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The approximation of logarithmic function by q-Bernstein polynomials in the case $q > 1$

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Abstract Since in the case $q > 1$, q-Bernstein polynomials are not positive linear operators on $C[0, 1]$, the study of their approximation properties is essentially more difficult than that for $0 < q < 1$. Despite the intensive research conducted in the area lately, the problem of describing the class of functions in $C[0, 1]$ uniformly approximated by their q-Bernstein polynomials ($q > 1$) remains open. It is known that the approximation occurs for functions admitting an analytic continuation into a disc $\{z : |z| < R\}$, $R > 1$. For functions without such an assumption, no general results on approximation are available. In this paper, it is shown that the function $f(x) = \ln(x + a)$, $a > 0$, is uniformly approximated by its q-Bernstein polynomials ($q > 1$) on the interval $[0, 1]$ if and only if $a \geq 1$.

Keywords q-integers · q-binomial coefficients · q-Bernstein polynomials · Uniform convergence

Mathematics Subject Classifications (2000) 41A10 · 30E10

1 Introduction

In 1912, S. N. Bernstein published his famous paper [1] containing a constructive proof of the Weierstrass Approximation Theorem. Using the Law of Large Numbers for a sequence of Bernoulli trials he defined, for any $f : [0, 1] \to \mathbf{C}$, a sequence of polynomials

$$B_n(f; x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1 - x)^{n-k}, \; n = 1, 2, \ldots \tag{1.1}$$

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and proved that if $f \in C[0, 1]$, then the sequence $\{B_n(f; x)\}$ converges uniformly to $f(x)$ on $[0, 1]$. Polynomials (1.1), called nowadays *Bernstein polynomials*, possess many remarkable properties, which made them an area of intensive research (see, for example, [4, 8] and references therein). Because of the importance of Bernstein polynomials, their new applications and generalizations are constantly under discovery.

Due to the intensive development of q-Calculus, generalizations of Bernstein polynomials related to q-Calculus have emerged.

The first step in this direction was made by A. Lupaş in 1987. In [9], he considered a q-analogue of the Bernstein operator and investigated its convergence and shape-preserving properties. However, the operators used by A. Lupaş are given by rational functions rather than polynomials.

In 1997, Phillips [17] (see also [16]) introduced q-Bernstein polynomials. While for $q = 1$ these polynomials coincide with the classical ones, for $q \neq 1$ we obtain new polynomials. These polynomials have attracted a lot of interest and have been studied by a number of authors from different perspectives lately. Surveys of results on q-Bernstein polynomials together with an extensive bibliography on the subject are given in [18], Ch. 7 (results obtained in 1997–1999) and [13] (results obtained in 2000–2004). The subject remains under intensive study and there are new papers constantly coming out (see, for example, [3, 14, 15, 20–24] published after [13]).

For the sequel, we need the following definitions (cf., e.g. [18], Ch. 8, Section 8.1):

Let $q > 0$. For any $n \in \mathbf{Z}_+$, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \ (n \in \mathbf{N}), \ \ [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \ (n = 1, 2, \ldots), \ \ [0]_q! := 1.$$

For integers $0 \le k \le n$, the *q-binomial,* or the Gaussian coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

Definition 1.1 (G. M. Phillips, [17]) Let $f : [0, 1] \to \mathbf{C}$. The *q-Bernstein polynomials* of f are given by

$$B_{n,q}(f; z) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q; z), \quad n \in \mathbf{N},$$

where

$$p_{nk}(q;z) := \begin{bmatrix} n \\ k \end{bmatrix}_q z^k \prod_{j=0}^{n-k-1} \left(1 - q^j z\right), \quad k = 0, 1, \ldots n. \tag{1.2}$$

Note that for $q = 1$, we recover classical Bernstein polynomials (1.1).

