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On divisibility of sums of Apéry polynomials

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ABSTRACT

For any positive integers m and α, we prove that

$$\sum_{k=0}^{n-1} \epsilon^k (2k+1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n},$$

where $\epsilon \in \{1, -1\}$ and the generalized Apéry polynomial

$$A_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}^\alpha \binom{n+k}{k}^\alpha x^k.$$

The key to our proof is to use q-congruences.

1. Introduction

The Apéry number A_n is defined by

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

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Those numbers play an important role in Apéry's ingredient proof [17] of the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$. The Apéry numbers have some interesting arithmetic properties. For example, Gessel [6] proved that

$$A_{pn} \equiv A_n \pmod{p^3}$$

for any positive integer n and prime $p \geq 5$. In 2000, Ahlgren and Ono [1] solved a conjecture of Beukers [4] and showed that for odd prime p,

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ is the Fourier coefficient of q^n in the modular form $\eta(2z)^4\eta(4z)^4$.

Recently, Sun [14] defined the Apéry polynomial as

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k,$$

and proved several new congruences for the sums of $A_n(x)$. For example,

$$\sum_{k=0}^{n-1} (2k+1)A_k(x) \equiv 0 \pmod{n}$$

for every positive integer n. In fact, he obtained a curious identity:

$$\frac{1}{n}\sum_{k=0}^{n-1} (2k+1)A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k} x^k.$$

In general, Sun proposed the following conjecture.

Conjecture 1.1. *For $m \in \{1, 2, 3, \ldots\}$,*

$$\sum_{k=0}^{n-1} \epsilon^k (2k+1)A_k(x)^m \equiv 0 \pmod{n}, \tag{1.1}$$

where $\epsilon \in \{1, -1\}$.

In [8], Guo and Zeng proved that (1.1) is true if $\epsilon = -1$ and $m = 1$. The key of their proof is the following complicated identity:

$$\begin{aligned}&\frac{1}{n}\sum_{k=0}^{n-1} (-1)^k (2k+1)A_k(x) \\ &\quad = (-1)^{n-1}\sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{j=0}^{k} \binom{k}{j}\binom{k+j}{j}\binom{n-1}{k+j}\binom{n+k+j}{k+j}.\end{aligned}$$

On the other hand, in the same paper, Sun also defined the central Delannoy polynomial

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k.$$

He showed that

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)D_k(x) = \sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k} x^k.$$

Sun [15] also conjectured that

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)D_k(x)^m$$

is also always an integer for any $m \geq 1$.

Motivated by [12] and [1, Eq. (1.7)], it is natural to define the generalized Apéry polynomial

$$A_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha}\binom{n+k}{k}^{\alpha} x^k,$$

where α is a positive integer. Such polynomial is also called the Schmidt polynomial [8]. In fact, Guo and Zeng proved that for each integer $\alpha \geq 1$, there exist explicit formulas for

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k^{(\alpha)}(x) \quad \text{and} \quad \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)A_k^{(\alpha)}(x).$$

Hence

$$\sum_{k=0}^{n-1}(2k+1)A_k^{(\alpha)}(x) \equiv \sum_{k=0}^{n-1}(-1)^k(2k+1)A_k^{(\alpha)}(x) \equiv 0 \pmod{n}.$$

So they conjectured that (1.1) still holds if $A_k(x)$ is replaced by $A_k^{(\alpha)}(x)$.

In this paper, we shall completely confirm those conjectures above.

Theorem 1.1. *For arbitrary positive integers n, m and α, we have*

$$\sum_{k=0}^{n-1}(2k+1)A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}, \tag{1.2}$$

and

$$\sum_{k=0}^{n-1}(-1)^k(2k+1)A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}. \tag{1.3}$$

Unfortunately, no explicit formula is known for

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k^{(\alpha)}(x)^m \quad \text{and} \quad \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)A_k^{(\alpha)}(x)^m$$

when $m \geq 2$. So we can't prove Theorem 1.1 by following the ways of Sun, Guo and Zeng.

On the other hand, in the last thirty years, the q-analogues of combinatorial congruences have been widely studied by several authors (cf. [3,11,2,7,10,16,13,18]). The method of q-analogues has its advantage for some divisibility problems (e.g., see [5] and [9]). So we shall use q-congruences to prove (1.2) and (1.3) respectively.

