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Modification and unification of the Apostol-type numbers and polynomials and their applications

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ABSTRACT

In this paper, we construct generating functions for modification and unification of the Apostol-type polynomials $Y^{(v)}_{n,\beta}(x;k,a,b)$ *of order* v. By using these generating functions, we derive many new identities related to the generalized Stirling type numbers of the second kind, array-type polynomials, Eulerian polynomials and the modification and unification of the Apostol-type polynomials and numbers. We give many applications related to these numbers and polynomials and PDEs.

1. Introduction, definitions and preliminaries

Throughout our paper, we use the following standard notation: $\mathbb{N} := \{1,2,3,\ldots\}$ denotes the set of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers. We also assume that $\log z$ denotes the principal branch of the multi-valued function $\log z$ with the imaginary part $\Im(\log z)$ constrained by $-\pi < \Im(\log z) \leqslant \pi$. For all $0 \leqslant k \leqslant n$, let $(n)_k = k!\binom{n}{k}$ (for example, see [31,32]).

Ozden [14] defined the following generating functions which are related to the unification of the Bernoulli, Euler and Genocchi polynomials:

$$g_\beta(x,t;k,a,b) := \frac{2^{1-k}t^k e^{tx}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;k,a,b)\frac{t^n}{n!}. \tag{1}$$

Note that for $x = 1$, Eq. (1) reduces to the generating functions for the unification of the Bernoulli, Euler and Genocchi numbers. In (1) we assume that if $\beta = a$, then $|t| < 2\pi$ and if $\beta \neq a, k \in \mathbb{N}_0, a, b \in \mathbb{C} \setminus \{0\}$, then $|t| < b\log\left(\frac{\beta}{a}\right)$. Also, by using the special values of a,b,k and β in (1), the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ provides us with a generalization and unification of the Apostol–Bernoulli polynomials, Apostol–Euler polynomials and Apostol–Genocchi polynomials, respectively:

$$\mathcal{B}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;1,1,1),$$

$$\mathcal{E}_n(x,\beta) = \mathcal{Y}_{n,\beta}(x;0,-1,1)$$

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and

$$\mathcal{G}_n(x,\beta)=\mathcal{Y}_{n,\beta}(x;1,-1,1)$$

(*cf.* [2–34]. Furthermore, for the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, we have

$$B_n(x)=\mathcal{B}_n(x,1),$$

$$E_n(x)=\mathcal{E}_n(x,1)$$

and

$$G_n(x)=\mathcal{G}_n(x,1)$$

(*cf.* [1–34] and the references cited in each of these earlier works). For $x=0$, we have the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n, we have

$$B_n=B_n(0),$$

$$E_n=E_n(0)$$

and

$$G_n=G_n(0)$$

(*cf.* [1–36] and the references cited in each of these earlier works).

In analogy with the generating functions introduced by Ozden [14] and also [34] for the unification of the Apostol-type numbers and polynomials, we define generating functions for these polynomials. Now, we modify and unify (1) as follows: Let $a,b\in\mathbb{R}^+$ $(a\neq b)$. Then *modification and unification of the Apostol-type polynomials*$Y_{n,\beta}^{(v)}(x;k,a,b)$ *of order v are defined by means of the following generating function:*

$$M_{k,v}(t,x,a,b;\beta)=\left(\frac{t^k2^{1-k}}{\beta b^t-a^t}\right)^v b^{xt}=\sum_{n=0}^{\infty}Y_{n,\beta}^{(v)}(x;k,a,b)\frac{t^n}{n!},\tag{2}$$

where

$$\left|t\ln\left(\frac{b}{a}\right)+\ln\beta\right|<2\pi$$

and $x\in\mathbb{R}$.

We observe that

$$Y_{n,\beta}^{(v)}(0,k,a,b)=Y_{n,\beta}^{(v)}(k,a,b),$$

which denotes *modification and unification of the Apostol-type numbers of orderv*. Therefore, the numbers $Y_{n,\beta}^{(v)}(k,a,b)$ are defined by means of the following generating functions:

$$\left(\frac{t^k2^{1-k}}{\beta b^t-a^t}\right)^v=\sum_{n=0}^{\infty}Y_{n,\beta}^{(v)}(k,a,b)\frac{t^n}{n!}.\tag{3}$$

Remark 1.1. By substituting $k=1$ into (2), we have known results of [33]

$$\mathcal{B}_n^{(v)}(x;\beta;k,a,b)=Y_{n,\beta}^{(v)}(x;1,a,b)$$

(*cf.* see also [27,34]).

2. Some properties of the numbers $Y_{n,\beta}^{(v)}(k,a,b)$ and the polynomials $Y_{n,\beta}^{(v)}(x;k,a,b)$

Here, by using generating functions, we give some properties of the numbers $Y_{n,\beta}^{(v)}(k,a,b)$ and the polynomials $Y_{n,\beta}^{(v)}(x;k,a,b)$.

