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NEW OPERATIONAL FORMULAS AND GENERATING FUNCTIONS FOR LAGUERRE POLYNOMIALS

by K. R. PATIL*, *Department of Mathematics, A.S.C. College, Ramanandnagar, Dist. Sangli (Maharashtra)*

and

N. K. THAKARE,† *Department of Mathematics, Shivaji University, Kolhapur 416004*

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In this paper we have obtained various operational formulas for the Laguerre polynomials in the generalized form. The results are obtained by introducing differential operator

$$\theta = \lambda x^k + x^{k+1}\frac{d}{dx}\cdot$$

For $\lambda = k = 1$ we get the operator considered by Al-Salam (1964); for $\lambda = 0$ we get the operator discussed by Chak (1956); and for $\lambda = 0$, $k = k - 1$ the operator of Shrivastava (1974) is obtained.
In the sequel various properties of this operator, as they are needed in establishing generating functions for the Laguerre polynomials, are obtained. Bilinear generating functions for the Laguerre polynomials are also obtained.

Our approach also results in obtaining a generalization of Hardy-Hille formula as well as Weisner's formula for the Laguerre polynomials.

1. INTRODUCTION

Operational formulas are being used in special function theory to obtain new results or to give short proofs of known formulas, Burchnall (1941, 1951) asserted that she could find a generating function for the generalized Bessel polynomials only on account of the operational formulas. Al-Salam (1964), Gould and Hopper (1962), Carlitz (1960), Singh (1965), Chatterjea (1963 *a, b*; 1964), Das (1967) and Thakare and Karande (1973) gave operational formulas for the classical orthogonal polynomials. Chatterjea (1966, 1968) and Karande and Thakare (1975) also gave in the unified form the operational formulas for the classical orthogonal polynomials, namely Jacobi polynomials, Laguerre polynomials and Hermite polynomials and also for the Bessel polynomials. In their investigation all these authors use the operators $D \equiv d/dx$ and/or $\delta = xD$.

Present address : * Science College, Satara, (Maharashtra)
† Department of Mathematics and Statistics, Marathwada University, Aurangabad 431 004.

Al-Salam (1964) investigated operational representations for the Laguerre and other polynomials by considering the operator $\theta = x(1 + xD)$. Very recently Srivastava and Singhal (1971) considered a class of polynomials defined by a generalized Rodrigues formulas in which they used the differential operator $\theta = x^{k+1}D$, following Chak (1956). An operator $\theta = x^k D$ was also used to study a generalized function by Shrivastava (1974).

In this paper we introduce a differential operator

$$\theta = \lambda x^k + x^{k+1}D. \tag{1.1}$$

It may be remarked that for $\lambda = k = 1$, we get the differential operator considered by Al-Salam (1964); for $\lambda = 0$ that of Chak (1956).

The introduction of this operator leads us to generalization of many well-known results for the Laguerre polynomials. In the course of our investigation we obtain new operational formulas for the Laguerre polynomials, a new generating function, a bilateral generating function for the Laguerre polynomials. It incidentally also yields a different generalization of Hardy-Hille formula; such generalizations have also been given by Srivastava and Singhal (1971) and Carlitz (1971).

2. General Properties

Let $F(x)$ be a function which has a Taylor series expansion; then we have the following formal shift rules for θ given by (1.1):

$$F(\theta)\{x^\alpha f(x)\} = x^\alpha F(\theta + \alpha x^k) f(x) \quad ...(2.1)$$

$$F(\theta)\{e^{g(x)} f(x)\} = e^{g(x)} F(\theta + g'(x)\, x^{k+1}) f(x). \quad ...(2.2)$$

The proofs of these results follow by using the method of induction.

Again by induction we can prove that,

$$\theta^n = x^{kn} \prod_{j=1}^{n} \{xD + \lambda + (j-1)k\} \quad ...(2.3)$$

$$\theta^n \{x^{\alpha+s}\} = k^n \left(\frac{\alpha + \lambda + s}{k}\right)_n x^{\alpha+s+nk} \quad ...(2.4)$$

where 's' is an integer, 'n' a non-negative integer and α is arbitrary.

As a consequence of our approach we can also obtain a Leibnitz formula for this operator which runs as

$$\theta^n \{x^\lambda uv\} = x^\lambda \sum_{r=0}^{n} \binom{n}{r} \theta^{n-r} v\, \theta^r u. \quad ...(2.5)$$

More generally (2.5) also implies,

$$e^{t\theta} \{x^\lambda uv\} = x^\lambda (e^{t\theta} v)(e^{t\theta} u). \quad ...(2.6)$$

By using (2.4) we can show,

$$e^{t\theta}\{x^{\alpha+s}\} = \frac{x^{\alpha+s}}{(1-tkx^k)^{(\alpha+\lambda+s)/k}}. \qquad \ldots(2.7)$$