It has been shown that some features of classical Bernstein polynomials (1.1) are inherited by q-Bernstein polynomials. For example, they possess the end-point interpolation property, leave invariant linear functions, and are degree-reducing on polynomials (see [18], Ch. 7, formulae (7.52), (7.55), (7.56) and the discussion on p. 268). The fact that q-Bernstein polynomials are degree-reducing follows from the representation of q-Bernstein polynomials with the help of q-differences obtained by G. M. Phillips (see [18], Ch. 7, Theorem 7.3.1.) For $q = 1$, this yields the known representation of Bernstein polynomials via forward differences. Furthermore, q-Bernstein polynomials are generated by the generalized de Casteljau algorithm (see [18], Ch. 7, Algorithm 7.3.1 and Theorem 7.3.5) which reduces to the de Casteljau algorithm when $q = 1$.

In the case $0 < q < 1$, the similarity between the classical Bernstein and q-Bernstein polynomials extends much further. We mention the following important facts, mainly obtained by G. M. Phillips and his collaborators. For $0 < q < 1$, q-Bernstein polynomials are *positive* linear operators on $C[0, 1]$ with $\|B_{n,q}\| = 1$ and they are variation-diminishing (see [18], Theorem 7.5.6). The latter implies that if a function f is increasing (decreasing) on [0,1], then $B_{n,q}(f; x)$ $(0 < q < 1)$ is also increasing (decreasing) on [0,1]; and if f is convex (concave) on [0, 1], then so is $B_{n,q}(f; x)$ $(0 < q < 1)$ (see, [18], Theorems 7.5.8 and 7.5.9). These shape-preserving properties stipulate the importance of q-Bernstein polynomials for the computer-aided geometric design. In addition, q-Bernstein polynomials of a convex function f in the case $0 < q < 1$ possess the same monotonicity properties as those of the classical Bernstein polynomials, namely,

$$B_{n-1,q}(f; x) \geq B_{n,q}(f; x) \geq f(x), \quad x \in [0, 1], \; n = 2, 3, \ldots$$

(see [18], Ch. 7, Theorems 7.3.3 and 7.3.4).

Apart from that, in the case $0 < q \leq 1$, q-Bernstein basic polynomials (1.2) admit a probabilistic interpretation: $p_{nk}(q; x)$ equals the probability of exactly k successes in n trials for the stochastic process constructed by Il'inskii in [6], Theorem 2.1. For $q = 1$, this reduces to the sequence of Bernoulli trials, where the probability of k successes in n experiments is given by the basic Bernstein polynomials. A quantitative estimate of the difference between $B_{n,q}(f; x)$, $0 < q < 1$, and $B_n(f; x)$ is given by Videnskii in [21], formula (20).

However, the investigation of convergence properties of q-Bernstein polynomials demonstrates that these properties are essentially different from those of the classical ones. Moreover, the cases $0 < q < 1$ and $q > 1$ are not similar to each other. The convergence of q-Bernstein polynomials has been studied in several papers, starting from [17]. Mostly, these papers deal with the case

$0 < q < 1$, see [7, 11, 14, 17, 18, 20–24] (it should be mentioned that the proof of Theorem 2.3 related to the convergence in the case $0 < q < 1$ in [11] is not correct, see [7], Theorem 6 for a more accurate analysis). Such a great number of results for $0 < q < 1$ is certainly attributed to the fact that, in this case, q-Bernstein polynomials are *positive* linear operators - a crucial point in the investigation of convergence. We notice here that despite being positive linear operators for $0 < q < 1$, q-Bernstein polynomials do not satisfy the conditions of Korovkin's Theorem, because

$$B_{n,q}(t^2; x) = x^2 + \frac{x(1-x)}{[n]_q} \to x^2 + (1-q)x(1-x) \neq x^2, \quad n \to \infty$$

(see [17], formula (15)). However, they satisfy the conditions of H. Wang's Korovkin-type Theorem ([22], Theorem 2) and serve as a leading example for this theorem. H. Wang's theorem guarantees the existence of the limit operator $B_{\infty,q}$ for the sequence $\{B_{n,q}\}$ which, unlike the situation in the classical case, is not the identity operator. Results related to the properties of $B_{\infty,q}$ may be found in the references listed above.