By using (1) and (2), we get

$$M_{k,1}(t,x,a,b;\beta)=\left(\ln\left(\frac{b}{a}\right)\right)^{-k}g_\beta\left(\frac{x\ln b-\ln a}{\ln\left(\frac{b}{a}\right)},t\ln\left(\frac{b}{a}\right);k,1,1\right).$$

Thus, we get

$$Y_{n,\beta}(x;k,a,b)=\left(\ln\left(\frac{b}{a}\right)\right)^{n-k}\mathcal{Y}_{n,\beta}\left(\frac{x\ln b-\ln a}{\ln\left(\frac{b}{a}\right)};k,1,1\right),$$

where $\mathcal{Y}_{n,\beta}(x;k,a,b)$ denotes the Apostol-type polynomials (*cf.* [14,16,32]; see also the references cited in each of these earlier works).

$$M_{1,v}(t,x,1,e;1)=\sum_{n=0}^{\infty}\mathcal{B}_n^{(v)}(x;\beta)\frac{t^n}{n!},$$

where

$$\mathcal{B}_n^{(v)}(x;\beta)=Y_{n,\beta}(x;1,1,e)$$

denotes the Apostol–Bernoulli polynomials of order v (*cf.* [2–34]and the references cited in each of these earlier works).

$$M_{0,v}(t,x,1,e;-1)=\sum_{n=0}^{\infty}(-1)^vE_n^{(v)}(x)\frac{t^n}{n!},\tag{4}$$

where

$$(-1)^vE_n^{(v)}(x)=Y_{n,-1}(x;k,1,e)$$

denotes the Euler polynomials of order v (*cf.* [2–34]; see also the references cited in each of these earlier works).

Substituting $x=0$ and $v=1$ into (2) and using the Umbral Calculus convention, we derive a recurrence relation for the modification and unification of the Apostol-type numbers, $Y_{n,\beta}(k,a,b)$. Therefore, we set

$$t^k2^{1-k}=\left(\beta e^{t\ln b}-e^{t\ln a}\right)e^{Y_\beta(k,a,b)t}.$$

From the above equation, we obtain

$$t^k2^{1-k}=\left(\beta\sum_{n=0}^{\infty}\frac{(t\ln b)^n}{n!}-\sum_{n=0}^{\infty}\frac{(t\ln a)^n}{n!}\right)\sum_{n=0}^{\infty}Y_{n,\beta}(k,a,b)\frac{t^n}{n!}.$$

Therefore

$$t^k2^{1-k}=\sum_{n=0}^{\infty}\left(\beta\sum_{j=0}^{n}\binom{n}{j}(\ln b)^{n-j}Y_{j,\beta}(k,a,b)-\sum_{j=0}^{n}\binom{n}{j}(\ln a)^{n-j}Y_{j,\beta}(k,a,b)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of t^n on both sides of the above equation, we arrive at the following theorem:

Theorem 2.1. *The following recurrence relation holds true:*

$$\beta(Y_\beta(k,a,b)+\ln b)^n-(Y_\beta(k,a,b)+\ln a)^n=\begin{cases}2^{1-k}k! & n=k,\\ 0 & n\neq k.\end{cases}$$

where $(Y_\beta(k,a,b))^m$ *is replaced conventionally by* $Y_{m,\beta}(k,a,b)$.

By substituting $n=0$ into Theorem 2.1, we find that

$$Y_{0,\beta}(0,a,b)=\frac{2}{\beta-1}$$

and

$$Y_{0,\beta}(1,a,b)=0,$$

where $\beta\neq 1$. By substituting $n=1$ into Theorem 2.1, we can easily calculate the numbers $Y_{1,\beta}(k,a,b)$ as follows:

If $k=1$, we have

$$Y_{1,\beta}(1,a,b)=\frac{1}{\beta-1}.$$

If $k=0$, we have

$$Y_{1,\beta}(0,a,b)=\frac{2(\beta-1)-2(\beta\ln b-\ln a)}{(\beta-1)^2}.$$

Consequently, applying Theorem 2.1, we can easily calculate all the numbers $Y_{n,\beta}(k,a,b)$.

By using (3), we get

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(v+p)}(k,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(k,a,b)\frac{t^n}{n!}\sum_{n=0}^{\infty} Y_{n,\beta}^{(p)}(k,a,b)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(v+p)}(k,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} Y_{j,\beta}^{(v)}(k,a,b)Y_{n-j,\beta}^{(p)}(k,a,b)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 2.2. *The following recurrence relation holds true:*

$$Y_{n,\beta}^{(v+p)}(k,a,b) = \sum_{j=0}^{n}\binom{n}{j} Y_{j,\beta}^{(v)}(k,a,b)Y_{n-j,\beta}^{(p)}(k,a,b).$$

By applying Theorem 2.2, we can calculate all the numbers $Y_{n,\beta}^{(v+p)}(k,a,b)$. For example, if we substitute $n=1$ and $v=p=1$ into Theorem 2.2, we get

$$Y_{1,\beta}^{(2)}(k,a,b) = \sum_{j=0}^{1}\binom{1}{j} Y_{j,\beta}(k,a,b)Y_{1-j,\beta}(k,a,b).$$

If $k=1$, we have

$$Y_{1,\beta}^{(2)}(1,a,b) = 0.$$

If $k=0$, we have

$$Y_{1,\beta}^{(2)}(0,a,b) = \frac{8(\beta-1) - 8(\beta\ln b - \ln a)}{(\beta-1)^3}.$$

Thus we are ready to generalize Theorem 2.2 as follows:

Theorem 2.3. *Let* $n, j_1, j_2, \ldots, j_r \in \mathbb{N}_0$. *Then we have*

$$Y_{n,\beta}^{(v_1+v_2+\cdots+v_r)}(k,a,b) = \sum_{j_1,j_2,\ldots,j_r=0}^{j_1+j_2+\cdots+j_r=n}\frac{(j_1+j_2+\cdots+j_r)!}{j_1!j_2!\cdots j_r!}\prod_{c=1}^{r} Y_{j_c,\beta}^{(v_c)}(k,a,b).$$