2. Proof of (1.2)

For an integer n, define the q-integer

$$[n]_q = \frac{1-q^n}{1-q}.$$

Clearly $\lim_{q\to 1}[n]_q = n$. For a non-negative integer k, the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{\prod_{1\leq j\leq k}[n-j+1]_q}{\prod_{1\leq j\leq k}[j]_q}.$$

In particular, $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$. Also, we set $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k < 0$. It is easy to see that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q with integral coefficients, since

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q.$$

Below we introduce the notion of q-congruences. Suppose that a, b, n are integers and $a \equiv b \pmod{n}$. Then over the polynomial ring $\mathbb{Q}(q)$, we have

$$\frac{1-q^a}{1-q} - \frac{1-q^b}{1-q} = q^a \cdot \frac{1-q^{b-a}}{1-q} \equiv 0 \quad \left(\text{mod } \frac{1-q^n}{1-q}\right),$$

i.e., $[a]_q \equiv [b]_q \pmod{[n]_q}$. Furthermore, for the q-binomial coefficients, we have the following q-Lucas congruence (cf. [11, Theorem 2.2]).

Lemma 2.1. *Suppose that $d > 1$ is a positive integer. Suppose that a, b, h, l are integers with $0 \le b, l \le d-1$. Then*

$$\begin{bmatrix} ad+b \\ hd+l \end{bmatrix}_q \equiv \binom{a}{h} \begin{bmatrix} b \\ l \end{bmatrix}_q \pmod{\Phi_d(q)},$$

where $\Phi_d(q)$ is the d-th cyclotomic polynomial.

Define the generalized q-Apéry polynomial

$$A_k^{(\alpha)}(x;q) = \sum_{j=0}^{k} q^{\alpha(\binom{j}{2}-jk)} \begin{bmatrix} k \\ j \end{bmatrix}_q^{\alpha} \begin{bmatrix} k+j \\ j \end{bmatrix}_q^{\alpha} x^j.$$

In order to prove (1.2), it suffices to show that

Theorem 2.1.

$$\sum_{k=0}^{n-1} q^{n-1-k}[2k+1]_q A_k^{(\alpha)}(x;q)^m \equiv 0 \pmod{[n]_q}. \tag{2.1}$$

Let us explain why (2.1) implies (1.2). Since $[n]_q$ is a primitive polynomial (i.e., the greatest divisor of all coefficients of $[n]_q$ is 1), by a well-known lemma of Gauss, there exists a polynomial $H(x,q)$ with integral coefficients such that

$$\sum_{k=0}^{n-1} q^{n-1-k}[2k+1]_q A_k^{(\alpha)}(x;q)^m = [n]_q H(x,q). \tag{2.2}$$

Substituting $q=1$ in (2.2), we get (1.2).

It is well-known that

$$[n]_q = \prod_{\substack{d|n \\ d>1}} \Phi_d(q).$$

The advantage of q-congruences is that we only need to prove

$$\sum_{k=0}^{n-1} q^{n-1-k}[2k+1]_q A_k^{(\alpha)}(x;q)^m \equiv 0 \pmod{\Phi_d(q)}$$

for every divisor $d > 1$ of n. Note that

$$\begin{aligned}\begin{bmatrix} k+j \\ j \end{bmatrix}_q &= \frac{(1-q^{k+1})(1-q^{k+2})\cdots(1-q^{k+j})}{(1-q)(1-q^2)\cdots(1-q^j)} \\ &= (-1)^j \frac{q^{jk+\binom{j+1}{2}}(1-q^{-k-1})(1-q^{-k-2})\cdots(1-q^{-k-j})}{(1-q)(1-q^2)\cdots(1-q^j)} \\ &= (-1)^j q^{jk+\binom{j+1}{2}} \begin{bmatrix} -k-1 \\ j \end{bmatrix}_q .\end{aligned}$$

So

$$A_k^{(\alpha)}(x;q) = \sum_{-\infty<j<+\infty} (-1)^{\alpha j} q^{\alpha j^2} \begin{bmatrix} k \\ j \end{bmatrix}_q^\alpha \begin{bmatrix} -k-1 \\ j \end{bmatrix}_q^\alpha x^j.$$