At the same time, the case $q > 1$, when the positivity fails, has not been researched that much. The investigation of convergence for $q > 1$, shows this case to be far more difficult. The fact that there is no analogue of G. M. Phillips' convergence Theorem ([18], Ch. 7, Theorem 7.3.2) in the case $q \downarrow 1$ (see [15], Theorem 2.1) demonstrates the stark distinction between the two cases clearly. As a result, in contrast to a great number of studies for $0 < q < 1$, there is only one paper [12] dealing systematically with the convergence of q-Bernstein polynomials in this case. The results of [12] show, nevertheless, that for $q > 1$, the approximation with q-Bernstein polynomials may be *faster* than the one with the classical Bernstein polynomials (see [12], Theorem 6). For instance, the rate of approximation in $C[0,1]$ for functions analytic in $\{z : |z| < q + \varepsilon\}$ is q^{-n} versus $1/n$ for the classical Bernstein polynomials. On the other hand, for some infinitely differentiable functions, their sequences of q-Bernstein polynomials ($q > 1$) may be divergent (see [12], Theorem 2). This situation is totally impossible for $0 < q \leq 1$.

It is exactly an unexpected behavior of q-Bernstein polynomials with respect to convergence that makes the study of their convergence properties interesting and challenging.

It has been shown in [[12], Theorem 7] that if $q > 1$ and f possesses an analytic continuation into a disc $\{z : |z| < R\}$ with $R > 1$, then $B_{n,q}(f; z) \to f(z)$, $n \to \infty$ uniformly in the closed unit disc $\{z : |z| \leq 1\}$. If we modify the methods applied in [12], this statement can be strengthened as follows:

Theorem 1.2 *Let $q > 1$. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ with $\sum_{k=0}^{\infty} |a_k| < \infty$, then*

$$B_{n,q}(f; z) \to f(z), \quad as \quad n \to \infty,$$

uniformly in the closed unit disc $\{z : |z| \leq 1\}$.

The proof of Theorem 1.2 is given in Appendix at the end of this paper.

In situations that are not addressed by Theorem 1.2, for example, for functions analytic only on $[0, 1]$, no general conclusions concerning the approximation by q-Bernstein polynomials $(q > 1)$ have been reached. What is yet to be described is the class of functions in $C[0, 1]$ which are uniformly approximated by their q-Bernstein polynomials in the case $q > 1$.

One of the reasons that causes such an unpredictable behavior of q-Bernstein polynomials in the case $q > 1$, is a fast increase of basic polynomials (1.2) near $x = 1$. If we take, for example, $x_0 \in (1/\sqrt{q}, 1)$, then

$$p_{n0}(q; x_0) = (1 - x_0)(1 - qx_0) \ldots (1 - q^{n-1}x_0)$$

and

$$|p_{n0}(q; x_0)| = q^{n(n-1)/2} x_0^n \left(1 - \frac{1}{x_0}\right)\left(1 - \frac{1}{qx_0}\right) \ldots \left(1 - \frac{1}{q^{n-1}x_0}\right)$$

$$= q^{\frac{n^2}{2}} \left(x_0/\sqrt{q}\right)^n \prod_{j=0}^{n-1} \left(1 - \frac{1}{q^j x_0}\right) \geq C_{q,x_0} q^{\frac{n^2}{2} - n}.$$

Similarly, for any fixed $k \in \mathbf{Z}_+$, we have:

$$|p_{nk}(q; x_0)| \geq C_{k,q,x_0} q^{\frac{n^2}{2} - n} \tag{1.3}$$

while $\sum_{k=0}^{n} p_{nk}(q; x) \equiv 1$. Such a tremendous increase of the magnitude of the basic q-Bernstein polynomials along with the sign oscillation creates a serious obstacle for numerical experiments with q-Bernstein polynomials in the case $q > 1$.

In this paper, we use an example of the one-parametric family of logarithmic functions $\ln(a + x)$, $a > 0$, to show that, in general, the analyticity on $[0, 1]$ does not imply the possibility of the uniform approximation by the q-Bernstein polynomials. To be specific, we prove that for $f(x) = \ln(a + x)$ with $0 < a < 1$, the sequence $\{B_{n,q}(f; x)\}$ is not even uniformly bounded on $[0, 1]$. On the other hand, we demonstrate that the uniform approximation is possible for functions whose Taylor series converges on $[0, 1]$ uniformly but not absolutely, that is, the conditions of Theorem 1.2 are excessive for the uniform approximation. We show that while q-Bernstein polynomials $(q > 1)$ uniformly approximate the logarithmic function $\ln(a + x)$, $a > 0$, on any interval in $\{z : |z| < a\}$, they are not even uniformly bounded on any interval outside of this disc.