Proof. By using (3), we get

$$\left(\frac{t^k 2^{1-k}}{\beta b^t - a^t}\right)^{v_1+v_2+\cdots+v_r} = \sum_{n=0}^{\infty} Y_{n,\beta}^{(v_1+v_2+\cdots+v_r)}(k,a,b)\frac{t^n}{n!}.$$

By using Theorem 2.2 and mathematical induction, we complete proof of theorem. □

By using (2) and (3), we easily arrive at the following theorem:

Theorem 2.4. *Let* $n \in \mathbb{N}_0$. *The we have*

$$Y_{n,\beta}^{(v)}(x,k,a,b) = \sum_{j=0}^{n}\binom{n}{j} x^{n-j}(x\ln b)^{n-j} Y_{j,\beta}^{(v)}(k,a,b).$$

3. PDEs for the generating functions

In this section, we give PDEs for the generating functions. By using these equations, we derive some derivative formulas and recurrence relations of the *modification and unification of the Apostol-type polynomials of order* v. Taking derivative of (2), with respect to x, we obtain the following PDE for the generating function:

$$\frac{\partial^m}{\partial x^m} M_{k,v}(t,x,k,a,b;\beta) = (t\ln b)^m M_{k,v}(t,x,k,a,b;\beta). \tag{5}$$

Taking derivative of (2), with respect to β, we obtain the following PDE for the generating function:

$$\frac{\partial}{\partial \beta} M_{k,v}(t,x,k,a,b;\beta) = -\frac{v}{2^{1-k}t^k} M_{k,v+1}(t,x+1,k,a,b;\beta) \tag{6}$$

or

$$\frac{\partial}{\partial \beta} M_{k,v}(t,x,k,a,b;\beta) = -\frac{v}{2^{1-k}t^k} M_{k,v-1}(t,x+1,k,a,b;\beta) M_{k,2}(t,k,a,b;\beta). \tag{7}$$

Taking derivative of (2), with respect to t, we obtain the following PDE for the generating function:

$$\begin{aligned} \frac{\partial}{\partial t} M_{k,v}(t,x,k,a,b;\beta) &= \frac{kv}{t} M_{k,v}(t,x,k,a,b;\beta) - \frac{\beta \ln b}{2^{1-k}t^k} M_{k,v+1}(t,x+1,k,a,b;\beta) \\ &\quad + \frac{\ln a}{2^{1-k}t^k} a^t M_{k,v+1}(t,x,k,a,b;\beta) + x \ln b M_{k,v}(t,x,k,a,b;\beta). \end{aligned} \tag{8}$$

By using the above PDEs, we derive the following theorems.

Theorem 3.1. *Let m and n be positive integers with $n \geqslant m$. Then we have*

$$\frac{\partial^m}{\partial x^m} Y_{n,\beta}^{(v)}(x,k,a,b) = m! \binom{n}{m} (\ln b)^m Y_{n-m,\beta}^{(v)}(x,k,a,b)$$

Proof. By using (5) with (2), we obtain

$$\sum_{n=0}^{\infty} \frac{\partial^m}{\partial x^m} Y_{n,\beta}^{(v)}(x,k,a,b) \frac{t^n}{n!} = \sum_{n=0}^{\infty} m! \binom{n}{m} (\ln b)^m Y_{n-m,\beta}^{(v)}(x,k,a,b) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result. □

Theorem 3.2. *Let v and n be positive integers. Then we have*

$$\frac{\partial}{\partial \beta} Y_{n,\beta}^{(v)}(x,k,a,b) = -\frac{2^{1-k} v}{\binom{n+k}{k}} Y_{n+k,\beta}^{(v+1)}(x+1,k,a,b)$$

or

$$\frac{\partial}{\partial \beta} Y_{n,\beta}^{(v)}(x,k,a,b) = -\frac{2^{1-k} v}{\binom{n+k}{k}} \sum_{j=0}^{n+k} \binom{n+k}{j} Y_{j,\beta}^{(v-1)}(x+1,k,a,b) Y_{n+k-j,\beta}^{(2)}(k,a,b). \tag{9}$$

Proof. By using (6) with (2), we obtain

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \beta} Y_{n,\beta}^{(v)}(x,k,a,b) \frac{t^n}{n!} = -\frac{v}{2^{1-k}} \sum_{n=0}^{\infty} Y_{n,\beta}^{(v+1)}(x+1,k,a,b) \frac{t^{n-k}}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{\partial}{\partial \beta} Y_{n,\beta}^{(v)}(x,k,a,b) \frac{t^n}{n!} = -2^{1-k} v \sum_{n=0}^{\infty} \binom{n+k}{k}^{-1} Y_{n+k,\beta}^{(v+1)}(x+1,k,a,b) \frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result. □

By using (7), we easily obtain the assertion (9) of Theorem 3.2.