As a continuation of this we have the general identity

$$e^{t\theta}(x^\alpha f(x)) = \frac{x^\alpha}{(1-tkx^k)^{(\alpha+\lambda)/k}} f\left(\frac{x}{(1-tkx^k)^{1/k}}\right). \qquad \ldots(2.8)$$

In particular, we can write

$$e^{t\theta}(f(x)) = (1-tkx^k)^{-\lambda/k} f\{x(1-tkx^k)^{-1/k}\}. \qquad \ldots(2.9)$$

The formula (2.4) also gives the general identity in terms of the generalized hypergeometric function which runs as

$${}_pF_q\left[\begin{matrix}(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\; t\theta\right] x^r = x^r\, {}_{p+1}F_q\left[\begin{matrix}(\alpha_p),\ \dfrac{r+\lambda}{k}\,; \\ (\beta_q)\,;\end{matrix}\; tkx^k\right]. \qquad \ldots(2.10)$$

In a similar manner, we can also obtain

$${}_pF_q\left[\begin{matrix}(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\; t\theta\right] x^r e^{-\mu x^r} = x^\alpha \sum_{s=0}^{\infty} \frac{(-\mu)^s}{s!} x^{rs}$$

$$\times\, {}_{p+1}F_q\left[\begin{matrix}(\alpha_p),\ \dfrac{\alpha+\lambda+rs}{k}\,; \\ (\beta_q)\,;\end{matrix}\; tkx^k\right]. \qquad \ldots(2.11)$$

If we put $\lambda = 0$, $\mu = -\mu$ we obtain the result due to Srivastava and Singhal (1971),

$${}_pF_q\left[\begin{matrix}(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\; t\theta\right] x^\alpha e^{\mu x^r}$$

$$= x^\alpha \sum_{s=0}^{\infty} \frac{(\mu)^s}{s!} x^{rs}\, {}_{p+1}F_q\left[\begin{matrix}(\alpha_p),\ \dfrac{\alpha+rs}{k}\,; \\ (\beta_q)\,;\end{matrix}\; tkx^k\right]. \qquad \ldots(2.12)$$

The use of (2.4) could be made to write

$$\frac{1}{\theta^r}\{x^{-\alpha}\} = \frac{(-1)^r}{k^r\left(\dfrac{\alpha-\lambda+k}{k}\right)_r} x^{-\alpha-rk} \qquad \ldots(2.13)$$

where $1/\theta$ is the inverse of the operator θ; here we have used the well-known result,

$$\frac{\Gamma(1-\alpha-n)}{\Gamma(1-\alpha)} = \frac{(-1)^n}{(\alpha)_n}.$$

Now let us define the operator

$$\phi = \lambda y^k + y^{k+1}\frac{d}{dy} \qquad \dots(2.14)$$

and thus by using (2.4) and (2.13) we obtain

$$\left(\frac{\phi}{\theta}\right)^r \left\{\frac{y^\beta}{x^{\alpha+1}}\right\} = \frac{(-1)^r \left(\frac{\beta+\lambda}{k}\right)_r}{\left(\frac{\alpha+k-\lambda+1}{k}\right)_r} \frac{y^{\beta+rk}}{x^{\alpha+rk+1}} . \qquad \dots(2.15)$$

Thus we have

$$\begin{aligned} &{}_pF_q\left[\begin{matrix}(\alpha_p)\,; \\ (\beta_q)\,;\end{matrix}\; t\frac{\phi}{\theta}\right]\left\{\frac{y^\beta}{x^{\alpha+1}}\right\} \\ &= \frac{y^\beta}{x^{\alpha+1}}\,{}_{p+1}F_{q+1}\left[\begin{matrix}(\alpha_p), \dfrac{\beta+\lambda}{k}\,; \\ (\beta_q), \dfrac{\alpha+k-\lambda+1}{k}\,;\end{matrix}\; -\frac{ty^k}{x^k}\right] . \end{aligned} \qquad \dots(2.16)$$

We have, in particular, for $p = 1$, $q = 0$ the results,

$$\begin{aligned} &\left(1-t\frac{\phi}{\theta}\right)^{-c}\left\{\frac{y^\beta}{x^{\alpha+1}}\right\} \\ &= \frac{y^\beta}{x^{\alpha+1}}\,{}_2F_1\left[\begin{matrix}c, \dfrac{\beta+\lambda}{k}\,; \\ \dfrac{\alpha+k-\lambda+1}{k}\,;\end{matrix}\; -\frac{ty^k}{x^k}\right] . \end{aligned} \qquad \dots(2.17)$$

$$\begin{aligned} &\left(1-t\frac{\phi}{\theta}\right)^{n}\left\{\frac{y^\beta}{x^{\alpha+1}}\right\} \\ &= \frac{y^\beta}{x^{\alpha+1}}\,{}_2F_1\left[\begin{matrix}-n, \dfrac{\beta+\lambda}{k}\,; \\ \dfrac{\alpha+k-\lambda+1}{k}\,;\end{matrix}\; -\frac{ty^k}{x^k}\right] . \end{aligned} \qquad \dots(2.18)$$

$$\left(1-t\frac{\phi}{\theta}\right)^{\frac{-\alpha-k+\lambda-1}{k}}\left\{\frac{y^\beta}{x^{\alpha+1}}\right\} = \frac{y^\beta}{x^{\alpha+1}}\left(1+\frac{ty^k}{x^k}\right)^{-\frac{\beta+\lambda}{k}} . \qquad \dots(2.19)$$