Our study reveals the following astonishing phenomenon: if q-Bernstein polynomials approximate the logarithmic function on some interval in $\mathbf{C}$, they have to approximate it on all intervals that are closer to the origin.

We expect that these results on the approximation of the logarithmic function will help to enlighten the general situation concerning the approximation of continuous functions by the q-Bernstein polynomials in the case $q > 1$.

Our examination is based on writing explicitly the q-Bernstein polynomials of the logarithmic function. It should be pointed out that, up to now, q-Bernstein polynomials used to be known explicitly only for monomials x^m, $m \in \mathbf{Z}_+$ (cf. [5], formulae (2.4) and (2.5)). Using the expression for the

q-Bernstein polynomials of $\ln(a+x), a>0$, we prove that this function is uniformly approximated by its q-Bernstein polynomials ($q>1$) on the interval [0, 1] if and only if $a\geq 1$. Our methods are also applicable to the functions of a more general form (see Remark 2.8 below) and we are positive that they can be useful for wider classes of functions.

The paper is organized as follows. Section 2 includes statements of the obtained results along with relevant comments. Proofs of Lemma 2.1 and Theorems 2.2, 2.4, 2.6 are given in Section 3. Appendix at the end of the paper contains the proof of Theorem 1.2.

2 Statement of results

The reasonings within this paper are based on the following lemma, which gives explicitly the coefficients of the q-Bernstein polynomials of the logarithmic function. In the sequel, whenever we consider the logarithmic function $\ln(a+z)$, $a>0$, we mean the branch of $\ln(a+z)$ analytic in $\mathbf{C}\setminus(-\infty,-a]$ such that $\ln(a+x)$ is real for $x>-a$.

Lemma 2.1 *Let*

$$B_{n,q}(f;z)=\sum_{k=0}^{n} c_{kn}z^k,\quad n\in\mathbf{N},$$

be q-Bernstein polynomials of $f(z)=\ln(a+z)$, $a>0$. Then:

$$c_{0n}=\ln a,\quad c_{kn}=(-1)^{k+1}\lambda_{kn}\int_a^\infty \frac{dx}{x\left(x+\frac{1}{[n]_q}\right)\dots\left(x+\frac{[k]_q}{[n]_q}\right)},\quad k=1,\dots,n, \tag{2.1}$$

where

$$\lambda_{0n}=\lambda_{1n}=1,\quad \lambda_{kn}=\left(1-\frac{1}{[n]_q}\right)\dots\left(1-\frac{[k-1]_q}{[n]_q}\right),\ k=2,\dots,n. \tag{2.2}$$

Remark 2.2 It has been shown in [11] (formulae (2.1), (2.6), and (3.5)) that λ_{kn} ($k=0,1,\dots,n$) are eigenvalues of the lower triangular stochastic $(n+1)\times(n+1)$ matrix $\mathbf{A}$, whose entries a_{ij} are the coefficients of $B_{n,q}(t^i;.)$ with respect to the standard basis in the space of polynomials of degree not greater than n, whence λ_{kn} are eigenvalues of the q-Bernstein operator $B_{n,q}$. For $q=1$, we recover eigenvalues of the Bernstein operator, whose eigenstructure is described in [2]. Some results of the latter paper have been extended for the q-Bernstein polynomials in [12].

The proof of Lemma 2.1 is based on the representation of q-Bernstein polynomials via divided differences, see formula (3.1) below. It is worth mentioning that representation of q-Bernstein polynomials using q-differences was first obtained in [17].

Our first result refers to the possibility of the uniform approximation of the logarithmic function by its q-Bernstein polynomials ($q>1$).

