# A FAMILY OF FIBONACCI-LIKE CONDITIONAL SEQUENCES

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**Abstract**

Let $\{a_{i,j}\}$ be real numbers for $0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant 2$, and define a sequence $\{v_n\}$ with initial conditions $v_0$, $v_1$ and conditional linear recurrence relation $v_n = a_{t,1}v_{n-1} + a_{t,2}v_{n-2}$ where $n \equiv t \pmod r$ $(n \geqslant 2)$. The sequence $\{v_n\}$ can be viewed as a generalization of many well-known integer sequences, such as Fibonacci, Lucas, Pell, Jacobsthal, etc. We find explicitly a linear recurrence equation which is satisfied by $\{v_n\}$, generating functions, matrix representations and extended Binet's formulas for $\{v_n\}$ in terms of a generalized continuant.

## 1. Introduction

The Fibonacci sequence is a sequence of integers in which each subsequent term is the sum of the two preceding it starting with 0 and 1. The so-called Fibonacci numbers appeared in the solution of a problem by Leonardo Pisano, in his book Liber Abaci (1202), concerning reproduction patterns of rabbits. Now, this sequence appears in many areas of mathematics. The Fibonacci sequence has been generalized in several directions, namely, by changing the initial values, by mixing two sequences, by demanding that the numbers in the sequences not be integers, and by having more than two parameters; see [6, 8, 10, 11, 12, 13, 14, 15, 16, 20, 21].

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Yet another extension of the Fibonacci sequence is given in [4] as follows. Let $a_0, a_1, \ldots, a_{r-1}$ be real numbers; define a sequence $\{q_n\}$, with initial values $q_0 = 0$ and $q_1 = 1$, and for $n \geqslant 2$,

$$q_n = \begin{cases} a_0 q_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod{r}, \\ a_1 q_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod{r}, \\ \vdots & \vdots \\ a_{r-1} q_{n-1} + q_{n-2}, & \text{if } n \equiv r-1 \pmod{r}. \end{cases}$$

It remained as an open problem in [4] to find a closed form for the generating function, and to obtain a Binet-like formula for $\{q_n\}$. Later on, the sequence $\{q_n\}$ was studied in [3], where it was called *$k$-periodic Fibonacci sequences* (here $k$ corresponds to our $r$), and the open problem was solved. Independently, in [17] and [19], one of the authors of this paper solved this open problem by using continuants while also introducing a matrix representation for the sequence $\{q_n\}$.

We now introduce a new sequence $\{v_n\}$ that is a generalization of the sequence $\{q_n\}$. Let $\{a_{i,j}\}$ be real numbers for $0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant 2$, and define a sequence $\{v_n\}$ with initial conditions $v_0$, $v_1$, and for $n \geqslant 2$,

$$v_n = \begin{cases} a_{0,1} v_{n-1} + a_{0,2} v_{n-2}, & \text{if } n \equiv 0 \pmod{r}, \\ a_{1,1} v_{n-1} + a_{1,2} v_{n-2}, & \text{if } n \equiv 1 \pmod{r}, \\ \vdots & \vdots \\ a_{r-1,1} v_{n-1} + a_{r-1,2} v_{n-2}, & \text{if } n \equiv r-1 \pmod{r}. \end{cases} \tag{1}$$

We call $\{v_n\}$ a *Fibonacci-like conditional sequence.*

We can get a great number of different sequences by providing the values of $r, a_{i,j}$ and the initial values in the sequence $\{v_n\}$. The following are some examples of sequences which are special cases of $\{v_n\}$. We note that some of the following examples are given with the a reference number to Sloane's *On-Line Encyclopedia of Integer Sequences.* Let us assume initial values $v_0 = 0$ and $v_1 = 1$.

1. If we take $a_{i,j} = 1$ for $0 \leqslant i \leqslant r-1$ and $1 \leqslant j \leqslant 2$, we get the classical Fibonacci numbers.

2. If we take $a_{i,1} = 1$ and $a_{i,2} = 1$ for $0 \leqslant i \leqslant r-1$, we get Pell's sequence.

3. If we take $a_{i,1} = k$ and $a_{i,2} = 1$ for $0 \leqslant i \leqslant r-1$, we get $k$-Fibonacci numbers.

4. If we take $r = 2$, $a_{0,1} = a_{1,2} = 1$ and $a_{1,1} = a_{0,2} = 2$, we get [$A005824$]

5. If we take $r = 2$, $a_{0,1} = 2$ and $a_{0,2} = a_{1,1} = a_{1,2} = 1$, we get [$A048788$]. The terms of the sequence are numerators of the continued fraction convergent to $\sqrt{3} - 1$.

6. If we take $r = 2$, $a_{0,1} = a_{0,2} = 1$ and $a_{1,1} = a_{1,2} = 2$, we get [$A001045$] (Jacobsthal sequence).

7. If we take $r = 3$ and $(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = (1, 1, 0, 1, 1, 1)$, we get [$A097564$].

8. If we take $r = 3$ and $(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = (1, 1, 1, 0, 0, 0)$, we get [$A117567$] which gives Riordan arrays.