If $F(y)$ is a function which has a Taylor series expansion, the formula (2.19) gives

$$\begin{aligned} &\left(1-t\frac{\phi}{\theta}\right)^{\frac{-\alpha-k+\lambda-1}{k}}\{F(y)\,x^{-\alpha-1}\} \\ &= x^{-\alpha-1}\left(1+\frac{ty^k}{x^k}\right)^{-\frac{\lambda}{k}} F\left\{\left(\frac{x^k y^k}{x^k+ty^k}\right)^{\frac{1}{k}}\right\} . \end{aligned} \qquad \dots(2.20)$$

and using (2.13) we get

$$e^{-t|\theta}\{x^{-\alpha-1}\} = x^{-\alpha-1}\,{}_0F_1\left[\begin{matrix} \text{---} & ; \\ \dfrac{\alpha+1-\lambda-k}{k} & ; \dfrac{t}{kx^k}\end{matrix}\right]. \qquad \ldots(2.21)$$

Also by (2.9) we have

$$e^{t\theta}\{f_1(x)f_2(x)\ldots\} = [(1-tkx^k)^{-\frac{\lambda}{k}} f_1\{x(1-tkx^k)^{-\frac{1}{k}}\}]\,.$$

$$\cdot\,[(1-tkx^k)^{-\frac{\lambda}{k}} f_2\{x(1-tkx^k)^{-\frac{1}{k}}\}]\;\ldots\,. \qquad \ldots(2.22)$$

It is observed that the results from (2.1) to (2.22) give for $\lambda = k = 1$ the results of Al-Salam (1964).

3. Operational Formulas

In this section we obtain some operational formulas for the Laguerre polynomials defined by

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1[-n;\alpha+1;x]\,. \qquad \ldots(3.1)$$

The Rodrigue's formula for these polynomials is

$$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}e^x D^n(x^{\alpha+n}e^{-x}). \qquad \ldots(3.2)$$

In order to arrive at the required result we put $x^k = u$ in our operator $\theta = \lambda x^k + x^{k+1}D$. *Then we have, $\theta_1 = \theta/k = (\lambda/ku + u^2 D_1)$ with $D_1 \equiv d/dy$. This can also be written as, $\theta_1 = u(\lambda/k + \delta_1)$ with $\delta_1 = uD_1$.
With this the formula (2.3) becomes

$$\theta_1^n = u^n \prod_{j=1}^{n}\left(\delta_1 + \frac{\lambda}{k} + (j-1)\right). \qquad \ldots(3.3)$$

And also the shift rules are transformed to

$$\left.\begin{aligned} F(\theta_1)\{u^\alpha f(u)\} &= u^\alpha F(\theta_1+\alpha u)f(u) \\ F(\theta_1)\{e^{g(u)}f(u)\} &= e^{g(u)}F(\theta_1+u^2g'(u)f(u)\end{aligned}\right\} \qquad \ldots(3.4)$$

with $u = x^k$, $\theta_1 = \theta/k$, and these can be proved by induction.

We now get, with the help of (3.3), the following equivalent forms:

$$\left(\theta_1 + \frac{\alpha}{k}\right)^n f(u) = u^n \prod_{j=1}^{n}\left(\delta_1 + \frac{\alpha}{k} + \frac{\lambda}{k} + j - 1\right)f(u) \qquad \ldots(3.5)$$

* *Added in Proof:* The operator θ_1 is essentially the same as the operator T_c of Mittal, H. B. [Glasnik Matematicki, 6 (1971), 45–53]

$$\left(\theta_1 + \frac{\alpha}{k} u - u^2\right)^n \cdot 1 = u^n \prod_{j=1}^{n} \left(\delta_1 + \frac{\alpha}{k} - u + \frac{\lambda}{k} + j - 1\right) \cdot 1 \quad \ldots(3.6)$$

with $u = x^k$ and $\theta_1 = \dfrac{\theta}{k}$.

Proofs of (3.5) and (3.6) follow by induction.