Theorem 3.3 (*Convolution recurrence relation*). *Let $n \in \mathbb{N}_0$. The following relationship hold true:*

$$\begin{aligned} &\frac{n+1-vk}{n+1} Y_{n+1,\beta}^{(v)}(x,k,a,b) + \frac{2^{k-1} \beta \ln b}{\binom{n+k}{k}} Y_{n+k,\beta}^{(v+1)}(x+1,k,a,b) \\ &\quad = x \ln b Y_{n,\beta}^{(v)}(x,k,a,b) + \frac{2^{k-1}}{\binom{n+k}{k}} \sum_{j=0}^{n+k} \binom{n+k}{j} (\ln a)^{n+k+1-j} Y_{j,\beta}^{(v+1)}(x+1,k,a,b). \end{aligned} \tag{10}$$

Proof. By using (8) with (2), after some elementary calculations, we obtain

$$\sum_{n=1}^{\infty} Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^{n-1}}{(n-1)!} - x\ln b\sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!} = vk\sum_{n=0}^{\infty} Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^{n-1}}{n!} - 2^{k-1}\beta\ln b\sum_{n=0}^{\infty} Y_{n,\beta}^{(v+1)}(x,k,a,b)\frac{t^{n-k}}{n!}$$
$$+\sum_{n=0}^{\infty}\frac{2^{k-1}}{\binom{n+k}{k}}\sum_{j=0}^{n+k}\binom{n+k}{j}(\ln a)^{n+k+1-j} Y_{j,\beta}^{(v+1)}(x+1,k,a,b)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result. □

Substituting $a = k = v = 1$ and $b = e$ into (10), we obtain the following corollary:

Corollary 3.1. *Let $n \in \mathbb{N}_0$. The following relationship hold true:*

$$\beta\mathcal{B}_{n+1}^{(2)}(x+1,\beta) = (n+1)x\mathcal{B}_n(x,\beta) - n\mathcal{B}_{n+1}(x,\beta). \tag{11}$$

Substituting $\beta = 1$ into (11) and using Theorem 2.2, we obtain the following corollary:

Corollary 3.2. *Let $n \in \mathbb{N}_0$. The following relationship hold true:*

$$\sum_{j=0}^{n+1}\binom{n+1}{j}(jx^{j-1} + B_j(x))B_{n+1-j} = x(n+1)B_n(x) - nB_{n+1}(x). \tag{12}$$

Substituting $x = 0$ into (12), one can easily arrive at the following convolution recurrence relation for the Bernoulli numbers:

Corollary 3.3. *Let $n \in \mathbb{N}_0$. The following relationship hold true:*

$$B_{n+1} + \frac{1}{2}B_n + \frac{1}{n+1}\sum_{j=2}^{n+1}\binom{n+1}{j}B_jB_{n+1-j} = 0, \tag{13}$$

where $B_0 = 1$.

Remark 3.1. A convolution recurrence relation in (13) give us modification of the Euler's convolution recurrence relation for the Bernoulli numbers:

$$\frac{1}{n}\sum_{j=1}^{n}\binom{n}{j}B_jB_{n-j} = -B_n - B_{n-1},$$

where $n \geqslant 1$ (*cf.* [7,8,6,27,30]).

Substituting $a = v = 1, k = 0, \beta = -1$ and $b = e$ into (10), we obtain the following corollary:

Corollary 3.4.

$$E_n^{(2)}(x+1) = 2E_{n+1}(x) - 2xE_n(x). \tag{14}$$

By (14), we have

$$\sum_{j=0}^{n}\binom{n}{j}E_j(x+1)E_{n-j} = 2E_{n+1}(x) - 2xE_n(x).$$

Substituting the following well-known identity

$$E_n(x+1) = 2x^n - E_n(x)$$

into the above equation, we easily obtain the following convolution recurrence relation for the Euler polynomials:

$$\sum_{j=0}^{n}\binom{n}{j}(2x^n - E_n(x))E_{n-j} = 2E_{n+1}(x) - 2xE_n(x).$$

Substituting $x = 0$ into the above equation, we arrive at the known result due to convolution recurrence relation for the Euler numbers:

$$\sum_{j=1}^{n}\binom{n}{j}E_jE_{n-j} = E_n - 2E_{n+1}$$

(*cf.* [7,8,6,27,30]).

4. Identities related to the polynomials $Y_{n,\beta}^{(v)}(x;k,a,b)$ and β-Stirling type numbers

By using an approach similar to that of Srivastava [27] as well as some well-known properties of the Stirling numbers of the second kind, we can derive some known and some new identities and formulas for the *modification and unification of the Apostol-type polynomials* $Y_{n,\beta}^{(v)}(x;k,a,b)$ *of order v and the* β-Stirling type numbers.

We need the following generating function for generalized β-Stirling type numbers of the second kind, related to nonnegative positive real parameters.

Definition 4.1 (*cf.* [25, p. 3, Definition 2.1]). Let $a,b\in\mathbb{R}^+$ $(a\neq b)$, $\beta\in\mathbb{C}$ and $v\in\mathbb{N}_0$. The generalized β-Stirling type numbers of the second kind $\mathcal{S}(n,v;a,b;\beta)$ are defined by means of the following generating function:

$$f_{S,v}(t;a,b;\beta)=\frac{\left(\beta b^t-a^t\right)^v}{v!}=\sum_{n=0}^{\infty}\mathcal{S}(n,v;a,b;\beta)\frac{t^n}{n!}. \tag{15}$$

By setting $a=1$ and $b=e$ in (15), we have the β-Stirling numbers of the second kind

$$\mathcal{S}(n,v;1,e;\lambda)=S(n,v;\lambda),$$

which are defined by means of the following generating function:

$$\frac{(\beta e^t-1)^v}{v!}=\sum_{n=0}^{\infty}S(n,v;\beta)\frac{t^n}{n!},$$

(*cf.* [9,27,30]). Substituting $\beta=1$ into above equation, we have the Stirling numbers of the second kind $S(n,v;1)=S(n,v)$ (cf. [7,9,19,27,29,30]).

