Turkish Journal of Analysis and Number Theory, 2013, Vol. 1, No. 1, 59-62
Available online at http://pubs.sciepub.com/tjant/1/1/12

DOI:10.12691/tjant-1-1-12

The Hyper-Geometric Daehee Numbers and Polynomials

Jin-Woo Park[1], Seog-Hoon Rim[1], Jongkyum Kwon[2,*]

[1]Department of Mathematics Education, Kyungpook National University, Taegu, Republic of Korea
[2]Department of Mathematics, Kyungpook National University, Taegu, Republic of Korea
*Corresponding author: mathkjk26@hanmail.net

Received November 07, 2013; Revised November 20, 2013; Accepted December 04, 2013

Abstract We consider the hyper-geometric Daehee numbrers and polynomials and investigate some properties of those numbers and polynomials.

Keywords: *Daehee numbers, Hyper-geometric Daehee numbers and polynomials*

Cite This Article: Jin-Woo Park, Seog-Hoon Rim, and Jongkyum Kwon, "The Hyper-Geometric Daehee Numbers and Polynomials." *Turkish Journal of Analysis and Number Theory* 1, no. 1 (2013): 59-62. doi: 10.12691/tjant-1-1-12.

1. Introduction

As is known, the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!},\qquad(1.1)$$

(see [5,6,7,9,10,11,12]).

In the special case, $x=0, D_n=D_n(0)$ are called the Daehee numbers.

Let $\mathbb{Z}_p,\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the rings of p-adic integers, the fields of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p=\frac{1}{p}$. Let $(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f\in UD(\mathbb{Z}_p)$, the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I(f)=\int_{\mathbb{Z}_p}f(x)d\mu_0(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x),\quad(1.2)$$

(see [7,8]).

Let f_1 be the translation of f with $f_1(x)=f(x+1)$. Then, by (1.2), we get

$$I(f_1)=I(f)+f'(0), where\ f'(0)=\frac{df(x)}{dx}|_{x=0}.\quad(1.3)$$

As is known, the Stirling number of the first kind is defined by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,\qquad(1.4)$$

and the Stirling number of the second kind is given by the generating function to be

$$\left(e^t-1\right)^m=m!\sum_{l=m}^{\infty}S_2(l,m)\frac{t^l}{l!},\qquad(1.5)$$

(see [2,3,4]) .

For $\alpha\in\mathbb{N}$, the Bernoulli polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!},\qquad(1.6)$$

(see [1,2,9]).

When $x=0, B_n^{(\alpha)}=B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order α.

A hyper-geometric series $\sum_k c_K$ is a series for which $c_0=1$ and the ratio of consecutive terms is a rational function of the summation index k, i.e., one for which

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)},$$

with $P(k)$ and $Q(k)$ polynomials. In this case, c_k is called a hyper-geometric term. The functions generated by hyper-geometric series are called generalized hyper-geometric functions. If the polynomials are completely factored, the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_p)(k+1)}\quad(1.7)$$

(see [13]),

where the factor of $k+1$ in the denominator is present for historical reasons of notation, and the resulting generalized hyper-geometric function is written

$$_pF_q\left[\begin{matrix}a_1 & a_2 & \cdots & a_p\\ b_1 & b_2 & \cdots & b_q\end{matrix}\middle|x\right]=\sum_{k=0}c_kx^k\qquad(1.8)$$

(see [13]).

If $p=2$ and $q=1$, the function becomes a traditional hyper-geometric function ${}_2F_1(a,b;c;x)$. Many sums can be written as generalized hyper-geometric functions by inspections of the ratios of consecutive terms in the generating hyper-geometric series.

We introduce the hyper-geometric Daehee numbers and polynomials. From our definition, we can derive some interesting properties related to the hyper-geometric Daehee numbers and polynomials.

2. The Hyper-Geometric Daehee Numbers and Polynomials

First, we consider the following integral representation associated with falling factorial sequences :

$$\int_{\mathbb{Z}_p}(x)_n\,d\mu_0(x),\ \textit{where } n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}. \tag{2.1}$$

By (2.1), we get

$$\begin{aligned}&\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x)_n\,d\mu_0(x)\frac{t^n}{n!}\\&=\int_{\mathbb{Z}_p}\sum_{n=0}^{\infty}\binom{x}{n}t^n\,d\mu_0(x)\\&=\int_{\mathbb{Z}_p}(1+t)^x\,d\mu_0(x),\end{aligned}\tag{2.2}$$

(see [6]), where $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$.