The left-hand side of (3.6) can also be written as

$$u^{-\frac{\alpha}{k}} e^u \theta_1^n (u^{\frac{\alpha}{k}} e^{-u}) f(u). \quad \ldots(3.7)$$

Recall the following result due to Carlitz (1960) :

$$\prod_{j=1}^{n} (xD - x + \alpha + j)\ Y = n! \sum_{r=0}^{n} \frac{1}{r!} x^r L_{n-r}^{\alpha+r}(x) D^r Y.$$

This can be put in the form

$$\prod_{j=1}^{n} \left(\delta_1 - u + \frac{\alpha + \lambda - k}{k} + j\right) Y$$

$$= n! \sum_{p=0}^{n} \frac{u^p}{p!} L_{n-p}^{\frac{\alpha+\lambda-k}{k}+p}(u) D^p Y. \quad \ldots(3.8)$$

From (3.6), (3.7) and (3.8) we get the following identity:

$$\theta_1^n \left(u^{\frac{\alpha}{k}} e^{-u} f(u)\right) = u^{\frac{\alpha}{k}+n} e^{-u} n! \sum_{p=0}^{n} \frac{u^p}{p!} L_{n-p}^{\frac{\alpha+\lambda-k}{k}+p}(u) D_1^p f(u) \quad \ldots(3.9)$$

with $u = x^k$ and $\theta_1 = \dfrac{\theta}{k}$.

The special case of (3.9) for $f(u) = 1$ is worth noting

$$\theta_1^n (u^{\frac{\alpha}{k}} e^{-u}) = u^{\frac{\alpha}{k}+n} e^{-u} n! L_n^{\frac{\alpha+\lambda-k}{k}}(u). \quad \ldots(3.10)$$

Or

$$\theta^n (x^\alpha e^{-x^k}) = x^\alpha (kx^k)^n e^{-x^k} n! L_n^{\frac{\alpha+\lambda-k}{k}}(x^k). \quad \ldots(3.11)$$

The change of variables also leads us to

$$\theta^n (x^\alpha e^{-\beta x^k}) = x^\alpha (kx^k)^n e^{-\beta x^k} n! L_n^{\frac{\alpha+\lambda-k}{k}}(\beta x^k). \quad \ldots(3.12)$$

Similarly by a simple change of variable, (3.6) could be transcribed as

$$\left(\theta_1 + \frac{\alpha}{k} u - \beta u^2\right)^n f(u) = u^n \prod_{j=1}^{n} \left(\delta_1 - \beta u + \frac{\alpha}{k} + \frac{\lambda}{k} + j - 1\right) f(u). \quad ...(3.13)$$

For $f(u) = 1$, we get the special case

$$(\theta + \alpha x^k - \beta k x^{2k})^n \cdot 1 = n!\, (kx^k)^n L_n^{\frac{\alpha+\lambda-k}{k}} (\beta x^k). \quad ...(3.14)$$

Now to obtain the addition formula for Laguerre polynomials; put $\beta = 1$ and $n = m + n$ in (3.14) and we have,

$$(\theta + \alpha x^k - kx^{2k})^{m+n} \{1\} = (m+n)!\, (kx^k)^{m+n} L_{m+n}^{\frac{\alpha+\lambda-k}{k}} (x^k)$$

$$= n!\, (kx^k)^n (\theta + (a + nk)\, x^k - kx^{2k})^m$$

$$\times \left\{ \frac{\left(\frac{\alpha+\lambda}{k}\right)_n}{n!} \sum_{p=0}^{n} \frac{(-n)_p}{\left(\frac{\alpha+\lambda}{k}\right)_p p!} x^{pk} \right\}$$

$$= \left(\frac{\alpha+\lambda}{k}\right)_n (kx^k)^n \sum_{p=0}^{n} \frac{(-n)_p}{\left(\frac{\alpha+\lambda}{k}\right)_p p!} \{\theta + (\alpha + nk + pk)\, x^k - kx^{2k}\}^m$$

$$= \left(\frac{\alpha+\lambda}{k}\right)_n (kx^k)^{m+n} m! \sum_{p=0}^{n} \frac{(-n)_p}{\left(\frac{\alpha+\lambda}{k}\right)_p p!} x^{pk} L_m^{\frac{\alpha+nk+pk+\lambda-k}{k}} (x^k)$$

We have thus obtained the addition formula for the Laguerre polynomials

$$\frac{(m+n)!}{m!} L_{m+n}^{\frac{\alpha+\lambda-k}{k}} (x^k)$$

$$= \left(\frac{\alpha+\lambda}{k}\right)_n \sum_{p=0}^{n} \frac{(-n)_p}{\left(\frac{\alpha+\lambda}{k}\right)_p p!} x^{pk} L_m^{\frac{\alpha+nk+pk+\lambda-k}{k}} (x^k). \quad ...(3.15)$$

Again from (3.10) we also have,

$$\theta_1^m \{u^{\frac{\alpha}{k}+n} e^{-u} L_n^{\frac{\alpha+\lambda-k}{k}} (u)\} = \theta_1^m \left\{\frac{1}{n!} \theta_1^n (u^{\frac{\alpha}{k}} e^{-u})\right\}$$

$$= \frac{(m+n)!}{n!} u^{\frac{\alpha}{k}+m+n} e^{-u} L_{m+n}^{\frac{\alpha+\lambda-k}{k}} (u).$$