In [25, p. 3, Theorem 2.2], Simsek gave the following formulas for the generalized β-Stirling type numbers of the second kind $\mathcal{S}(n,v;a,b;\beta)$:

Theorem 4.1. *We have*

$$\mathcal{S}(n,v;a,b;\beta)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^j\binom{v}{j}\beta^{v-j}(j\ln a+(v-j)\ln b)^n \tag{16}$$

and

$$\mathcal{S}(n,v;a,b;\beta)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\beta^{j}(j\ln b+(v-j)\ln a)^n. \tag{17}$$

Note that by setting $a=1$ and $b=e$ in the assertions (16) of Theorem 4.1, we have the following result:

$$S(n,v;\beta)=\frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}\beta^{v-j}(-1)^j(v-j)^n.$$

The above relation has been studied by Srivastava [27] and Luo and Srivastava [9]. By setting $\beta=1$ in the above equation, we have the following well-known result for the classical Stirling numbers of the second kind:

$$S(n,v)=\frac{1}{v!}\sum_{j=0}^{v}\binom{v}{j}(-1)^j(v-j)^n$$

(*cf.* [1,6,5,7,9,19,22,25,27,30]).

In [25], Simsek constructed generating function of the *generalized array type polynomials* as follows: Let $a,b\in\mathbb{R}^+$, $(a\neq b)$, $\lambda\in\mathbb{C}$ and $v\in\mathbb{N}_0$.

$$g_v(x,t;a,b;\beta)=\frac{1}{v!}\left(\beta b^t-a^t\right)^v b^{xt}=\sum_{n=0}^{\infty}\mathcal{S}_v^n(x;a,b;\beta)\frac{t^n}{n!}. \tag{18}$$

The following definition provides a natural generalization and unification of the array polynomials:

Definition 4.2 [25]. Let $a,b\in\mathbb{R}^+$ $(a\neq b)$, $x\in\mathbb{R}$, $\lambda\in\mathbb{C}$ and $v\in\mathbb{N}_0$. The generalized array type polynomials $\mathcal{S}_v^n(x;a,b;\lambda)$ can be defined by

$$\mathcal{S}_v^n(x;a,b;\lambda)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}\lambda^j\left(\ln\left(a^{v-j}b^{x+j}\right)\right)^n. \tag{19}$$

Remark 4.1. The polynomials $\mathcal{S}_v^n(x;a,b;\lambda)$ may be also called generalized λ-array type polynomials. By substituting $x=0$ into (19), we arrive at (17):

$$\mathcal{S}_v^n(0;a,b;\lambda)=\mathcal{S}(n,v;a,b;\lambda).$$

Setting $a=\lambda=1$ and $b=e$ in (19), we have

$$S_v^n(x)=\frac{1}{v!}\sum_{j=0}^{v}(-1)^{v-j}\binom{v}{j}(x+j)^n,$$

a result due to Chang and Ha [3, Eq-(3.1)], Simsek [22]. It is easy to see that

$$S_0^0(x)=S_n^n(x)=1,$$

$$S_0^n(x)=x^n$$

and for $v>n$,

$$S_v^n(x)=0,$$

cf. [3, Eq-(3.1)].

By using (2) and (15), we derive the following functional equation:

$$\frac{2^{v(k-1)}v!}{t^{kv}}M_{k,v}(t,x,a,b;\beta)f_{S,v}(t;a,b;\beta)=b^{tx}. \tag{20}$$

Theorem 4.2. *Let $n,k,v\in\mathbb{N}$. Thus we have*

$$\sum_{l=0}^{n+kv}\binom{n+kv}{l}\mathcal{S}(n+kv-l,v,a,b,\beta)Y_{l,\beta}^{(v)}(x,k,a,b)=\binom{n+kv}{kv}\frac{(x\ln b)^n}{2^{v(k-1)}v!}. \tag{21}$$

Proof. By using (20), we get

$$\frac{2^{v(1-k)}}{v!}\sum_{n=0}^{\infty}(x\ln b)^n\frac{t^n}{n!}=t^{-kv}\sum_{n=0}^{\infty}\mathcal{S}(n,v;a,b;\beta)\frac{t^n}{n!}\sum_{n=0}^{\infty}Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!}.$$

Therefore

$$\frac{2^{v(1-k)}}{v!}\sum_{n=0}^{\infty}(x\ln b)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=v}^{n+kv}\frac{\binom{n+kv}{l}}{\binom{n+kv}{kv}}\mathcal{S}(n+kv-l,v,a,b,\beta)Y_{l,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the result. □

Remark 4.2. Substituting $k=a=1, b=e$ into (21), we obtain known result due to [27, p. 420, Theorem 15] see also [9].