9. If we take $r = 3$ and $(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = (0, 1, 1, 1, 1, 1)$, we get [$A092550$] which is the two-steps-forward and one-step backwards Fibonacci-based switched sequence inspired by Per Bak's sand piles.

10. If we take $r = 3$ and $(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = (2, 0, 2, 1, 2, 0)$, we get [$A004647$].

11. If we take $r = 3$ and $(a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}) = (2, 0, 1, 1, 1, 1)$, we get [$A133335$].

12. If we take $r = 2$ with $a_{0,2} = a_{1,2} = 1$ and any nonzero real numbers $a_{0,1}$ and $a_{1,1}$ we get *generalized Fibonacci sequences* [4].

13. If we take $a_{i,2} = 1$ for $0 \leqslant i \leqslant r - 1$ with any nonzero real numbers $a_{i,1}$, we get the sequences studied in [3, 17, 19].

14. If we take $r = 2$ and real numbers $a_{i,j}$, not all zeros, we get the sequences defined in [18].

Next we give the structure of the paper. In Section 2, we introduce the concept of the generalized continuant and the generalized continued fraction, we make a connection between them and we find explicitly a linear recurrence relation which is satisfied by $\{v_n\}$ for any given integer $r \geq 2$. In Sections 3 and 4, we find generating functions, matrix representations and extended Binet's formulas for $\{v_n\}$ in terms of a generalized continuant. Conclusions and further work are given in Section 5.

## 2. Linear Recurrence of $\{v_n\}$

Continuants are a sequence of polynomials extensively studied by Euler. They are defined by the following recurrence

$$\begin{aligned} K_0() &= 1; \\ K_1(x_1) &= x_1; \\ K_n(x_1, \ldots, x_n) &= K_{n-1}(x_1, \ldots, x_{n-1})x_n + K_{n-2}(x_1, \ldots, x_{n-2}). \end{aligned}$$

It is well known that continuants are the key to the study of continued fractions like

$$a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\ddots + \vdots \atop \cfrac{1}{a_0}}}} \ .$$

Indeed, the above continued fraction is equal to

$$\frac{K\left(a_n, a_{n-1}, \ldots, a_0\right)}{K\left(a_{n-1}, a_{n-2}, \ldots, a_0\right)} \ .$$

We refer to pages 301-309 in [5] for detailed information about the continuant.

Let us consider a generalization of continued fractions of the form

$$a_{n,1} + \cfrac{a_{n,2}}{a_{n-1,1} + \cfrac{a_{n-1,2}}{a_{n-2,1} + \cfrac{a_{n-2,2}}{\ddots + \vdots \atop \cfrac{a_{1,2}}{a_{0,1}}}}}$$

and denote it by $[a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}]_{\overrightarrow{b}}$ where $\overrightarrow{b} = [a_{n,2}, a_{n-1,2}, \ldots, a_{1,2}]$ for any positive integer $n$. If we take $\overrightarrow{b} = [1, 1, \ldots, 1]$ then we get the classical continued fraction. For example, let us take $[1, 2, 1, 1, 2, 1]_{\overrightarrow{b}}$ where $\overrightarrow{b} = [1, 1, 2, 1, 1]$; then

$$[1, 2, 1, 1, 2, 1]_{\overrightarrow{b}} = 1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{2}{1 + \cfrac{1}{2 + \cfrac{1}{1}}}}} = \frac{17}{12} \ .$$

We associate this generalized continued fraction with the following Fibonacci-like sequences

$$A_n = \begin{cases} A_{n-1} + 2A_{n-2}, & \text{if } n \equiv 0 \pmod 3, \\ 2A_{n-1} + A_{n-2}, & \text{if } n \equiv 1 \pmod 3, \\ A_{n-1} + A_{n-2}, & \text{if } n \equiv 2 \pmod 3, \end{cases} \qquad A_0 = 1 \text{ and } A_1 = 3,$$

and

$$B_n = \begin{cases} B_{n-1} + B_{n-2}, & \text{if } n \equiv 0 \pmod 3, \\ B_{n-1} + 2B_{n-2}, & \text{if } n \equiv 1 \pmod 3, \\ 2B_{n-1} + B_{n-2}, & \text{if } n \equiv 2 \pmod 3, \end{cases} \qquad B_0 = 1 \text{ and } B_1 = 1.$$

Here, $[1,2,1,1,2,1]_{\overrightarrow{b}} = \dfrac{17}{12} = \dfrac{A_5}{B_5}$ where $\overrightarrow{b} = [1,1,2,1,1]$. The sequences $\{A_n\}$ and $\{B_n\}$ correspond to a generalized continuant; explanation of the mathematical background of this example is forthcoming.

In order to study the general conditional sequences introduced in Section 1, we now introduce a generalized continuant with an input of two-dimensional arrays $\{a_{i,j} \mid i, j \geq 0\}$.