Hence

$$\theta^m \left\{x^{\alpha+nk}\ e^{-xk}\, L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)\right\}$$

$$= \frac{(m+n)!}{n!}(kx^k)^m\, x^{\alpha+nk}\, e^{-xk}\, L_{n+m}^{\frac{\alpha+\lambda-k}{k}}(x^k). \qquad \ldots(3.16)$$

We now consider the operator

$${}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ -t\theta\right].$$

Consider,

$${}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ -t\theta\right](x^{\alpha+nk})$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p\, t^p}{p!\left(\frac{\alpha+\lambda}{k}\right)_p}\ \theta^p\,(x^{\alpha+nk})$$

$$= x^{\alpha+nk}\ {}_1F_1\left[\frac{\alpha+\lambda+nk}{k}\,;\ \frac{\alpha+\lambda}{k}\,;\ -tkx^k\right]$$

$$= \frac{n!}{\left(\frac{\alpha+\lambda}{k}\right)_n}\ x^{\alpha+nk}\, e^{-tkxk}\, L_n^{\frac{\alpha+\lambda-k}{k}}(tkx^k).$$

Hence we get

$${}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ -t\theta\right]x^{\alpha+nk}$$

$$= \frac{n!}{\left(\frac{\alpha+\lambda}{k}\right)_n}\ x^{\alpha+nk}\, e^{-tkxk}\, L_n^{\frac{\alpha+\lambda-k}{k}}(tkx^k). \qquad \ldots(3.17)$$

Similarly we obtain

$$\left(1+\frac{t}{\theta}\right)^n \{x^{-\alpha-1}\} = x^{-\alpha-1}\,\frac{n!}{\left(\frac{\alpha+1-\lambda+k}{k}\right)_n}\ L_n^{\frac{\alpha+1-n}{k}}\left(\frac{t}{kx^k}\right) \qquad \ldots(3.18)$$

and

$${}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ t\theta\right]\{x^\alpha\,(1-x^k)^c\}$$

$$= x^\alpha\, e^{tkxk}\,(1-x^k)\ {}_1F_1\left[\begin{matrix}-c\,; \\ \frac{\alpha+\lambda}{k}\,;\end{matrix}\ \frac{tkx^{2k}}{1-x^k}\right]. \qquad \ldots(3.19)$$

If we put $u = x^{\beta}$, $v = x^{\alpha} e^{-x^k}$ in Leibnitz formula (2.5) we get, by using (2.4) and (3.11),

$$\theta^n \{x^{\lambda} (x^{\alpha} e^{-x^k}) x^{\beta}\}$$

$$= x^{\alpha+\beta+\lambda} e^{-x^k} \sum_{p=0}^{n} \binom{n}{p} \left(\frac{\beta+\lambda}{k}\right)_p$$

$$\times (kx^k)^n (n-p)!\, L_{n-p}^{\frac{\alpha+\lambda-k}{k}} (x^k).$$

Again by (3.11) we can write

$$\theta^n \{x^{\lambda} (x^{\alpha} e^{-x^k}) x^{\beta}\}$$

$$= x^{\alpha+\beta+\lambda} (kx^k)^n e^{-x^k} n!\, L_{n-p}^{\frac{\alpha+\beta+2\lambda-k}{k}} (x^k).$$

Hence

$$L_n^{\frac{\alpha+\beta+2\lambda-k}{k}} (x^k) = \sum_{p=0}^{n} \left(\frac{\beta+\lambda}{k}\right)_p \frac{1}{p!} L_{n-p}^{\frac{\alpha+\lambda-k}{k}} (x^k). \qquad \text{...(3.20)}$$

On the other hand, if we put $u = x^{\alpha} e^{-ax^k}$, $v = x^{\beta} e^{-bx^k}$ in (2.5) and then employing (3.12) as above, we get,

$$L_n^{\frac{\alpha+\beta+2\lambda-k}{k}} ((a+b) x^k) = \sum_{p=0}^{n} L_p^{\frac{\alpha+\lambda-k}{k}} (ax^k)\, L_{n-p}^{\frac{\beta+\lambda-k}{k}} (bx^k). \qquad \text{...(3.21)}$$

It is remarked that the results from (3.5) to (3.21) give for $\lambda = k = 1$ the results of Al-Salam (1964).

4. Generating Functions

In this section, we shall obtain some generating functions involving Laguerre polynomials.