Substituting $x=0$ into (21), we have the following results:

Corollary 4.1. *If $n=0$, then we have*

$$\sum_{l=0}^{kv}\binom{kv}{l}\mathcal{S}(kv-l,v,a,b,\beta)Y_{l,\beta}^{(v)}(k,a,b)=1$$

and if $n\neq 0$, then we obtain

$$\sum_{l=0}^{n+kv}\binom{n+kv}{l}\mathcal{S}(n+kv-l,v,a,b,\beta)Y_{l,\beta}^{(v)}(k,a,b)=0. \tag{22}$$

Remark 4.3. For $k=a=1, b=e$, Eq. (22) reduces to the known result given recently by Srivastava [27, p. 417, Theorem 13].

Corollary 4.2. *We have*

$$Y^{(v)}_{n+kv,\beta}(k,a,b)=\frac{-v!}{(\beta-1)^{v}}\sum_{l=0}^{n+kv-1}\binom{n+kv}{l}\mathcal{S}(n+kv-l,v,a,b,\beta)Y^{(v)}_{l,\beta}(k,a,b).$$

Proof. By (21), we have

$$0=\sum_{l=0}^{n+kv-1}\binom{n+kv}{l}\mathcal{S}(n+kv-l,v,a,b,\beta)Y^{(v)}_{l,\beta}(k,a,b)+Y^{(v)}_{n+kv,\beta}(k,a,b)\mathcal{S}(0,v,a,b,\beta).$$

By using the formula (16), we have

$$\mathcal{S}(0,v,a,b,\beta)=\frac{(\beta-1)^{v}}{v!}.$$

Substituting these numbers into the above equation, we obtain the desired result. □

Remark 4.4. By substituting $v=a=k=\beta=1, b=e$ into Corollary 4.2, we obtain

$$\mathcal{B}^{(v)}_{n+v}(\beta)=\frac{-v!}{(\beta-1)^{v}}\sum_{l=0}^{n+v-1}\binom{n+v}{l}S(n+v-l,v;\beta)\mathcal{B}^{(v)}_{l}(\beta)$$

(*cf.* [27, p. 418, Eq-(7.7)]). By same method of [27], for $v=a=k=\beta=1, b=e$ and $S(n,1)=1$, Eq. (22) reduces to the following well known results for the classical Bernoulli numbers: $B_0=1$ and

$$B_n=-\frac{1}{n+1}\sum_{l=0}^{n-1}\binom{n+1}{l}B_l$$

(*cf.* see for detail [27, p. 418, Eq-(7.8)]).

We now define modification and unification of the Apostol-type polynomials of order $-v$, $Y^{(-v)}_{n,\beta}(x,k,a,b)$ by means of the following generating function

$$M_{k,-v}(t,x,a,b;\beta)=\left(\frac{\beta b^{t}-a^{t}}{t^{k}2^{1-k}}\right)^{v}b^{xt}=\sum_{n=0}^{\infty}Y^{(-v)}_{n,\beta}(x,k,a,b)\frac{t^{n}}{n!}. \tag{23}$$

In work of Srivastava, we know that the Apostol-type polynomials of order $-v$ are related to the Stirling-type numbers [27, p. 417].

Theorem 4.3. *We have*

$$Y^{(-v)}_{n,\beta}(x,k,a,b)=\frac{v!2^{v(k-1)}}{(kv)!}\binom{n+kv}{kv}^{-1}\sum_{j=0}^{n+kv}\binom{n+kv}{j}\mathcal{S}(j,v,a,b,\beta)x^{n+kv-j}.$$

First Proof of Theorem 4.3. By combing (23) with (15), we get

$$\sum_{n=0}^{\infty}Y^{(-v)}_{n,\beta}(x,k,a,b)\frac{t^{n}}{n!}=v!2^{(k-1)v}t^{-kv}\sum_{n=0}^{\infty}\mathcal{S}(n,v,a,b,\beta)\frac{t^{n}}{n!}\sum_{n=0}^{\infty}(x\ln b)^{n}\frac{t^{n}}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}Y^{(-v)}_{n,\beta}(x,k,a,b)\frac{t^{n}}{n!}=\sum_{n=0}^{\infty}\left(\frac{v!2^{v(k-1)}}{(kv)!\binom{n+kv}{kv}}\right)\sum_{j=0}^{n+kv}\binom{n+kv}{j}\mathcal{S}(j,v,a,b,\beta)x^{n+kv-j}\frac{t^{n}}{n!}.$$

Thus by comparing the coefficients of t^n on both sides of the above equation, we arrive at the desired result. □

Theorem 4.4. *We have*

$$Y^{(-v)}_{n,\beta}(x,k,a,b)=\frac{v!2^{v(k-1)}}{(kv)!\binom{n+kv}{kv}}\mathcal{S}^{n+kv}_{v}(x;a,b;\lambda). \tag{24}$$

Proof. By combining (23) with (18), we get

$$\sum_{n=0}^{\infty} Y_{n,\beta}^{(-v)}(x,k,a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{v!2^{v(k-1)}}{(kv)!\binom{n+kv}{kv}} \mathcal{S}_v^{n+kv}(x;a,b;\lambda)\frac{t^n}{n!}.$$

Thus by comparing the coefficients of t^n on the both sides of the above equation, we arrive at the desired result. □

First Proof of Theorem 4.3. By combing (23) with (15), we get

$$Y_{n,\beta}^{(-v)}(x,k,a,b) = \frac{v!2^{v(k-1)}}{(kv)!\binom{n+kv}{kv}} \sum_{j=0}^{n+kv} \binom{n+kv}{j} \mathcal{S}(j,v;a,b;\lambda)(x\ln b)^{n+kv-j}.$$

Thus the proof is completed. □

Remark 4.5. If we substitute $a = k = 1, b = e$ into Theorem 4.3, we have known result in [27, p. 419, Theorem 14].

