for $s \in \langle -1, 0\rangle$. This means $g^*(s) = f^*(s) = -\frac{\pi}{s\sin\pi s}$ for $s \in \langle -2, -1\rangle$. Since $\zeta(-s)$ converges absolutely in this fundamental strip we have: $l^*(s) = -\zeta(-s)\frac{\pi}{s\sin\pi s}$ for $s \in \langle -2, -1\rangle$.

To obtain an asymptotic expansion at ∞ we must find a meromorphic continuation to the right of the fundamental strip. The Laurent expansion of the zeta function at 0 is $\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log 2\pi(s) + O(s^2)$. Taking note that there are double poles at $s = -1, s = 0$ and simple poles at the positive integers, we obtain the following singular expansion:

$$l^*(s) \asymp \left[\frac{1}{(s+1)^2} + \frac{1-\gamma}{s+1}\right] + \left[\frac{1}{2s^2} - \frac{\log\sqrt{2\pi}}{s}\right] + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}\zeta(-n)}{n(s+n)}, \; s \in \langle -2, \infty\rangle.$$

The Converse mapping theorem yields as $x \to \infty$:

$$l(x) \sim [x\log x + \gamma x - x] + \left[\frac{1}{2}\log x + \log\sqrt{2\pi}\right] + \sum_{n=1}^{\infty}\frac{\zeta(-n)(-1)^{n-1}}{n(-x^n)} \; .$$

Now $\Gamma(x+1) = x!$ and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\zeta(-n)(-1)^{n-1}}{n(-x^n)} &= \sum_{n=1}^{\infty}\frac{B_{n+1}(-1)^{n-1}}{(n+1)(n)x^n} \\ &= \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)x^{2n-1}} \; .\end{aligned}$$

The last line follows since we can replace n by $2n-1$ because $B_{2k+1} = 0$ for all $k \in \mathbb{N}$. Putting this all together we obtain Stirling's formula:

$$\log(x!) \sim \log\left(x^x e^{-x}\sqrt{2\pi x}\right) + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)(2n-1)x^{2n-1}} \; .$$

5 Appendix

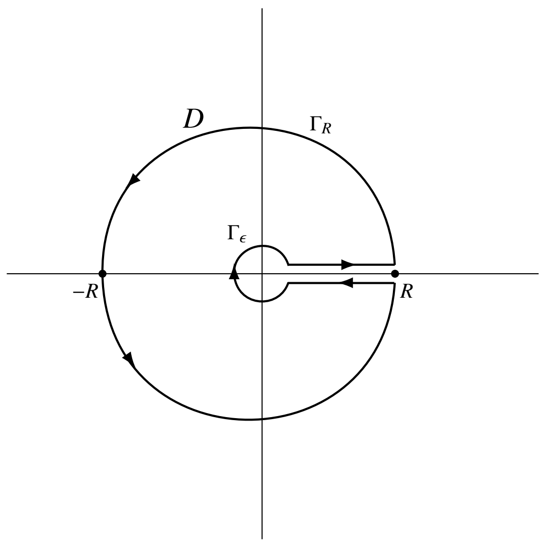


Figure 1: Domain D.

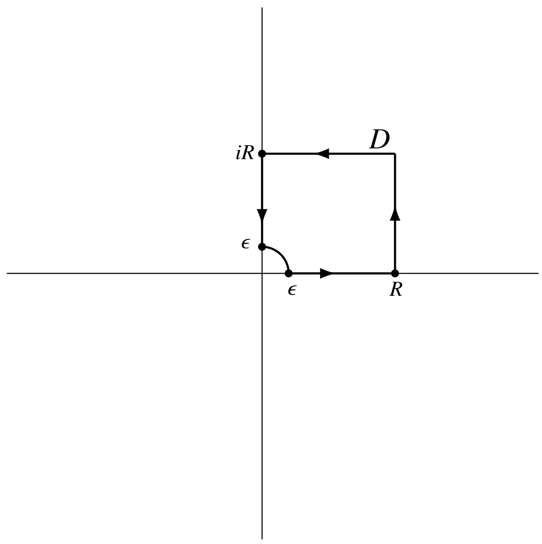


Figure 2: Domain D.

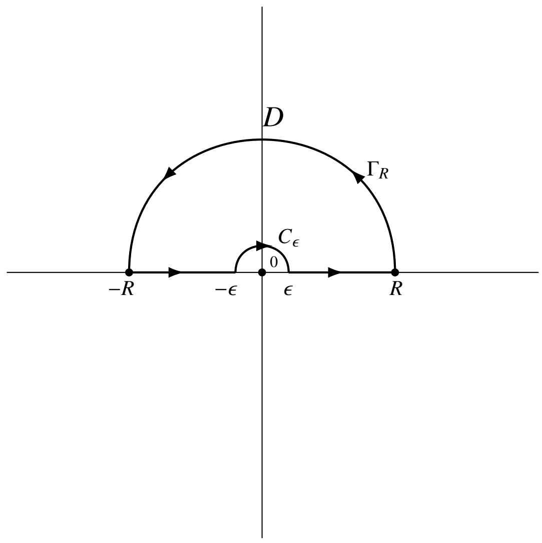


Figure 3: Domain D.

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