Suppose that $d>1$ is a divisor of n. Let $h=n/d$. Write $k=ad+b$ where $0\le b\le d-1$. Then by Lemma 2.1,

$$\begin{aligned}&A_{ad+b}^{(\alpha)}(x;q) \\ &\quad= \sum_{\substack{-\infty<s<+\infty\\ 0\le t\le d-1}} (-1)^{\alpha(sd+t)} q^{\alpha(sd+t)^2} \begin{bmatrix} ad+b \\ sd+t \end{bmatrix}_q^\alpha \begin{bmatrix} (-a-1)d+d-b-1 \\ sd+t \end{bmatrix}_q^\alpha x^{sd+t} \\ &\quad\equiv \sum_{\substack{-\infty<s<+\infty\\ 0\le t\le d-1}} (-1)^{\alpha(sd+t)} q^{\alpha t^2} \binom{a}{s}^\alpha \begin{bmatrix} b \\ t \end{bmatrix}_q^\alpha \binom{-a-1}{s}^\alpha \\ &\qquad\times \begin{bmatrix} d-b-1 \\ t \end{bmatrix}_q^\alpha x^{sd+t} \pmod{\Phi_d(q)}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k=0}^{n-1}[2k+1]_q q^{n-1-k} A_k^{(\alpha)}(x;q)^m &= \sum_{\substack{0\le a\le h-1\\ 0\le b\le d-1}} q^{hd-1-ad-b}[2ad+2b+1]_q A_{ad+b}^{(\alpha)}(x;q)^m \\ &\equiv \sum_{\substack{0\le a\le h-1\\ 0\le b\le d-1}} q^{-1-b}[2b+1]_q B_{a,b,d}^{(\alpha)}(x;q)^m \pmod{\Phi_d(q)},\end{aligned}$$

where

$$B_{a,b,d}^{(\alpha)}(x;q) = \sum_{\substack{-\infty<s<+\infty\\ 0\le t\le d-1}} (-1)^{\alpha(sd+t)} q^{\alpha t^2} \binom{a}{s}^\alpha \begin{bmatrix} b \\ t \end{bmatrix}_q^\alpha \binom{-a-1}{s}^\alpha \begin{bmatrix} d-b-1 \\ t \end{bmatrix}_q^\alpha x^{sd+t}.$$

Similarly, note that

$$k=ad+b \iff n-k-1=(h-a-1)d+(d-b-1)$$

and

$$B^{(\alpha)}_{a,b,d}(x;q) = B^{(\alpha)}_{a,d-b-1,d}(x;q).$$

We have

$$\begin{aligned}
&\sum_{k=0}^{n-1} q^k[2n-2k-1]_q A^{(\alpha)}_{n-k-1}(x;q)^m \\
&\quad\equiv \sum_{\substack{0\le a\le h-1\\0\le b\le d-1}} q^b[-2b-1]_q B^{(\alpha)}_{h-a-1,d-b-1,d}(x;q)^m \\
&\quad= \sum_{\substack{0\le a'\le h-1\\0\le b\le d-1}} q^b[-2b-1]_q B^{(\alpha)}_{a',b,d}(x;q)^m \quad \bigl(\mathrm{mod}\ \Phi_d(q)\bigr).
\end{aligned}$$

Clearly,

$$q^{-1-b}[2b+1]_q + q^b[-2b-1]_q = q^{-1-b} - q^b + q^b - q^{-b-1} = 0.$$

Therefore,

$$\begin{aligned}
&2\sum_{k=0}^{n-1} q^{n-1-k}[2k+1]_q A^{(\alpha)}_k(x;q)^m \\
&\quad= \sum_{k=0}^{n-1} q^{n-1-k}[2k+1]_q A^{(\alpha)}_k(x;q)^m + \sum_{k=0}^{n-1} q^k[2n-2k-1]_q A^{(\alpha)}_{n-1-k}(x;q)^m \\
&\quad\equiv 0 \quad \bigl(\mathrm{mod}\ \Phi_d(q)\bigr).
\end{aligned}$$

This concludes the proof of Theorem 2.1.

3. Proof of (1.3)

The proof of (1.3) is a little complicated.

Theorem 3.1.

$$\sum_{k=0}^{n-1} (-1)^k q^{n-1-k}[2k+1]_q A^{(\alpha)}_k\bigl(x;q^2\bigr)^m$$

is divisible by

$$\prod_{\substack{d\mid n\\ d>1\ \text{is odd}}} \Phi_d(q) \cdot \prod_{\substack{d\mid n\\ d\ \text{is even}}} \Phi_d\bigl(q^2\bigr).$$

In fact, we only need to prove that

$$\sum_{k=0}^{n-1}(-1)^k q^{n-1-k}[2k+1]_q A_k^{(\alpha)}\big(x;q^2\big)^m + \sum_{k=0}^{n-1}(-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}\big(x;q^2\big)^m$$

is divisible by $\Phi_d(q)$ for odd $d>1$ and by $\Phi_d(q^2)$ for even d respectively.