Theorem 2.3 *Let $q > 1$, $a > 0$ and $f(z) = \ln(a+z)$. Then for any compact set $K \subset \{z : |z| < a\}$,*

$$B_{n,q}(f; z) \to f(z) \ \text{as} \ n \to \infty,$$

uniformly on K.

Corollary 2.4 *If $q > 1$, $a > 0$, then for any $0 < c < a$,*

$$B_{n,q}(\ln(a+t); x) \to \ln(a+x), \ n \to \infty,$$

uniformly on $[0, c]$.

In particular, we obtain the following result contained in Theorem 7 of [12]: if $q > 1$, $a > 1$, then

$$B_{n,q}(\ln(a+t); x) \to \ln(a+x), \quad n \to \infty,$$

uniformly on $[0, 1]$.

The next statement shows that Theorem 2.3 is sharp in the following sense: we cannot approximate $\ln(a+x)$ with its q-Bernstein polynomials on an interval beyond $[-a, a]$. More precisely, the following assertion holds.

Theorem 2.5 *Let $q > 1$, $a > 0$, $f(z) = \ln(a+z)$. Then, for any Jordan arc $\mathcal{L} \subset \{z : c \le |z| \le d\}$, $a < c$, the sequence $\{B_{n,q}(f; z)\}$ is not uniformly bounded on $\mathcal{L}$.*

Taking $\mathcal{L} = [c, d]$, $a < c < d < 1$, we get:

Corollary 2.6 *If $q > 1$, $0 < a < 1$, then $\ln(a+x)$ cannot be uniformly approximated by its q-Bernstein polynomials on the interval* $[0, 1]$.

Finally comes the case $a = 1$. The previous results imply the uniform convergence of $\{B_{n,q}(\ln(1+t); x)\}$ to $\ln(1+x)$ on any interval $[0, c]$ with $0 < c < 1$ and pointwise convergence to $\ln(1+x)$ on $[0, 1]$. However, it turns out that the q-Bernstein polynomials *uniformly* tend to $\ln(x+1)$ on $[0, 1]$, as the following theorem shows.

Theorem 2.7 *If $q > 1$, then $\ln(1+x)$ is uniformly approximated by its q-Bernstein polynomials on the interval* $[0, 1]$.

Remark 2.8 Theorem 2.7 may be easily generalized for functions of the form:

$$\sum_{k=0}^{\infty}(-1)^k a_k x^k,$$

where $\sum_{k=0}^{\infty}(-1)^k a_k$ is a Leibnitz series, that is $a_k \to 0$ and $0 < a_{k+1} \le a_k$ for $k \ge k_0$.

3 Proofs of the theorems

Proof of Lemma 2.1 We use the following representation of q-Bernstein polynomials (see [12], formulae (6) and (7)):

$$B_{n,q}(f;z)=\sum_{k=0}^{n} c_{kn}z^k, \quad c_{kn}=\lambda_{kn} f\left[0;\frac{1}{[n]_q};\ldots;\frac{[k]_q}{[n]_q}\right], \tag{3.1}$$

where $f[x_0;x_1;\ldots;x_k]$ denotes the divided differences of f, that is

$$f[x_0]=f(x_0), \quad f[x_0;x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0},\ldots,$$
$$f[x_0;x_1;\ldots;x_k]=\frac{f[x_1;\ldots;x_k]-f\left[x_0;\ldots;x_{k-1}\right]}{x_k-x_0},$$

and λ_{kn} are given by (2.2).

If f is analytic on $[0,1]$, then by ([8], Section 2.7, page 44) we have:

$$f\left[0;\frac{1}{[n]_q};\ldots;\frac{[k]_q}{[n]_q}\right]=\frac{1}{2\pi i}\oint_L \frac{f(\zeta)\,d\zeta}{\zeta\left(\zeta-\frac{1}{[n]_q}\right)\ldots\left(\zeta-\frac{[k]_q}{[n]_q}\right)}, \tag{3.2}$$

where L is a contour encircling $[0,1]$ so that f is analytic on and within L. Since $f(z)=\ln(a+z)$ is analytic in $\mathbf{C}\setminus(-\infty,-a]$, and the integrand in (3.2) is $o(|\zeta|^{-1})$ as $\zeta\to\infty$, $k\geq 1$, we obtain by the Cauchy Theorem:

$$\begin{aligned}\oint_L \frac{f(\zeta)\,d\zeta}{\zeta\left(\zeta-\frac{1}{[n]_q}\right)\ldots\left(\zeta-\frac{[k]_q}{[n]_q}\right)} &= \int_{-\infty}^{-a}\frac{\ln|a+t|+i\pi}{t\left(t-\frac{1}{[n]_q}\right)\ldots\left(t-\frac{[k]_q}{[n]_q}\right)}dt \\ &\quad+\int_{-a}^{-\infty}\frac{\ln|a+t|-i\pi}{t\left(t-\frac{1}{[n]_q}\right)\ldots\left(t-\frac{[k]_q}{[n]_q}\right)}dt \\ &=2\pi i\int_{-\infty}^{-a}\frac{dt}{t\left(t-\frac{1}{[n]_q}\right)\ldots\left(t-\frac{[k]_q}{[n]_q}\right)} \\ &=2\pi i(-1)^{k+1}\int_a^{\infty}\frac{dx}{x\left(x+\frac{1}{[n]_q}\right)\ldots\left(x+\frac{[k]_q}{[n]_q}\right)}.\end{aligned}$$

Using (3.1) and (3.2), we derive (2.1) for $k\geq 1$. For $k=0$, (3.1) implies $c_{0n}=f(0)=\ln a$. □

Proof of Theorem 2.3 First, we prove the following estimate for the coefficients c_{kn} of the polynomials $B_{n,q}(f;z)$:

$$|c_{kn}| \leq \frac{C}{a^k}, \quad k = 0, 1, \ldots, n, \tag{3.3}$$

where $C = \max\{1, |\ln a|\}$.

Indeed, for $k = 0$, there is nothing to prove. For $k \geq 1$, we have by Lemma 2.1,

$$|c_{kn}| \leq \lambda_{kn} \int_a^\infty \frac{dx}{x^{k+1}\left(1 + \frac{1}{x[n]_q}\right) \ldots \left(1 + \frac{[k]_q}{x[n]_q}\right)}$$

$$\leq \int_a^\infty \frac{dx}{x^{k+1}} = \frac{1}{ka^k},$$

which proves (3.3).

It follows that the sequence $\{B_{n,q}(f;z)\}$ is uniformly bounded in any disc $\{z : |z| \leq \rho\}$ with $\rho < a$, because due to (3.3), we have:

$$\left|B_{n,q}(f;z)\right| \leq \sum_{k=0}^{n} |c_{kn}|\rho^k \leq \frac{C}{1 - \rho/a}, \quad |z| \leq \rho.$$

Apart from that, Lemma 1 of [12] says that the sequence $\{B_{n,q}(f;z)\}$ converges to $f(z)$ on the set $\{q^{-j}\}_{j=0}^{\infty}$. By the Vitali Theorem (cf., e.g., [19], Ch.V, Section 5.2), the sequence converges to $f(z)$ on any compact set in $\{z : |z| < a\}$. □

Proof of Theorem 2.5 Assume that the sequence $\{B_{n,q}(f;z)\}$ is uniformly bounded on a Jordan arc $\mathcal{L} \subset \{z : c \leq |z| \leq d\}$, $a < c$, that is, for some $M > 0$,

$$\left|B_{n,q}(f;z)\right| \leq M, \quad z \in \mathcal{L}. \tag{3.4}$$

Consider the auxiliary polynomials:

$$Q_n(f;z) := z^n B_{n,q}(f; a/z) = \sum_{k=0}^{n} c_{n-k,n} a^{n-k} z^k. \tag{3.5}$$

By virtue of (3.4), we conclude that

$$|Q_n(f;z)| \leq \left(\frac{a}{c}\right)^n \cdot M \ \text{ for } \ z \in \mathcal{L}_1,$$

where $\mathcal{L}_1$ is the image of $\mathcal{L}$ under the conformal mapping $z \mapsto \dfrac{a}{z}$.