**Definition 1 (Generalized Continuant).** Let $K\left(\right) = 1, K\left(a_{i,j}\right) = a_{i,j}$, where $i, j$ are nonnegative integers, and for $n \geqslant 2$,

$$\begin{aligned} &K\left(a_{n,j}, a_{n-1,j}, \ldots, a_{i+1,j}, a_{i,j}\right) \\ &\quad = \quad a_{i,j} K\left(a_{n,j}, a_{n-1,j}, \ldots, a_{i+1,j}\right) + a_{i+1,j+1} K\left(a_{n,j}, a_{n-1,j}, \ldots, a_{i+2,j}\right). \end{aligned}$$

For example, let us suppose that

$$[a_{0,1}, \ldots, a_{5,1}] = [1,2,1,1,2,1] \quad \text{and} \quad [a_{0,2}, \ldots, a_{5,2}] = [2,1,1,2,1,1],$$

then we can obtain $K(a_{5,1}, \ldots, a_{0,1}) = 34$ and $K(a_{4,1}, \ldots, a_{0,1}) = 24$. We emphasize that the definition of $K$ depends on both input sequences. For example, let $[b_{0,1}, \ldots, b_{5,1}] = [1,2,1,1,2,1]$ and $[b_{0,2}, \ldots, b_{5,2}] = [2,1,1,1,2,2]$. Although $[b_{5,1}, b_{4,1}, b_{3,1}] = [b_{2,1}, b_{1,1}, b_{0,1}] = [1,2,1]$, we still have $K(b_{5,1}, b_{4,1}, b_{3,1}) = 6 \neq K(b_{2,1}, b_{1,1}, b_{0,1}) = 4$ because $[b_{5,2}, b_{4,2}, b_{3,2}] \neq [b_{2,2}, b_{1,2}, b_{0,2}]$.

We note that this generalized continuant satisfies the following identity which can serve as an alternative definition.

**Theorem 2.** *For a positive integer $n \geqslant 1$, and $0 \leq k \leq n-1$, the generalized continuant satisfies*

$$K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{k,1}\right) = a_{n,1} K\left(a_{n-1,1}, \ldots, a_{k,1}\right) + a_{n,2} K\left(a_{n-2,1}, \ldots, a_{k,1}\right).$$

*Proof.* We shall prove the formula by induction on the length of the generalized continuant. By the definition of the continuant (Definition 1), we can write

$$\begin{aligned} K\left(a_{k+1,1}, a_{k,1}\right) &= a_{k,1} K\left(a_{k+1,1}\right) + a_{k+1,2} \\ &= a_{k,1} a_{k+1,1} + a_{k+1,2} = a_{k+1,1} K\left(a_{k,1}\right) + a_{k+1,2}. \end{aligned}$$

Hence it is true for the base case (length 2), that is simply the case $n = k+1$. Let us assume now that the formula holds for the generalized continuant of the above form of length less than or equal to $m$ (that is, $n = m + k - 1$). If we use the definition of continuant and the hypothesis of induction, we get

$$
\begin{aligned}
&K\left(a_{m+k,1}, a_{m+k-1,1}, \ldots, a_{k,1}\right)\\
&\quad= a_{k,1} K\left(a_{m+k,1}, a_{m+k-1,1}, \ldots, a_{k+1,1}\right)\\
&\qquad +a_{k+1,2} K\left(a_{m+k,1}, a_{m+k-1,1}, \ldots, a_{k+2,1}\right)\\
&\quad= a_{k,1}\left(a_{m+k,1} K\left(a_{m+k-1,1}, \ldots, a_{k+1,1}\right)+a_{m+k,2} K\left(a_{m+k-2,1}, \ldots, a_{k+1,1}\right)\right)\\
&\qquad +a_{k+1,2}\left(a_{m+k,1} K\left(a_{m+k-1,1}, \ldots, a_{k+2,1}\right)+a_{m+k,2} K\left(a_{m+k-2,1}, \ldots, a_{k+2,1}\right)\right)\\
&\quad= a_{m+k,1}\left(a_{k,1} K\left(a_{m+k-1,1}, \ldots, a_{k+1,1}\right)+a_{k+1,2} K\left(a_{m+k-1,1}, \ldots, a_{k+2,1}\right)\right)\\
&\qquad +a_{m+k,2}\left(a_{k,1} K\left(a_{m+k-2,1}, \ldots, a_{k+1,1}\right)+a_{k+1,2} K\left(a_{m+k-2,1}, \ldots, a_{k+2,1}\right)\right)\\
&\quad= a_{m+k,1} K\left(a_{m+k-1,1}, \ldots, a_{k,1}\right)+a_{m+k,2} K\left(a_{m+k-2,1}, \ldots, a_{k,1}\right).
\end{aligned}
$$

□

Similar to the classical continuant, by mathematical induction we can easily establish the following connection between the generalized continuant and the generalized continued fraction.

**Theorem 3.** *For any positive integer $n$,*

$$\left[a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right]_{\overrightarrow{b}}=\frac{K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right)}{K\left(a_{n-1,1}, a_{n-2,1}, \ldots, a_{0,1}\right)},$$

*where* $\overrightarrow{b}=\left[a_{n,2}, a_{n-1,2}, \ldots, a_{1,2}\right]$.