By using (24) and (19), we can compute some values of the polynomials $Y_{n,\beta}^{(-v)}(x,k,a,b)$ as follows (see [25]):

$$Y_{n,\beta}^{(0)}(x,k,a,b) = \mathcal{S}_0^n(x;a,b;\beta),$$

$$Y_{0,\beta}^{(-v)}(x,k,a,b) = 2^{v(k-1)}\mathcal{S}_v^0(x;a,b;\beta)$$

and

$$Y_{1,\beta}^{(-1)}(x,k,a,b) = \frac{2^{(k-1)}}{k!\binom{1+k}{k}}\mathcal{S}_1^1(x;a,b;\lambda) = \frac{2^{(k-1)}}{k!\binom{1+k}{k}}(-\ln a - x\ln b + \beta(x+1)\ln b).$$

If we substitute $x = 0$ into Theorem 4.3, we get the following result.

Corollary 4.3. *We have*

$$Y_{n,\beta}^{(-v)}(k,a,b) = \frac{v!2^{v(k-1)}}{(kv)!}\binom{n+kv}{kv}^{-1}\mathcal{S}(n+kv,v,a,b,\beta). \tag{25}$$

Remark 4.6. By substituting $v = k = n$, into (25) we get

$$Y_{n,\beta}^{(-n)}(n,a,b) = \frac{n!2^{n(n-1)}}{(n^2)!}\binom{n+n^2}{n^2}^{-1}\mathcal{S}(n+n^2,n,a,b,\beta).$$

By substituting $v = n, a = k = \beta = 1$ and $b = e$ into (25), we have

$$B_n^{(-n)} = \binom{2n}{n}^{-1}S(2n,n), \tag{26}$$

$S(2n,n)$ denotes the Stirling numbers of the second kind [27, p. 419, Eq-(7.18)].

Remark 4.7. In (26), we easily see that

$$S(2n,n) = (n+1)C_nB_n^{(-n)},$$

where C_n denotes the Catalan numbers (*cf.* [4]) which are defined by

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In [34], Srivastava et al. defined the generalized Apostol-type Genocchi polynomials as follows:

$$\left(\frac{2t}{\beta b^t + a^t}\right)^v c^{xt} = \sum_{n=0}^{\infty}\mathcal{G}_n^{(v)}(x;\beta;a,b,c)\frac{t^n}{n!},$$

(*cf*. see also [33], [30]). In the next theorem, we give relationship between the Stirling-type numbers and the generalized Apostol-type Genocchi polynomials of order $-v$.

Theorem 4.5. *We have*

$$\mathcal{S}\left(n+v,kv;a^2,b^2;\beta^2\right)=2^{-v}\binom{n+v}{v}\sum_{j=0}^{n}\binom{n}{j}\mathcal{S}(j+v,v;a,b;\beta)\mathcal{G}_{n-j}^{(-v)}(\beta;a,b).$$

Proof. By using (15), we get the following functional equation:

$$\frac{1}{(2t)^v}f_{S,v}(2t;a^2,b^2;\beta^2)=f_{S,v}(t;a,b;\beta)f_{G,-v}(t;a,b;\beta), \tag{27}$$

where

$$f_{G,-v}(t;a,b;\beta)=\left(\frac{2t}{\beta b^t+a^t}\right)^{-v}=\sum_{n=0}^{\infty}\mathcal{G}_n^{(-v)}(\beta;a,b)\frac{t^n}{n!},$$

$\mathcal{G}_n^{(-v)}(\beta,a,b)$ denotes β-Genocchi numbers of order $-v$, related to nonnegative real parameters.

By using (27), we obtain

$$\frac{1}{(2t)^v}\sum_{n=0}^{\infty}\mathcal{S}\left(n,v;a^2,b^2;\beta\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{S}(n,v;a,b;\beta)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{G}_n^{(-v)}(\beta;a,b)\frac{t^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}2^{-v}\binom{n+v}{v}^{-1}\mathcal{S}\left(n+v;kv,a^2,b^2;\beta^2\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\mathcal{S}(j+v,v;a,b;\beta)\mathcal{G}_{n-j}^{(-v)}(\beta;a,b)\frac{t^n}{n!}.$$

Thus by comparing the coefficients of t^n on both sides of the above equation, we arrive at the desired result. □

5. Identities related to the Eulerian type numbers and the Stirling type numbers

In this section, by using generating functions, we derive identities related to the Eulerian type numbers and the Stirling type numbers.