Consider,

$$x^{\alpha} e^{-x^k} e^{tkx^k} = \sum_{p=0}^{\infty} \frac{(tkx^k)^p}{p!} x^{\alpha} e^{-x^k}.$$

Operate θ^m on both sides, we then get the generating function

$$e^{tk} L_m^{\frac{\alpha+\lambda-k}{k}} (x^k - tk) = \sum_{p=0}^{\infty} \frac{(tk)^p}{p!} L_m^{\frac{\alpha+pk+\lambda-k}{k}} (x^k). \qquad \text{...(4.1)}$$

As a special case, $\lambda = k = 1$, we get the generating function (Buchholtz 1953, p. 142)

$$e^t L_m^{(\alpha)}(x-t) = \sum_{p=0}^{\infty} \frac{t^p}{p!} L_m^{\alpha+p}(x). \qquad \ldots(4.2)$$

Next, operating $e^{t\theta}$ on the expression $x^\alpha e^{-x^k}$, we have

$$e^{t\theta}\{x^\alpha e^{-x^k}\} = \sum_{p=0}^{\infty} \frac{t^p}{p!} \theta^p \{x^\alpha e^{-x^k}\}.$$

Applying (2.8) for left-hand side and (3.11) for right-hand side and putting $tk = t$, we have

$$(1-t)^{-\frac{\alpha+\lambda}{k}} \exp\left\{\frac{-tx^k}{1-t}\right\} = \sum_{p=0}^{\infty} t^p L_p^{\frac{\alpha+\lambda-k}{k}}(x^k). \qquad \ldots(4.3)$$

In particular, for $\lambda = k = 1$, we get the familiar generating function

$$(1-t)^{-\alpha-1} \exp\left\{\frac{-xt}{1-t}\right\} = \sum_{p=0}^{\infty} t^p L_p^{(\alpha)}(x). \qquad \ldots(4.4)$$

Similarly operating

$${}_0F_1\left[-;\ \frac{\alpha+\lambda}{k};\ t\theta\right]$$

on both sides of

$$x^\alpha (1-x^k)^{-\frac{\alpha+\lambda}{k}} = \sum_{p=0}^{\infty} \frac{\left(\frac{\alpha+\lambda}{k}\right)_p}{p!} x^{\alpha+pk},$$

it is possible to obtain the results (4.3). Let us consider

$${}_0F_1\left[-;\ \frac{\alpha+\lambda}{k};\ t\theta\right] x^\alpha e^{-x^k}$$

$$= x^\alpha e^{-x^k} \sum_{n=0}^{\infty} \frac{(tkx^k)^n}{\left(\frac{\alpha+\lambda}{k}\right)_n} L_n^{\frac{\alpha+\lambda-k}{k}}(x^k). \qquad \ldots(4.5)$$

Since we have the generating function

$$\sum_{n=0}^{\infty} L_n^{\frac{\alpha+\lambda-k}{k}}(x^k) \frac{t^n}{\left(\frac{\alpha+\lambda}{k}\right)_n} = e^t\, {}_0F_1\left[-;\ \frac{\alpha+\lambda}{k};\ -tx^k\right] \qquad \ldots(4.6)$$

we obtain, on account of (4.5), the equivalent relation

$$ {}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ t\theta\right] x^{\alpha}\, e^{-x^k} $$

$$ = x^{\alpha}\, e^{-x^k+tkx^k}\ {}_0F_1\left[-\,;\ \frac{\alpha+\lambda}{k}\,;\ -\,tkx^{2k}\right]. \qquad \ldots(4.7) $$

If in (4.6) we replace t by $t\phi$ and then operate on $y^{ck-\lambda}$ where

$$ \phi = \lambda y^k + y^{k+1}\frac{d}{dy} $$

and by using (2.4), (2.8) we get,

$$ (1-tky^k)^{-c}\ {}_1F_1\left[c\,;\ \frac{\alpha+\lambda}{k}\,;\ \frac{-\,tkx^k y^k}{1-tkx^k}\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(c)_n}{\left(\frac{\alpha+\lambda}{k}\right)_n}\, L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)\,(tky^k)^n. $$

Replace tky^k by t, then we obtain

$$ (1-t)^{-c}\ {}_1F_1\left[c\,;\ \frac{\alpha+\lambda}{k}\,;\ -\frac{tx^k}{1-t}\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(c)_n}{\left(\frac{\alpha+\lambda}{k}\right)_n}\, L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)\, t^n. \qquad \ldots(4.8) $$

In particular, for $\lambda = k = 1$, we get

$$ \sum_{n=0}^{\infty} \frac{(c)_n}{(\alpha+1)_n}\, t^n L_n^{(\alpha)}(x) = (1-t)^{-c}\ {}_1F_1\left[c;\ \alpha+1;\ -\frac{xt}{1-t}\right]. \qquad \ldots(4.9) $$

5. Bilinear Generating Functions for the Laguerre Polynomials

In this section we prove the following theorem :