*Proof.* First we observe that the result is true for $n=0$ and $n=1$,

$$a_{0,1}=\frac{K\left(a_{0,1}\right)}{K\left(\right)} \quad \text{and} \quad a_{1,1}+\frac{a_{1,2}}{a_{0,1}}=\frac{K\left(a_{1,1}, a_{0,1}\right)}{K\left(a_{0,1}\right)}$$

by the definition of the generalized continuant. Assume that the statement of the theorem holds for $n$. By using hypothesis of the induction and Theorem 2 (taking $k=0$), we get

$$
\begin{aligned}
&\left[a_{n+1,1}, a_{n,1}, \ldots, a_{0,1}\right]_{\overrightarrow{b_1}}\\
&\qquad= a_{n+1,1}+\frac{a_{n+1,2}}{\left[a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right]_{\overrightarrow{b_2}}}\\
&\qquad= a_{n+1,1}+\frac{a_{n+1,2}}{\dfrac{K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right)}{K\left(a_{n-1,1}, a_{n-2,1}, \ldots, a_{0,1}\right)}}\\
&\qquad= \frac{a_{n+1,1} K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right)+a_{n+1,2} K\left(a_{n-1,1}, a_{n-2,1}, \ldots, a_{0,1}\right)}{K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right)}\\
&\qquad= \frac{K\left(a_{n+1,1}, a_{n,1}, \ldots, a_{0,1}\right)}{K\left(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}\right)}
\end{aligned}
$$

where $\overrightarrow{b_1} = [a_{n+1,2}, a_{n,2}, \ldots, a_{1,2}]$ and $\overrightarrow{b_2} = [a_{n,2}, a_{n-1,2}, \ldots, a_{1,2}]$. $\square$

For a continued fraction $[a_{n,1}, a_{n-1,1}, \ldots, a_{0,1}]_{\overrightarrow{b}}$ with $\overrightarrow{b} = [a_{n,2}, a_{n-1,2}, \ldots, a_{1,2}]$, the convergents $\dfrac{K(a_{n,1}, a_{n-1,1}, \ldots, a_{0,1})}{K(a_{n-1,1}, a_{n-2,1}, \ldots, a_{0,1})}$ are

$$\frac{a_{0,1}}{1}, \frac{a_{0,1}a_{1,1} + a_{1,2}}{a_{0,1}}, \frac{a_{2,1}(a_{0,1}a_{1,1} + a_{1,2}) + a_{2,2}(a_{0,1})}{a_{1,1}(a_{0,1}) + a_{1,2}}, \ldots.$$

If successive convergents are found, with numerators $A_0, A_1, A_2, \ldots$ and denominators $B_0, B_1, B_2, \ldots$, then the relevant recursive relations are

$$A_0 = a_{0,1}, \;\; A_1 = a_{1,1}a_{0,1} + a_{1,2} \quad \text{and} \quad A_n = a_{n,1}A_{n-1} + a_{n,2}A_{n-2} \quad \text{for} \;\; n \geq 2,$$

and

$$B_0 = 1, \;\; B_1 = a_{0,1} \quad \text{and} \quad B_n = a_{n-1,1}B_{n-1} + a_{n-1,2}B_{n-2} \quad \text{for} \;\; n \geq 2.$$

Let us define, for given positive integer $r$,

$$\begin{aligned} K_1^{(i)} &= K(a_{i,1}, a_{i-1,1}, a_{i-2,1} \ldots, a_{1,1}, a_{0,1}, a_{r-1,1}, ..., a_{i+2,1}, a_{i+1,1}), \quad \text{and} \\ K_2^{(i)} &= K(a_{i-1,1}, a_{i-2,1}, a_{i-3,1}, \ldots, a_{1,1}, a_{0,1}, a_{r-1,1}, \ldots, a_{i+2,1}), \end{aligned}$$

where as usual $i$ is taken modulo $r$.

**Theorem 4.** *We have, for $i = 0, \ldots, r-1$,*

$$K_1^{(i)} + a_{i+1,2}K_2^{(i)} = K_1^{(i+1)} + a_{i+2,2}K_2^{(i+1)}.$$

*Proof.* By Theorem 2,

$$\begin{aligned} K_1^{(i+1)} &= K(a_{i+1,1}, a_{i,1}, \ldots, a_{i+3,1}, a_{i+2,1}) \\ &= a_{i+1,1}K(a_{i,1}, \ldots, a_{i+3,1}, a_{i+2,1}) + a_{i+1,2}K(a_{i-1,1}, a_{i-2,1}, \ldots, a_{i+2,1}). \end{aligned}$$

Since $K_2^{(i+1)} = K(a_{i,1}, a_{i-1,1}, \ldots, a_{i+4,1}, a_{i+3,1})$, we get

$$\begin{aligned} & K_1^{(i+1)} + a_{i+2,2}K_2^{(i+1)} \\ = \; & a_{i+1,1}K(a_{i,1}, \ldots, a_{i+3,1}, a_{i+2,1}) + a_{i+1,2}K(a_{i-1,1}, a_{i-2,1}, \ldots, a_{i+2,1}) \\ & + a_{i+2,2}K(a_{i,1}, a_{i-1,1}, \ldots, a_{i+4,1}, a_{i+3,1}). \end{aligned} \tag{2}$$