In [25,24, p.10], Simsek constructed generating functions of the Eulerian type polynomials of higher order $\mathcal{H}_n^{(m)}(x;u;a,b,c;\lambda)$ as follows:

$$F_{\lambda}^{(m)}(t,x;u,a,b,c)=\frac{(a^t-u)^m c^{xt}}{\left(\lambda b^t-u\right)^m}=\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}(x;u;a,b,c;\beta)\frac{t^n}{n!}, \tag{28}$$

where $m\in N$ and

$$\mathcal{H}_n^{(m)}(0;u;a,b,c;\beta)=\mathcal{H}_n^{(m)}(u;a,b;\beta),$$

denotes the Eulerian type numbers of higher order. By combining (15) with the above generating function, we derive the following functional equation:

$$\frac{\left(\frac{1}{u}-1\right)^m}{m!}b^{xt}=f_{S,m}\left(t;1,b;\frac{\beta}{u}\right)\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}(x;u;1,b,b;\beta)\frac{t^n}{n!}.$$

By using this functional equation, we get

$$\frac{\left(\frac{1}{u}-1\right)^m}{m!}\sum_{n=0}^{\infty}(x\ln b)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\mathcal{S}\left(n,m;1,b;\frac{\beta}{u}\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_n^{(m)}(x;u;1,b,b;\lambda)\frac{t^n}{n!}.$$

By using Cauchy product in the above equation, we obtain

$$\frac{\left(\frac{1}{u}-1\right)^m}{m!}\sum_{n=0}^{\infty}(x\ln b)^n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}\left(l,m;1,b;\frac{\beta}{u}\right)\mathcal{H}_{n-l}^{(m)}(x;u;1,b,b;\beta)\right)\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 5.1.

$$\frac{\left(\frac{1}{u}-1\right)^m}{m!}(x\ln b)^n=\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}\left(l,m;1,b;\frac{\beta}{u}\right)\mathcal{H}_{n-l}^{(m)}(x;u;1,b,b;\beta). \tag{29}$$

Substituting $b=e,\beta=1$ into (29), we have the following result:

Corollary 5.1.

$$\frac{\left(\frac{1}{u}-1\right)^m}{m!}x^n=\sum_{l=0}^{n}\binom{n}{l}S(l,m)H_{n-l}^{(m)}(x;u).$$

We now derive the following functional equations:

$$M_{k,v}(t,k,a,b;\beta)=(-1)^{v}F_1^{(v)}\left(t,x;u,a,\frac{a}{b},b\right)G_Y^{(v)}(t,a,\beta), \tag{30}$$

where

$$G_Y^{(v)}(t,a,\beta)=\left(\frac{1}{\beta a^t-1}\right)^v=\sum_{n=0}^{\infty}y_n^{(v)}(\beta,a)\frac{t^n}{n!}$$

and $a\geqslant 1$ (*cf.* [25, p-21, Eq-(37)])

$$M_{k,v}(t,x-v,a,b;\beta)=2^{-kv}M_{k,v}\left(2t,\frac{x}{2},\sqrt{ab},b;\beta\right) \tag{31}$$

and

$$M_{k,v}(t,v-x,a,b;\beta)=(-1)^{v(1-k)}\beta^{-v}a^{-tv}M_{k,v}(-t,x,a,b;\beta^{-1}). \tag{32}$$

Theorem 5.2. *Let $n-kv\geqslant 0$. The following identity holds true:*

$$\frac{(-1)^v2^{v(k-1)}}{\binom{n+kv}{kv}(kv)!}Y_{n,\beta}^{(v)}(x+v,k,a,b)=\sum_{l=0}^{n-kv}\binom{n-kv}{l}\mathcal{H}_l^{(v)}\left(x;\beta;a,\frac{a}{b},b;1\right)y_{n-kv-l}^{(v)}(\beta,a). \tag{33}$$

Proof. By combining (30) with (2) and (28), we get

$$\sum_{n=0}^{\infty}Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!}=(-1)^v2^{v(1-k)}t^{kv}\sum_{n=0}^{\infty}\mathcal{H}_n^{(v)}\left(x;\beta;a,\frac{a}{b},b;1\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}y_n^{(v)}(\beta,a)\frac{t^n}{n!}.$$

Therefore

$$(-1)^v2^{v(k-1)}\sum_{n=0}^{\infty}Y_{n,\beta}^{(v)}(x,k,a,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}\mathcal{H}_l^{(v)}\left(x;\beta;a,\frac{a}{b},b;1\right)y_{n-l}^{(v)}(\beta,a)\frac{t^{n-kv}}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result. □

Remark 5.1. Substituting $a=v=k=1$ and $b=e$ into (33), we have

$$\frac{-1}{n+1}\mathcal{B}_n(x+1,\beta)=\frac{1}{\beta-1}\sum_{l=0}^{n-1}\binom{n-1}{l}H_l(x;\beta).$$

By combining (31) with (2), after some elementary calculation, we arrive at the following theorem:

Theorem 5.3.

$$Y_{n,\beta}^{(v)}(x-v,k,a,b)=2^{n-kv}Y_{n,\beta}^{(v)}\left(\frac{x}{2},k,\sqrt{ab},b\right).$$

By combining (32) with (2), after some elementary calculation, we arrive at the following theorem:

Theorem 5.4. *The following identity holds true:*

$$Y_{n,\beta}^{(v)}(v-x,k,a,b)=\frac{(-1)^{v(1-k)+n}}{\beta^{v}}\sum_{j=0}^{n}\binom{n}{j}(v\ln a)^{n-j}Y_{j,\beta^{-1}}^{(v)}(x,k,a,b). \tag{34}$$

Remark 5.2. Substituting $a=v=k=1$ and $b=e$ into (34), we have

$$\mathcal{B}_n(1-x,\beta)=(-1)^n\beta^{-1}\mathcal{B}_n(x,\beta^{-1})$$

(cf. [33,30,34]). If we set $\beta=1$ into the above equation, we have the following well-known result:

$$B_n(1-x)=(-1)^nB_n(x)$$

(cf. [7,30]).

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