Lemma 3.1. *If $d>1$ is odd, then $\Phi_d(q)$ divides $\Phi_d(q^2)$. If d is even, then $\Phi_d(q^2)=\Phi_{2d}(q)$.*

Proof. We know that for $d>1$,

$$\Phi_d(q) = \prod_{\substack{\xi \text{ is } d\text{-th primitive} \\ \text{root of unity}}} (q-\xi).$$

Suppose that d is odd and ξ is an arbitrary d-th primitive root of unity. Then ξ^2 also is a d-th primitive root of unity, i.e., $\Phi_d(\xi^2)=0$. Hence $\Phi_d(q)$ divides $\Phi_d(q^2)$. Similarly, if d is even and ξ is a $2d$-th primitive root of unity, then ξ^2 is a d-th primitive root of unity. So $\Phi_{2d}(q)$ divides $\Phi_d(q^2)$. Note that now $\deg\Phi_{2d}=\phi(2d)=2\phi(d)=2\deg\Phi_d$, where ϕ is the Euler totient function. We must have $\Phi_d(q^2)=\Phi_{2d}(q)$. □

Suppose that $d>1$ is an odd divisor of n. Let $h=n/d$. Then

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^k q^{n-1-k}[2k+1]_q A_k^{(\alpha)}\big(x;q^2\big)^m \\
&\quad\equiv \sum_{\substack{0\le a\le h-1\\ 0\le b\le d-1}} (-1)^{ad+b} q^{hd-1-ad-b}\big[2(ad+b)+1\big]_q B_{a,b,d}^{(\alpha)}\big(x;q^2\big)^m \pmod{\Phi_d\big(q^2\big)} \\
&\quad\equiv \sum_{\substack{0\le a\le h-1\\ 0\le b\le d-1}} (-1)^{ad+b} q^{-1-b}[2b+1]_q B_{a,b,d}^{(\alpha)}\big(x;q^2\big)^m \pmod{\Phi_d(q)}.
\end{aligned}$$

And

$$\begin{aligned}
&\sum_{k=0}^{n-1}(-1)^{n-1-k}[2n-2k-1]_q q^k A_{n-1-k}^{(\alpha)}\big(x;q^2\big)^m \\
&\quad\equiv \sum_{\substack{0\le a\le h-1\\ 0\le b\le d-1}} (-1)^{hd-1-ad-b} q^{ad+b}\big[2hd-2(ad+b)-1\big]_q B_{h-a-1,d-b-1,d}^{(\alpha)}\big(x;q^2\big)^m \\
&\quad\equiv \sum_{\substack{0\le a'\le h-1\\ 0\le b\le d-1}} (-1)^{a'd+d-1-b}[-2b-1]_q q^b B_{a',b,d}^{(\alpha)}\big(x;q^2\big)^m \pmod{\Phi_d(q)}.
\end{aligned}$$

Since d is odd,

$$(-1)^{ad+b}q^{-1-b}[2b+1]_q+(-1)^{ad+d-1-b}q^{b}[-2b-1]_q=0.$$

So $\Phi_d(q)$ divides

$$\sum_{k=0}^{n-1}(-1)^k q^{n-1-k}[2k+1]_q A_k^{(\alpha)}\big(x;q^2\big)^m+\sum_{k=0}^{n-1}(-1)^{n-1-k}q^k[2n-2k-1]_q A_{n-1-k}^{(\alpha)}\big(x;q^2\big)^m.$$

Suppose that d is an even divisor of n. Then

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^k q^{n-1-k}[2k+1]_q A_k^{(\alpha)}\big(x;q^2\big)^m\\ &\quad\equiv\sum_{\substack{0\le a\le h-1\\0\le b\le d-1}}(-1)^{ad+b}q^{hd-1-(ad+b)}\big[2(ad+b)+1\big]_q B_{a,b,d}^{(\alpha)}\big(x;q^2\big)^m\\ &\quad\equiv\sum_{\substack{0\le a\le h-1\\0\le b\le d-1}}(-1)^{ad+b}q^{hd-ad-1-b}[2b+1]_q B_{a,b,d}^{(\alpha)}\big(x;q^2\big)^m\pmod{\Phi_d\big(q^2\big)}.\end{aligned}$$