Let

$$ F(x^k, t) = \sum_{p=0}^{\infty} a_p\, t^p\, L_p^{\frac{\alpha+\lambda-k}{k}}(x^k) \qquad \ldots(5.1) $$

where $a_p \neq 0$ be the given generating function. Then

$$ (1-t)^{-\frac{\alpha+\lambda}{k}} \exp\left(-\frac{tx^k}{1-t}\right) F\left[\frac{x^k}{1-t},\ \frac{ty^k}{1-t}\right] $$

$$ = \sum_{n=0}^{\infty} b_n(-y^k)\, L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)\, t^n, \qquad \ldots(5.2) $$

where

$$b_n(y^k) = \sum_{p=0}^{n} a_p \frac{(-n)_p (-1)^p}{p!} \cdot y^{pk}. \qquad \ldots(5.3)$$

In order to prove the above theorem, replace 't' in (5.1) by $tkx^k y^k$; multiply both sides by $x^\alpha e^{-x^k}$ and then operate by $e^{t\theta}$ upon them to get,

$$e^{t\theta}\{x^\alpha e^{-x^k} F(x^k, tkx^k y^k)\}$$

$$= \sum_{p=0}^{\infty} a_p (tky^k)^p \sum_{n=0}^{\infty} \frac{t^n}{n!} \theta^n \{x^{\alpha+pk} e^{-x^k} L_p^{\frac{\alpha+\lambda-k}{k}}(x^k)\}. \qquad \ldots(5.4)$$

On account of (3.16) the right-hand side of (5.4) becomes

$$\sum_{p=0}^{\infty} a_p (tky^k)^p \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{(p+n)!}{p!} (kx^k)^n x^{\alpha+pk} e^{-x^k} L_{n+p}^{\frac{\alpha+\lambda-k}{k}}(x^k)$$

$$= x^\alpha e^{-x^k} \sum_{n=0}^{\infty} L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)(tkx^k)^n \sum_{p=0}^{n} a_p y^{pk} \frac{(-n)_p}{p!} (-1)^p.$$

Applying (2.8) left-hand side of (5.4) gets reduced to

$$\frac{x^\alpha}{(1-tkx^k)^{\frac{\alpha+\lambda}{k}}} \exp\left(-\frac{x^k}{1-tkx^k}\right) F\left[\frac{x^k}{1-tkx^k}, \frac{tkx^k y^k}{1-tkx^k}\right].$$

Hence we have in view of (5.4)

$$(1-tkx^k)^{-\frac{\alpha+\lambda}{k}} \exp\left(\frac{tkx^{2k}}{1-tkx^k}\right) F\left[\frac{x^k}{1-tkx^k}, \frac{tkx^k y^k}{1-tkx^k}\right]$$

$$= \sum_{n=0}^{\infty} b_n(y^k) L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)(tkx^k)^n,$$

where $b_n(y^k)$ is given by (5.3). Replace tkx^k by 't' and y^k by $-y^k$ to get

$$(1-t)^{-\frac{\alpha+\lambda}{k}} e^{-\frac{tx^k}{1-t}} F\left(\frac{x^k}{1-t}, \frac{-ty^k}{1-t}\right)$$

$$= \sum_{n=0}^{\infty} b_n(-y^k) L_n^{\frac{\alpha+\lambda-k}{k}}(x^k) t^n,$$

where $b_n(-y^k)$ will be given by (5.3). This completes the proof.

As an application of this theorem, let

$$a_p = \frac{(c_p)}{\left(\frac{\alpha+\lambda}{k}\right)_p}.$$

Thus $F\left[\frac{x^k}{1-t}, \frac{-ty^k}{1-t}\right]$

$$= \left(1 + \frac{ty^k}{1-t}\right)^{-c} {}_1F_1\left[c; \frac{\alpha+\lambda}{k}; \frac{ty^k x^k}{(1-t)(1-t+yt)}\right].$$

Thus (5.2) becomes,

$$(1-t)^{-\frac{\alpha+\lambda}{k}} e^{-\frac{tx^k}{1-t}} \left(1 + \frac{ty^k}{1-t}\right)^{-c}$$

$$\times {}_1F_1\left[c; \frac{\alpha+\lambda}{k}; \frac{ty^k x^k}{(1-t)(1-t+yt)}\right],$$

$$= \sum_{n=0}^{\infty} b_n(-y^k) L_n^{\frac{\alpha+\lambda-k}{k}}(x^k) t^v,$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{(c)_p (-n_p)}{\left(\frac{\alpha+\lambda}{k}\right)_p p!} y^{pk} L_n^{\frac{\alpha+\lambda-k}{k}}(x^k) t^n.$$

Thus

$$(1-t)^{c-\frac{\alpha+\lambda}{k}} e^{-\frac{tx^k}{1-t}} (1-t+ty^k)^{-c}$$

$$\times {}_1F_1\left[c; \frac{\alpha+\lambda}{k}; \frac{ty^k x^k}{(1-t)(1-t+yt)}\right]$$

$$= \sum_{n=0}^{\infty} {}_2F_1\left[-n, c; \frac{\alpha+\lambda}{2}; y^k\right] L_n^{\frac{\alpha+\lambda-k}{k}}(x^k) \cdot t^n. \qquad \ldots(5.5)$$

This result is a generalization of Weisner's formula (cf., *e.g.* Rainville 1967, p. 213) which can be obtained from our result if we put $\lambda = k = 1$.