For $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$, let us take $f(x)=(1+t)^x$. Then, from (1.3), we have

$$\int_{\mathbb{Z}_p}(1+t)^x d\mu_0(x)=\frac{\log(1+t)}{t}. \tag{2.3}$$

By (1.1) and (2.3), we see that

$$\begin{aligned}\sum_{n=0}^{\infty}D_n\frac{t^n}{n!}&=\frac{\log(1+t)}{t}\\&=\int_{\mathbb{Z}_p}(1+t)^x d\mu_0(x)\\&=\sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}(x)_n\,d\mu_0(x)\frac{t^n}{n!}.\end{aligned}\tag{2.4}$$

(see [6]).

Therefore, by (2.4), we obtain the following Lemma.

Lemma 1. *For $n\ge 0$, we have*

$$\int_{\mathbb{Z}_p}(x)_n d\mu_0(x)=D_n.$$

For $n\in\mathbb{Z}$, it is known that

$$\left(\frac{t}{\log(1+t)}\right)^n(1+t)^{x-1}=\sum_{k=0}^{\infty}B_k^{(k-n+1)}(x)\frac{t^k}{k!}, \tag{2.5}$$

(see [4,5,6]).

Thus, by (2.5), we get

$$D_k=\int_{\mathbb{Z}_p}(x)_k d\mu_0(x)=B_k^{(k+2)}(1),(k\ge 0), \tag{2.6}$$

where $B_k^{(n)}(x)$ are the Bernoulli polynomials of order n.

In the special case, $x=0, B_k^{(n)}=B_k^{(n)}(0)$ are called the n-th Bernoulli numbers of order n.

From (2.4), we note that

$$\begin{aligned}(1+t)^x\int_{\mathbb{Z}_p}(1+t)^y d\mu_0(y)&=\left(\frac{\log(1+t)}{t}\right)(1+t)^x\\&=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

(see [6]).

Thus, by (2.7), we get

$$\int_{\mathbb{Z}_p}(x+y)_n d\mu_0(y)=D_n(x),(n\ge 0), \tag{2.8}$$

and, from (2.5), we have

$$D_n(x)=B_n^{(n+2)}(x+1). \tag{2.9}$$

(see [6]).

Therefore, by (2.8) and (2.9), we obtain the following Lemma.

Lemma 2. *For $n\ge 0$, we have*

$$D_n(x)=\int_{\mathbb{Z}_p}(x+y)_n d\mu_0(y),$$

and

$$D_n(x)=B_n^{(n+2)}(x+1).$$

By Lemma 1, we easily see that

$$D_n=\sum_{l=0}^{n}S_1(n,l)B_l, \tag{2.10}$$

(see [6]), where B_l are the ordinary Bernoulli numbers.

From Lemma 2, we have

$$\begin{aligned}D_n(x)&=\int_{\mathbb{Z}_p}(x+y)_n d\mu_0(y)\\&=\sum_{l=0}^{n}S_1(n,l)B_l(x),\end{aligned}\tag{2.11}$$

(see [6]), where $B_l(x)$ are the Bernoulli polynomials defined by generating function to be

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

Therefore, by (2.10) and (2.11), we obtain the following corollary.

Corollary 3. *For $n\ge 0$, we have*

$$D_n(x)=\sum_{l=0}^{n}S_1(n,l)B_l(x).$$

In (2.4), we have

$$\begin{aligned}\frac{t}{e^t-1} &= \sum_{n=0}^{\infty} D_n \frac{1}{n!}\left(e^t-1\right)^n \\ &= \sum_{n=0}^{\infty} D_n \frac{1}{n!} n! \sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \qquad (2.12)\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} D_n S_2(m,n)\right)\frac{t^m}{m!}\end{aligned}$$

and

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}. \qquad (2.13)$$

(see [6]).

Therefore, by (2.12) and (2.13), we obtain the following Lemma.

Lemma 4. *For* $n \geq 0$, *we have*

$$B_m = \sum_{n=0}^{m} D_n S_2(m,n).$$

In particular,

$$\int_{\mathbb{Z}_p} x^m d\mu_0(x) = \sum_{n=0}^{m} D_n S_2(m,n).$$

Remark. For $m \geq 0$, by (2.11), we have

$$\int_{\mathbb{Z}_p} (x+y)^m d\mu_0(y) = \sum_{n=0}^{m} D_n(x) S_2(m,n).$$

(see [6]).

Now, we define the hyper-geometric Daehee polynomials

$$F\left(\begin{matrix} a & b \\ & c\end{matrix}\middle| x\right) = \sum_{n=0}^{\infty} \frac{\binom{-a}{n}\binom{-b}{n}}{\binom{-c}{n}}(-x)^n, \qquad (2.14)$$

where $\binom{x}{n} = \frac{x(x+1)\cdots(x+n-1)}{n!}$.