By the definition of the continuant,

$$\begin{aligned} K_1^{(i)} &= K(a_{i,1}, a_{i-1,1}, \ldots, a_{i+2,1}, a_{i+1,1}) \\ &= a_{i+1,1}K(a_{i,1}, a_{i-1,1}, \ldots, a_{0,1}, a_{r-1,1}, \ldots, a_{i+2,1}) \\ &\quad + a_{i+2,2}K(a_{i,1}, a_{i-1,1}, \ldots, a_{0,1}, a_{r-1,1}, \ldots, a_{i+3,1}), \end{aligned}$$

and since $K_2^{(i)} = K\left(a_{i-1,1}, a_{i-2,1}, \ldots, a_{i+3,1}, a_{i+2,1}\right)$, we get

$$\begin{aligned} & K_1^{(i)} + a_{i+1,2} K_2^{(i)} \\ = \; & a_{i+1,1} K\left(a_{i,1}, a_{i-1,1}, \ldots, a_{i+2,1}\right) + a_{i+2,2} K\left(a_{i,1}, a_{i-1,1}, \ldots, a_{i+3,1}\right) \\ & + a_{i+1,2} K\left(a_{i-1,1}, a_{i-2,1}, \ldots, a_{i+3,1}, a_{i+2,1}\right). \end{aligned} \tag{3}$$

Thus, using (2) and (3), we get $K_1^{(i)} + a_{i+1,2}K_2^{(i)} = K_1^{(i+1)} + a_{i+2,2}K_2^{(i+1)}$, for $i = 0, \ldots, r-1$. □

Let us define $K_1 = K_1^{(0)}$ and $K_2 = K_2^{(0)}$. By Theorem 4, we can write

$$K_1^{(i)} + a_{i+1,2} K_2^{(i)} = K_1 + a_{1,2} K_2, \quad \text{for } i = 0, \ldots, r-1.$$

The following theorem extends results in [2]. The author of [2] provides a combinatorial description; here, we give our results in terms of the generalized continuant.

**Theorem 5.** *For any $n \geq 2r$, the sequence $\{v_n\}$ satisfies the following $2r$-order linear recurrence*

$$v_n = \left(K_1 + a_{1,2}K_2\right) v_{n-r} + (-1)^{r+1} \left(a_{0,2}a_{1,2}a_{2,2} \ldots a_{r-1,2}\right) v_{n-2r}.$$

*Proof.* By the definition of the sequence $\{v_n\}$, we have

$$v_{mr+i} = a_{i,1} v_{mr+i-1} + a_{i,2} v_{mr+i-2} = K\left(a_{i,1}\right) v_{mr+i-1} + a_{i,2} K\left(\right) v_{mr+i-2}.$$

If we substitute $v_{mr+i-1} = a_{i-1,1}v_{mr+i-2} + a_{i-1,2}v_{mr+i-3}$ in the above equality and rearrange it, we get

$$\begin{aligned} v_{mr+i} &= (a_{i,1}a_{i-1,1} + a_{i,2}) v_{mr+i-2} + a_{i-1,2} a_{i,1} v_{mr+i-3} \\ &= K\left(a_{i,1}, a_{i-1,1}\right) v_{mr+i-2} + a_{i-1,2} K\left(a_{i,1}\right) v_{mr+i-3}. \end{aligned}$$

If we proceed in this way, we obtain

$$\begin{aligned} v_{mr+i} &= K\left(a_{i,1}\right) v_{mr+i-1} + a_{i,2} K\left(\right) v_{mr+i-2} \\ &= K\left(a_{i,1}, a_{i-1,1}\right) v_{mr+i-2} + a_{i-1,2} K\left(a_{i,1}\right) v_{mr+i-3} \\ &= K\left(a_{i,1}, a_{i-1,1}, a_{i-2,1}\right) v_{mr+i-3} + a_{i-2,2} K\left(a_{i,1}, a_{i-1,1}\right) v_{mr-4} \\ &\;\vdots \quad \vdots \\ &= K\left(a_{i,1}, a_{i-1,1}, \ldots, a_{i+2,1}, a_{i+1,1}\right) v_{mr+i-r} \\ &\quad + a_{i+1,2} K\left(a_{i,1}, a_{i-1,1}, \ldots, a_{i+2,1}\right) v_{mr+i-r-1} \\ &= K_1^{(i)} v_{mr+i-r} + a_{i+1,2} K\left(a_{i,1}, a_{i-1,1}, \ldots, a_{i+2,1}\right) v_{mr+i-r-1}. \end{aligned} \tag{4}$$

Now by the definition of the sequence, since $mr + i - 2r + 2 \equiv i + 2 \pmod{r}$, we have

$$\begin{aligned} a_{i+2,2} v_{mr+i-2r} &= v_{mr+i-2r+2} - a_{i+2,1} v_{mr+i-2r+1} \\ &= K\left(\right) v_{mr+i-2r+2} - K\left(a_{i+2,1}\right) v_{mr+i-2r+1}. \end{aligned}$$