Al-Salam (1964) also obtains Weisner's formula but it may be mentioned that there are misprints in his result.

If we select $a_p = \frac{1}{\left(\frac{\alpha+\lambda}{k}\right)_p}$

in (5.2) we shall get one more special case namely,

$$(1-t)^{-\frac{\alpha+\lambda}{k}} \exp\left(-\frac{(x^k+y^k)\,t}{(1-t)}\right) {}_0F_1\left[-\,;\frac{\alpha+\lambda}{k}\,;\frac{t y^k x^k}{(1-t)^2}\right]$$

$$=\sum_{n=0}^{\infty} \frac{n\,!}{\left(\frac{\alpha+\lambda}{k}\right)_n} L_n^{\frac{\alpha+\lambda-k}{k}}(y^k)\, L_n^{\frac{\alpha+\lambda-k}{k}}(x^k)\, t^n. \qquad \ldots(5.6)$$

If we put $\lambda = k = 1$, we get Hardy-Hille formula: (See Rainville 1967; Al-Slam 1964, pp. 135).

6. Remark

It is remarked that in subsequent work we have studied the polynomials

$$P_n^{(\alpha)}(x, r, s, p, k, \lambda) = x^\alpha e^{px^r} \theta^n (x^{\alpha+sn} e^{-px^r}) \qquad \ldots(6.1)$$

where

$$\theta = \lambda x^k + x^{k+1} D.$$

This polynomial gives as a special case all the generalizations of the Hermite and Laguerre polynomials.

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References

Al-Slam, W. A. (1963). The Bessel polynomials. *Duke. Math. J.*, **30**, 51–62.

———— (1964). Operational representations for the Laguerre and other polynomials. *Duke Math. J.*, **31**, 127–42.

Buchholtz, H. (1953). Die Konfluents Hypergeometrische Funktionen. Berlin.

Burchnall, J. L. (1951). The Bessel polynomials. *Can. J. Math.*, **3**, 62–68.

———— (1941). A note on the polynomials of Hermite. *Quart. Jour. Math., Oxford series* 2, **12**, 9–11.

Carlitz, L. (1960). A note on the Laguerre polynomials. *Mich. Math. J.*, **7**, 219–23.

———— (1971). A note on the Hardy-Hille formula. *Glasnik Mat.*, **6** (26), No. 1, 55–58.

Chak, A. M. (1956). A class of polynomials and a generalization of Stirling numbers. *Duke Math. J.*, **23**, 45–56.

Chatterjea, S. K. (1963*a*). Operational formulae for certain classical polynomials. *Quart. Jour. Math. Series* 2 (*Oxford*), **14**, 241–46.

———— (1963*b*). Operational formulae for certain classical polynomials—II. *Rend. Sem. Mat. Univ. Padova*, **33**, 163–69.

———— (1964). On a genralization of Laguerre polynomials. *Rend. Sem. Mat. Univ. Padova*, **34**, 180–90.

———— (1966). Some operational formulas connected with a function defined by a generalized Rodrigues' formula. *Acta Math. Acad. Scient. Hungaricae*, Tomus 17 (3–4), 379–85.

Chattrjea, S. K. (1968). An operational formula for a function defined by a generalized Rodrigues' formula. *Boll. U.M.I.*, **4**, 508–13.

Das, M. K. (1967). Operational representation for the Laguerre polynomials. *Acta Math. Acad. Scient. Hungaricae*, Tomus **18**, (3–4), 335–38.

Gould, H. W., and Hopper, A. T. (1962). Operational formulas connected with two generalizations of Hermite polynomials. *Duke Math. J.*, **29**, 51–64.

Karande, B. K., and Thakare, N. K. (1975). Operational formulas in the unified form for the classical orthogonal polynomials (Communicated).

Rainville, E. D. (1967). Special Functions. MacMillan & Co., New York.

Shrivastava P. N. (1972). Certain operational formulae. *J. Indian math. Soc.*, **36**, 133–141.

———— (1974). Some operational formulas and a generalized Generating Function. *Math. Edu.*, **8**, (1), 19–22.

Singh, R. P. (1965). Operational formulae for Jacobi and other polynomials. *Rend. San. Mat. Univ. Padova*, **35**, 237–44.

Srivastava, H. M. and Singhal, J. P. (1971). A class of polynomials defined by generalized Roddrigues' formula. *Ann. Mat. pura applic.*, (IV). Vol. XC 75–86.

Thakare, N. K. and Karande, B. K. (1973). Remarks on operational formulas for orthogonal polynomials. *Math. Vesnik*, **10** (25), 139–44.

Weisner, L. (1955). Group-theoretic origin of certain generating functions. *Pacific J. Math.*, **5**, 1033–39.