At the same time, estimate (3.3) implies that

$$|Q_n(f;z)| \leq \frac{C}{1 - \rho} =: C_1 \ \text{ for } \ |z| \leq \rho < 1.$$

We fix $\rho \in (a/c, 1)$ and apply the Two-constants Theorem (cf., e.g., [10], p.41) to obtain an estimate for $Q_n(f; z)$ in $\{z : |z| < \rho\}$. As a result, we obtain:

$$|Q_n(f;z)| \le \left[M\left(\frac{a}{c}\right)^n\right]^{\omega(z)} \cdot C_1^{1-\omega(z)}, \ z \in \{z : |z| < \rho\} \setminus \mathcal{L}_1, \tag{3.6}$$

where $\omega(z)$ is the harmonic measure of $\mathcal{L}_1$ with respect to $\{z : |z| < \rho\} \setminus \mathcal{L}_1$. We note that $0 \in \{z : |z| < a/d\} \subset \{z : |z| < \rho\} \setminus \mathcal{L}_1$. Therefore, $\omega(0) > 0$ and (3.6) implies that

$$|Q_n(f;0)| \le C_1^{1-\omega(0)} \cdot M^{\omega(0)}\left(\frac{a}{c}\right)^{n\omega(0)} =: M_1\left(\frac{a}{c}\right)^{n\omega(0)}. \tag{3.7}$$

On the other hand, it follows directly from (3.5) that

$$|Q_n(f;0)| = |c_{nn}|a^n. \tag{3.8}$$

By Lemma 2.1,

$$|c_{nn}| = \lambda_{nn}\int_a^\infty \frac{dx}{x^{n+1}\left(1+\frac{1}{x[n]_q}\right)\left(1+\frac{[2]_q}{x[n]_q}\right)\dots\left(1+\frac{[n-1]_q}{x[n]_q}\right)\left(1+\frac{[n]_q}{x[n]_q}\right)}$$
$$\ge \frac{\lambda_{nn}}{\left(1+\frac{1}{a[n]_q}\right)\dots\left(1+\frac{1}{a}\right)} \cdot \frac{1}{na^n}.$$

Since

$$\frac{[n-j]_q}{[n]_q} \le \frac{1}{q^j} \quad \text{for} \quad j = 0, 1, \dots, n,$$

it follows that

$$\lambda_{nn} \ge \prod_{j=1}^{n-1}\left(1-\frac{[j]_q}{[n]_q}\right) \ge \prod_{j=1}^{n-1}\left(1-\frac{1}{q^j}\right) \ge \prod_{j=1}^{\infty}\left(1-\frac{1}{q^j}\right) =: \lambda > 0.$$

Likewise,

$$\left(1+\frac{1}{a[n]_q}\right)\dots\left(1+\frac{1}{a}\right) \le \prod_{j=0}^{\infty}\left(1+\frac{1}{aq^j}\right) =: C_2 > 0.$$

Therefore,

$$|c_{nn}| \ge \frac{\lambda}{C_2 na^n}$$

and (3.8) implies that

$$|Q_n(f;0)| \ge \frac{\lambda}{C_2 n} =: \frac{C_3}{n},$$

which contradicts (3.7) because $a < c$. □

Proof of Theorem 2.6 It is known that the Taylor series

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k$$

converges uniformly on [0, 1] and its remainder is estimated by

$$\left| \sum_{k=N}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k \right| \le \frac{1}{N}.$$

Since for each fixed $n \in \mathbf{N}$, the operator $B_{n,q}$ is bounded on $C[0,1]$, we can apply it term by term, that is,

$$B_{n,q}\,(\ln(1+t); x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\, B_{n,q}(t^k; x).$$

Given $\varepsilon > 0$, we choose $N \in \mathbf{N}$ such that $1/N < \varepsilon$. We write:

$$\begin{aligned} \left| B_{n,q}(\ln(1+t); x) - \ln(1+x) \right| &= \left| \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left[B_{n,q}(t^k; x) - x^k \right] \right| \\ &\le \left| \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k} \left[B_{n,q}(t^k; x) - x^k \right] \right| \\ &\quad + \left| \sum_{k=N+1}^{\infty} \frac{(-1)^{k+1}}{k}\, B_{n,q}(t^k; x) \right| \\ &\quad + \left| \sum_{k=N+1}^{\infty} \frac{(-1)^{k+1}}{k}\, x^k \right|. \end{aligned}$$