Multiplying the above equation by $a_{i+3,2}$, substituting

$$a_{i+3,2}v_{mr+i-2r+1} = v_{mr+i-2r+3} - a_{i+3,1}v_{mr+i-2r+2}$$

in the above equality and arranging it, we obtain

$$\begin{aligned}
& a_{i+2,2}a_{i+3,2}v_{mr+i-2r} \\
= \; & a_{i+3,2}v_{mr+i-2r+2} - a_{i+2,1}a_{i+3,2}v_{mr+i-2r+1} \\
= \; & -K\left(a_{i+2,1}\right)v_{mr+i-2r+3} + \left(a_{i+2,1}a_{i+3,1} + a_{i+3,2}\right)v_{mr+i-2r+2} \\
= \; & -K\left(a_{i+2,1}\right)v_{mr+i-2r+3} + K\left(a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-2r+2}.
\end{aligned}$$

If we proceed in this way, we get

$$\begin{aligned}
a_{i+2,2}v_{mr+i-2r} &= K\left(\right)v_{mr+i-2r+2} - K\left(a_{i+2,1}\right)v_{mr+i-2r+1} \\
a_{i+2,2}a_{i+3,2}v_{mr+i-2r} &= -K\left(a_{i+2,1}\right)v_{mr+i-2r+3} \\
&\quad +K\left(a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-2r+2} \\
a_{i+2,2}a_{i+3,2}a_{i+4,2}v_{mr+i-2r} &= K\left(a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-2r+4} \\
&\quad -K\left(a_{i+4,1}, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-2r+3}
\end{aligned}$$

and so on up to

$$\begin{aligned}
& a_{i+2,2}a_{i+3,2}\dots a_{i+r,2}v_{mr+i-2r} \\
= \; & (-1)^r K\left(a_{i-1,1}, a_{i-2,1}, \dots, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-r} \\
& + (-1)^{r+1} K\left(a_{i,1}, a_{i-1,1}, \dots, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-r-1}.
\end{aligned}$$

Multiplying the last equation by $(-1)^r a_{i+1,2}$, reducing the first subscript of the coefficients $a_{i,j}$ modulo $r$ and rearranging the expression, we get

$$\begin{aligned}
& (-1)^r a_{0,2}a_{1,2}a_{2,2}\dots a_{r-1,2}v_{mr+i-2r} \qquad (5)\\
= \; & a_{i+1,2}K\left(a_{i-1,1}, a_{i-2,1}, \dots, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-r} \\
& -a_{i+1,2}K\left(a_{i,1}, a_{i-1,1}, \dots, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-r-1} \\
= \; & a_{i+1,2}K_2^{(i)}v_{mr+i-r} \\
& -a_{i+1,2}K\left(a_{i,1}, a_{i-1,1}, \dots, a_{i+3,1}, a_{i+2,1}\right)v_{mr+i-r-1}. \qquad (6)
\end{aligned}$$

Let $n = mr + i$; summing (4) and (6), we get

$$v_n = \left(K_1^{(i)} + a_{1,2}K_2^{(i)}\right)v_{n-r} + (-1)^{r+1}\left(a_{0,2}a_{1,2}a_{2,2}\dots a_{r-1,2}\right)v_{n-2r}.$$

By Theorem 4, we get the result. □

## 3. The Generating Function of the Sequence $\{v_n\}$

The generating function for $\{v_n\}$ is

$$G(x) = \sum_{m=0}^{\infty} v_m x^m.$$

If we define

$$G_i(x) = \sum_{m=0}^{\infty} v_{mr+i} x^{mr+i}, \quad i = 0, 1, \ldots, r-1,$$

then we can write

$$G(x) = \sum_{i=0}^{r-1} G_i(x). \tag{7}$$

**Theorem 6.** *The generating function of the sequence* $\{v_n\}$ *is*

$$G(x) = \frac{\sum_{i=0}^{r-1} \left(v_i x^i + (v_{r+i} - Av_i) x^{r+i}\right)}{1 - Ax^r + (-1)^r Bx^{2r}},$$

*where*

$$A = K_1 + a_{1,2}K_2 \quad \text{and} \quad B = \prod_{i=0}^{r-1} a_{i,2}.$$

*Proof.* Multiplying $G_i(x)$ by $1 - Ax^r + (-1)^r Bx^{2r}$ for $i = 0, \ldots, r-1$ and using Theorem 5 and Theorem 25.5.1 in [1], we get for $i = 0, 1, \ldots, r-1$,

$$G_i(x) = \frac{v_i x^i + (v_{r+i} - Av_i) x^{r+i}}{1 - Ax^r + (-1)^r Bx^{2r}}. \tag{8}$$