For example, we have

$$\begin{aligned}F\left(\begin{matrix} 1 & 1 \\ & 2\end{matrix}\middle| x\right) &= \sum_{n=0}^{\infty} \frac{\binom{-1}{n}\binom{-1}{n}}{\binom{-2}{n}}(-x)^n \\ &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1} \qquad (2.15)\\ &= \frac{1}{x}\log(1+x) \\ &= \sum_{n=0}^{\infty} D_n \frac{x^n}{n!}.\end{aligned}$$

Thus the hyper-geometric Daehee number are defined by

$$2F\left(\begin{matrix} 1 & N \\ & N+1\end{matrix}\middle| -t\right) = \sum_{n=0}^{\infty} D_{N,n}\frac{t^n}{n!}. \qquad (2.16)$$

Note that $D_{1,N} = D_n$ is the Daehee number.

$$F\left(\begin{matrix} 1 & N \\ & N+1\end{matrix}\middle| -x\right) = \sum_{n=0}^{\infty} \frac{(1)_\infty (N)_n}{(N+1)_n}\frac{(-x)^n}{n!}, \qquad (2.17)$$

where $(a)_n = a(n+1)\cdots(a_n-1)$.

$$\begin{aligned}&\sum_{n=0}^{\infty} D_{1,N}\frac{(-x)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{n!N(N+1)\cdots(N+n-1)(-x)^n}{(N+1)\cdots(N+n)n!} \\ &= \frac{N!}{(N-1)!}\sum_{n=0}^{\infty}\frac{(N+n-1)!}{(N+n)!}(-x)^n \qquad (2.18)\\ &= \frac{N!}{(N-1)!}\sum_{n=0}^{\infty}\frac{1}{(N+n)}(-x)^n.\end{aligned}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 5. *For* $n \geq 0$, *we have*

$$\frac{D_{1,N}}{n!} = \frac{N}{N+n}(-1)^n.$$

In (2.17), we have

$$\begin{aligned}&N(-1)^{N-1}\sum_{n=N}^{\infty}\frac{(-1)^{n-1}}{n}x^{n-N} \\ &= \frac{(-1)^{N-1}N}{x^N}\sum_{n=N}^{\infty}\frac{(-1)^{n-1}}{n}x^n \\ &= \frac{(-1)^{N-1}N}{x^N}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n - \sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n}x^n\right\} \qquad (2.19)\\ &= (-1)^{N-1}\frac{N}{x^N}\left\{\log(1+x) - \sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n}x^n\right\}\end{aligned}$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 6. *For* $n \geq 0$, *we have*

$$\begin{aligned}&(-1)^{N-1}\frac{x^N}{N}F\left(\begin{matrix} 1 & N \\ & N+1\end{matrix}\middle| -x\right) \\ &= \log(1+x) - \sum_{n=1}^{N-1}\frac{(-1)^{n-1}}{n}x^n.\end{aligned}$$

References

[1] S. Araci, M. Acikgoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math. 22 (2012), no. 3, 399-406.

[2] A. Bayad, *Special values of Lerch zeta function and their Fourier expansions*, Adv. Stud. Contemp. Math. 21 (2011), no. 1, 1-4.

[3] L. Carlitz, *A note on Bernoulli and Euler polynomials of the second kind*, Scripta Math. 25 (1961), 323-330.

[4] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[5] H. W. Gould, *Explicit formulas for Bernoulli numbers*, Amer. Math. Monthly 79(1972), 44-51.

[6] D. S. Kim,T. Kim, *Daehee Numbers and Polynomials*, Applied Mathematical Sciences, Vol. 7, 2013, no. 120, 5969-5976

[7] T. Kim, D. S. Kim, T. Mansour, S. H. Rim, M. Schork, *Umbral calculus and Sheffer sequence of polynomials*, J. Math. Phys. 54, 083504 (2013) : http://dx.doi.org/10.1063/1.4817853 (15 pages).

[8] T. Kim, *An invariant p-adic integral associated with Daehee numbers*, Integral Transforms Spec. Funct. 13 (2002), no. 1, 65-69.

[9] V. Kurt, *Some relation between the Bernstein polynomials and second kind Bernoulli polynomials,* Adv. Stud. Contemp. Math. 23 (2013), no. 1, 43-48.

[10] H. Ozden, I. N. Cangul, Y. Simsek, *Remarks on q-Bernoulli numbers associated with Daehee numbers*, Adv. Stud. Contemp. Math. 18 (2009), no. 1, 41-48.

[11] Y. Simsek, S-H. Rim, L. -C. Jang, D.-J. Kang, J.-J. Seo, *A note q-Daehee sums*, Proceedings of the 16th International Conference of the Jangjeon Mathematical Society, 159-166, Jangjeon Math. Soc., Hapcheon, 2005.

[12] Y. Simsek, Special functions related to Dedekind-type DC-sums and their applications, Russ. J. Math. Phys. 17 (2010), no. 4, 495-508.

[13] Koepf. W, Hypergeometric Summation. An Algorithmic Approach to Summation and Special Function Identities, Braunschweig, Germany, Vieweg, 1998.