It can be derived easily from (3.1) that all coefficients of $B_{n,q}(t^k; z)$ are non-negative, whence

$$0 \le B_{n,q}(t^k; x) \le 1, \quad \text{for all} \;\; n \in \mathbf{N}, \;\; k \in \mathbf{Z}_+, \;\; x \in [0,1].$$

Now we apply V. S. Videnskii's recurrence formula (see [20], formula (3.1)):

$$B_{n,q}(t^{k+1}; x) = B_{n,q}(t^k; x) - \frac{[n-1]_q^k}{[n]_q^k}(1-x)\, B_{n-1,q}(t^k; qx).$$

It implies that

$$B_{n,q}(t^{k+1}; x) \le B_{n,q}(t^k; x), \quad \text{for all} \;\; n \in \mathbf{N}, \;\; k \in \mathbf{Z}_+, \;\; x \in [0,1].$$

Therefore

$$\left| \sum_{k=N+1}^{\infty} \frac{(-1)^{k+1}}{k}\, B_{n,q}(t^k; x) \right| \le \frac{1}{N+1} < \varepsilon.$$

Consequently,

$$\left|B_{n,q}(\ln(t+1);x)-\ln(x+1)\right| \le \left|\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k}\left[B_{n,q}(t^k;x)-x^k\right]\right| + 2\varepsilon.$$

This is true for all $n \in \mathbf{N}$. Now, we choose n_0 in such a way that for $x \in [0,1]$,

$$\left|B_{n,q}(t^k;x)-x^k\right| < \frac{\varepsilon}{N} \quad \text{for} \quad k = 1, 2, \ldots, N \quad \text{and} \quad \text{all} \quad n > n_0.$$

Such a choice is possible because for $q > 1$,

$$B_{n,q}(t^k;z) \to z^k, \quad k \in \mathbf{Z}_+$$

uniformly on any compact set in $\mathbf{C}$ (see [12], Corollary 5). Then, for $n > n_0$, we obtain:

$$\left|\sum_{k=1}^{N}\frac{(-1)^{k+1}}{k}\left[B_{n,q}(t^k;x)-x^k\right]\right| < \frac{\varepsilon}{N}\cdot\sum_{k=1}^{N}\frac{1}{k} < \varepsilon.$$

□

Appendix

Proof of Theorem 1.2 Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, where $\sum_{k=0}^{\infty}|a_k| < \infty$. Since for each fixed $n \in \mathbf{N}$, $B_{n,q}$ is a bounded linear operator in the space of continuous functions on $\{z : |z| \le 1\}$, we may write:

$$B_{n,q}(f;z) = \sum_{k=0}^{\infty} a_k B_{n,q}(t^k;z).$$

We notice that, according to formula (3.1), $B_{n,q}(t^k;z)$ has all coefficients non-negative. Hence for $|z| \le 1$,

$$\left|B_{n,q}(t^k;z)\right| \le B_{n,q}(t^k;1) = 1.$$

Now, given $\varepsilon > 0$, we choose $N \in \mathbf{N}$ such that $\sum_{k=N+1}^{\infty}|a_k| < \varepsilon$. Then we have:

$$\left|B_{n,q}(f;z)-f(z)\right| \le \sum_{k=0}^{N}|a_k|\cdot\left|B_{n,q}(t^k;z)-z^k\right| + 2\varepsilon.$$

By Corollary 5 of [12], $B_{n,q}(t^k;z) \to z^k$, $n \to \infty$ uniformly on any compact subset of $\mathbf{C}$. Therefore we can choose n_0 in such a way that for $|z| \le 1$,

$$\left|B_{n,q}(t^k;z)-z^k\right| < \frac{\varepsilon}{N} \quad \text{when} \quad n > n_0, \; k = 0, 1, \ldots, N.$$

Thus, for $n > n_0$, we obtain

$$\left|B_{n,q}(f; z) - f(z)\right| \le 3\varepsilon, \quad |z| \le 1,$$

which proves the theorem. □

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