Hence, using (7) we get the desired result. □

**Example 7.** For $r = 3$, the sequence $\{v_n\}$ satisfies $v_0 = 0, v_1 = 1$ and for $n \geq 2$

$$v_n = \begin{cases} a_{0,1}v_{n-1} + a_{0,2}v_{n-2}, & \text{if } n \equiv 0 \pmod 3, \\ a_{1,1}v_{n-1} + a_{1,2}v_{n-2}, & \text{if } n \equiv 1 \pmod 3, \\ a_{2,1}v_{n-1} + a_{2,2}v_{n-2}, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Let us find the generating function of the sequence $\{v_n\}$ for the given integer $r = 3$. Since $r = 3$, using Definition 1 we have

$$\begin{aligned} K_1 &= K(a_{0,1}, a_{2,1}, a_{1,1}) = a_{1,1}K(a_{0,1}, a_{2,1}) + a_{2,2}K(a_{0,1}) \\ &= a_{1,1}(a_{2,1}K(a_{0,1}) + a_{0,2}K()) + a_{2,2}K(a_{0,1}) \\ &= a_{0,1}a_{1,1}a_{2,1} + a_{0,2}a_{1,1} + a_{0,1}a_{2,2}, \end{aligned}$$

and

$$K_2 = K(a_{2,1}) = a_{2,1}.$$

Thus, we have

$$A = K_1 + a_{1,2}K_2 = a_{0,1}a_{1,1}a_{2,1} + a_{0,2}a_{1,1} + a_{0,1}a_{2,2} + a_{1,2}a_{2,1}.$$

By Theorem 6, we get

$$G(x) = \frac{\sum_{i=0}^{2} \left(v_i x^i + (v_{3+i} - Av_i) x^{3+i}\right)}{1 - Ax^3 + (-1)^3 Bx^6}.$$

Since $v_0 = 0$, $v_1 = 1$, $v_2 = a_{2,1}$, $v_3 = a_{0,1}a_{2,1} + a_{0,2}$, $v_4 = a_{0,1}a_{1,1}a_{2,1} + a_{0,2}a_{1,1} + a_{1,2}a_{2,1}$ and $v_5 = a_{0,1}a_{1,1}a_{2,1}^2 + a_{0,2}a_{1,1}a_{2,1} + a_{1,2}a_{2,1}^2 + a_{0,1}a_{2,1}a_{2,2} + a_{0,2}a_{2,2}$, we finally obtain

$$G(x) = \frac{x + a_{2,1}x^2 + (a_{0,1}a_{2,1} + a_{0,2})\, x^3 - a_{0,1}a_{2,2}x^4 + a_{0,2}a_{2,2}x^5}{1 - (a_{0,1}a_{1,1}a_{2,1} + a_{0,2}a_{1,1} + a_{0,1}a_{2,2} + a_{1,2}a_{2,1})\, x^3 - a_{0,2}a_{1,2}a_{2,2}x^6}.$$

## 4. The Extended Binet's Formula of the Sequence $\{v_n\}$

By Theorem 5, we can consider the conditional sequence $\{v_n\}$ as a constant coefficient $2r$-order linear recurrence for any positive integer $s$. So, we can find the matrix representation of the sequence $\{v_n\}$ and we can use methods for linear recurrences to obtain the extended Binet's formula for the conditional sequence $\{v_n\}$.

Let us define the $2r \times 2r$ matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ (-1)^r \prod_{i=0}^{r-1} a_{i,2} & 0 & 0 & \dots & (K_1 + a_{1,2}K_2) & \dots & 0 & 0 \end{bmatrix},$$

where $K_1 + a_{1,2}K_2$ is the entry in the $(2r)$-th row and $(r+1)$-th column. In fact, $N$ is the companion matrix for the polynomial

$$q(x) = x^{2r} - (K_1 + a_{1,2}K_2)\, x^r - (-1)^r \prod_{i=0}^{r-1} a_{i,2}.$$

The polynomial $q(x)$ is the characteristic polynomial of $N$. By using an inductive argument, we can give the matrix representation of the conditional sequence $\{v_n\}$:

$$N^n \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{2r-1} \end{bmatrix} = \begin{bmatrix} v_n \\ v_{n+1} \\ \vdots \\ v_{n+2r-1} \end{bmatrix}. \tag{9}$$

This matrix representation is important since it may be used to derive many interesting properties of the conditional sequence $\{v_n\}$.

Let $\mu_1, \mu_2, \dots, \mu_{2r}$ be the eigenvalues of the matrix $N$. Let us assume that the polynomial $q(x)$ has no multiple roots, so these eigenvalues are all distinct. In this case, $N$ can be diagonalized using the Vandermonde matrix

$$V(\mu_1, \mu_2 \ldots, \mu_{2r}) = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ \mu_1 & \mu_2 & \ldots & \mu_{2r-1} & \mu_{2r} \\ \mu_1^2 & \mu_2^2 & \ldots & \mu_{2r-1}^2 & \mu_{2r}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mu_1^{2r-1} & \mu_2^{2r-1} & \ldots & \mu_{2r-1}^{2r-1} & \mu_{2r}^{2r-1} \end{bmatrix}.$$

We can give the following theorem using the matrix methods in [9].

**Theorem 8.** *Assume $q(x)$ has roots $\mu_1, \mu_2, \ldots, \mu_{2r}$ and also assume the polynomial $q(x)$ has no multiple roots. Then, the Binet-like formula for the conditional sequence $\{u_n\}$ is $v_n = \sum_{i=1}^{2r} \frac{\mu_i^n}{q'(\mu_i)}$, where $q'$is the derivative of $q$.*

We can also give the following general extended Binet's formula for the sequence $\{v_n\}$ independent of whether $q(x)$ has multiple roots or not.