And

$$\begin{aligned}&\sum_{k=0}^{n-1}(-1)^{n-1-k}q^k[2n-2k-1]_q A_{n-1-k}^{(\alpha)}\big(x;q^2\big)^m\\ &\quad\equiv\sum_{\substack{0\le a\le h-1\\0\le b\le c-1}}(-1)^{hd-1-(ad+b)}q^{ad+b}\big[2hd-2(ad+b)-1\big]_q B_{h-a-1,d-b-1,d}^{(\alpha)}\big(x;q^2\big)^m\\ &\quad\equiv\sum_{\substack{0\le a'\le h-1\\0\le b\le d-1}}(-1)^{a'd+d-1-b}q^{hd-a'd-d+b}[-2b-1]_q B_{a',b,d}^{(\alpha)}\big(x;q^2\big)^m\pmod{\Phi_d\big(q^2\big)}.\end{aligned}$$

Note that $\Phi_d(q^2)=\Phi_{2d}(q)$ divides $1+q^d=(1-q^{2d})/(1-q^d)$, i.e.,

$$q^d\equiv-1\pmod{\Phi_d\big(q^2\big)}.$$

We have

$$\begin{aligned}&(-1)^{ad+b}q^{hd-ad-1-b}[2b+1]_q+(-1)^{ad+d-1-b}q^{hd-ad-d+b}[-2b-1]_q\\ &\quad\equiv(-1)^{ad+b}q^{hd-ad}\big(q^{-1-b}[2b+1]_q+q^b[-2b-1]_q\big)\\ &\quad=0\pmod{\Phi_d\big(q^2\big)}.\end{aligned}$$

That is, $\Phi_d(q^2)$ divides

$$\sum_{k=0}^{n-1}(-1)^k q^{n-1-k}[2k+1]_q A_k^{(\alpha)}\big(x;q^2\big)^m + \sum_{k=0}^{n-1}(-1)^{n-1-k} q^k [2n-2k-1]_q A_{n-1-k}^{(\alpha)}\big(x;q^2\big)^m.$$

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References

[1] S. Ahlgren, K. Ono, A Gaussian hypergeometric series evaluation and Apéry number congruences, J. Reine Angew. Math. 518 (2000) 187–212.
[2] G.E. Andrews, q-Analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, Discrete Math. 204 (1999) 15–25.
[3] G.E. Andrews, D. Foata, Congruences for the q-secant numbers, European J. Combin. 1 (1980) 283–287.
[4] F. Beukers, Another congruence for the Apéry numbers, J. Number Theory 25 (1987) 201–210.
[5] N.J. Calkin, Factors of sums of powers of binomial coefficients, Acta Arith. 86 (1998) 17–26.
[6] I. Gessel, Some congruences for Apéry numbers, J. Number Theory 14 (1982) 362–368.
[7] Victor J.W. Guo, J. Zeng, Some arithmetic properties of the q-Euler numbers and q-Salié numbers, European J. Combin. 27 (2006) 884–895.
[8] V.J.W. Guo, J. Zeng, Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials, J. Number Theory 132 (2012) 1731–1740.
[9] V.J.W. Guo, J. Zeng, New congruences for sums involving Apery numbers or central Delannoy numbers, Int. J. Number Theory 8 (2012) 2003–2016.
[10] H. Pan, H.-Q. Cao, A congruence involving products of q-binomial coefficients, J. Number Theory 121 (2006) 224–233.
[11] Bruce E. Sagan, Congruence properties of q-analogs, Adv. Math. 95 (1992) 127–143.
[12] A.L. Schmidt, Generalized q-Legendre polynomials, J. Comput. Appl. Math. 49 (1993) 243–249.
[13] A. Schultz, R. Walker, A generalization of the Gaussian formula and a q-analog of Fleck's congruence, J. Number Theory 133 (2013) 3717–3738.
[14] Z.-W. Sun, On sums of Apéry polynomials and related congruences, J. Number Theory 132 (2012) 2673–2699.
[15] Z.-W. Sun, Congruences involving generalized central trinomial coefficients, Sci. China Math. (2014), in press.
[16] R. Tauraso, q-Analogs of some congruences involving Catalan numbers, Adv. in Appl. Math. 48 (2012) 603–614.
[17] A. van der Poorten, A proof that Euler missed ... Apéry's proof of the irrationality of $\zeta(3)$, Math. Intelligencer 1 (1978/1979) 195–203.
[18] J. Zhao, On q-analog of Wolstenholme type congruences for multiple harmonic sums, Integers 13 (2013), A23, 11 pp.