**Theorem 9.** *The terms of the sequence $\{v_n\}$ satisfy*

$$v_{mr+i} = (-1)^{r(m+1)} \left[ \frac{\alpha^m - \beta^m}{\alpha - \beta} v_{r+i} - B \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} v_i \right],$$

*where*

$$\alpha = \frac{(-1)^r A + \sqrt{A^2 - 4(-1)^r B}}{2} \quad \text{and} \quad \beta = \frac{(-1)^r A - \sqrt{A^2 - 4(-1)^r B}}{2},$$

*that is, $\alpha$ and $\beta$ are the roots of the polynomial $p(z) = z^2 - (-1)^r Az + (-1)^r B$, where $A = K_1 + a_{1,2}K_2$ and $B = \prod_{i=0}^{r-1} a_{i,2}$.*

*Proof.* By (8), the generating function for the subsequence $\{v_{mr+i}\}$ is given by

$$G_i(x) = \frac{v_i x^i + (v_{r+i} - Av_i) x^{r+i}}{1 - Ax^r + (-1)^r Bx^{2r}}.$$

Using the identities $\alpha + \beta = (-1)^r A$, $\alpha - \beta = \sqrt{A^2 - 4(-1)^r B}$, $\alpha\beta = (-1)^r B$, $(-1)^r A\alpha^m - \alpha^{m+1} = \beta\alpha^m$ and $(-1)^r A\beta^m - \beta^{m+1} = \alpha\beta^m$, we get

$$\begin{aligned} G_i(x) &= \frac{v_i x^i + (v_{r+i} - Av_i) x^{r+i}}{1 - Ax^r + (-1)^r Bx^{2r}} \\ &= \frac{x^i [v_i + (v_{r+i} - Av_i) x^r]}{\alpha - \beta} \left[ \frac{\alpha}{1 - (-1)^r \alpha x^r} - \frac{\beta}{1 - (-1)^r \beta x^r} \right] \end{aligned}$$

$$
\begin{aligned}
&= x^i \left[v_i + \left(v_{r+i} - Av_i\right) x^r\right] \sum_{m=0}^{\infty} \frac{(-1)^{mr}\left(\alpha^{m+1} - \beta^{m+1}\right) x^{mr}}{\alpha - \beta} \\
&= x^i \left[\sum_{m=0}^{\infty} \frac{(-1)^{mr}\left(\alpha^{m+1} - \beta^{m+1}\right) v_i}{\alpha - \beta} x^{mr}\right. \\
&\qquad \left. + \sum_{m=0}^{\infty} \frac{(-1)^{mr}\left(\alpha^{m+1} - \beta^{m+1}\right)\left(v_{r+i} - Av_i\right)}{\alpha - \beta} x^{mr+r}\right] \\
&= x^i \left[v_i + \sum_{m=1}^{\infty} \frac{(-1)^{mr}\left(\alpha^{m+1} - \beta^{m+1}\right) v_i}{\alpha - \beta} x^{mr}\right. \\
&\qquad \left. + (-1)^r \sum_{m=1}^{\infty} \frac{(-1)^{mr}\left(\alpha^{m} - \beta^{m}\right)\left(v_{r+i} - Av_i\right)}{\alpha - \beta} x^{mr}\right] \\
&= x^i v_i + \sum_{m=1}^{\infty} (-1)^{mr} \left[(-1)^r \frac{\left(\alpha^m - \beta^m\right)}{\alpha - \beta} v_{r+i}\right. \\
&\qquad \left. - \frac{\left((-1)^r A\alpha^m - \alpha^{m+1}\right) - \left((-1)^r A\beta^m - \beta^{m+1}\right)}{\alpha - \beta} v_i\right] x^{mr+i} \\
&= \sum_{m=0}^{\infty} (-1)^{mr} \left[(-1)^r \frac{\left(\alpha^m - \beta^m\right)}{\alpha - \beta} v_{r+i} - \frac{\beta\alpha^m - \alpha\beta^m}{\alpha - \beta} v_i\right] x^{mr+i} \\
&= \sum_{m=0}^{\infty} (-1)^{mr+r} \left[\frac{\left(\alpha^m - \beta^m\right)}{\alpha - \beta} v_{r+i} - B\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} v_i\right] x^{mr+i}.
\end{aligned}
$$

Thus, we obtain

$$
v_{mr+i} = (-1)^{r(m+1)} \left[\frac{\alpha^m - \beta^m}{\alpha - \beta} v_{r+i} - B\frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} v_i\right].
$$

□

**Corollary 10.** *If we take the values $a_{0,2} = a_{1,2} = \cdots = a_{r-1,2} = 1$ in the sequence $\{v_n\}$ we get a special case the $k$-*periodic Fibonacci sequence *in [3]. Also, substituting $B = 1$ in Theorem 6 and 9, we get Theorem 13 and Theorem 16 in [3], respectively.*

## 5. Conclusion

In this paper we consider the Fibonacci-like conditional sequences $\{v_n\}$ as given in Equation (1). We define the concept of generalized continuant and use it to show that $\{v_n\}$ satisfies a linear recurrence relation. We also derive generating functions, matrix representations and extended Binet's formulas for $\{v_n\}$ in terms of the generalized continuant. It would be interesting to find more useful applications of the generalized continuant and generalized fractions.

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