Cumulants and classical umbral calculus

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Techniques

Classical umbral calculus was introduced in 1994 by Rota and Taylor [RT]. We refer the setting developed by Di Nardo and Senato [DNS].

[DNS] E. Di Nardo, D. Senato, *Umbral nature of Poisson random variable*, in: H. Crapo and D. Senato eds., Algebraic combinatorics and computer science, Springer Verlag, Italia, (2001), 245-266.

[RT] G.-C. Rota, B.D. Taylor, *The classical umbral calculus*, SIAM J. Math. Anal. **25** (1994), 694-71.

Results

1. we show how generalized umbral Abel polynomials $A_n^{(k)}(x,\alpha) = x(x + k.\alpha)^{n-1}$ encode the formulae connecting a sequence of moments to its classical cumulants, free cumulants and boolean cumulants,
2. we prove that the convolutions $a \star b$ (classical), $a \boxplus b$ (free) and $a \uplus b$ (boolean) are represented by umbrae $\alpha_{(k)}\gamma$ such that

$$A_n^{(k)}(\alpha_{(k)}\gamma) = A_n^{(k)}(\alpha) + A_n^{(k)}(\gamma).$$

[DNPS] E. Di Nardo, P. Petrullo, D. Senato, *Cumulants, convolutions and volume polynomial*, preprint.

[P] P. Petrullo, *A symbolic treatment of Abel polynomials*, preprint.

Classical cumulants

Consider $a = (a_n)_{n\geq 1}$ and $k_a = (k_n)_{n\geq 1}$ and their exponential generating functions

$$M(z) = 1 + \sum_{n\geq 1} a_n \frac{z^n}{n!}, \quad K(z) = 1 + \sum_{n\geq 1} k_n \frac{z^n}{n!}.$$

If we have

$$M(z) = e^{K(z)-1},$$

then $k_n(a) = k_n$ is the n-th (formal) classical cumulant of a.

Free cumulants and boolean cumulants

Consider $a = (a_n)_{n \geq 1}$, $r_a = (r_n)_{n \geq 1}$ and $s_a = (s_n)_{n \geq 1}$ with ordinary generating functions

$$M(z) = 1 + \sum_{n \geq 1} a_n z^n, \quad R(z) = 1 + \sum_{n \geq 1} r_n z^n \text{ and } S(z) = \sum_{n \geq 1} s_n z^n,$$

such that

$$M(z) = \frac{1}{1 - S(z)} = \frac{1}{z}[zR(z)]^{<-1>},$$

then $r_n(a) = r_n$ is the n-th (formal) free cumulants of a, and $s_n(a) = s_n$ is the n-th (formal) boolean cumulants of a.

Convolutions

Cumulants linearize convolutions. Given $a = (a_n)_{n\geq 1}$ and $b = (b_n)_{n\geq 1}$, we denote by $a \star b$, $a \boxplus b$ and $a \uplus b$ the sequences such that

$$k_n(a \star b) = k_n(a) + k_n(b) \quad \text{(classical convolution)},$$

$$r_n(a \boxplus b) = r_n(a) + r_n(b) \quad \text{(free convolution)},$$

$$s_n(a \uplus b) = s_n(a) + s_n(b) \quad \text{(boolean convolution)}.$$

Cumulants via Moebius inversion formula

We have (see [S] and [SW])

$$a_n = \sum_{\pi \in \Pi_n} k_\pi(a) \Longleftrightarrow k_n(a) = \sum_{\pi \in \Pi_n} \mu_\Pi(\pi, 1_n) a_\pi,$$

$$a_n = \sum_{\pi \in NC_n} r_\pi(a) \Longleftrightarrow r_n(a) = \sum_{\pi \in NC_n} \mu_{NC}(\pi, 1_n) a_\pi,$$

$$a_n = \sum_{\pi \in I_n} s_\pi(a) \Longleftrightarrow s_n(a) = \sum_{\pi \in I_n} \mu_I(\pi, 1_n) a_\pi.$$

[S] R. Speicher, *Free probability theory and noncrossing partitions*, Sém. Loth. Combin., (1997) **B39C**, 38pp.
[SW] R. Speicher, R. Woroudi, *Boolean convolution*, in: Free Probability Theory (Waterloo, ON, 1995), American Mathematical Society, Providence, RI, 1997, 267-279.

The classical umbral calculus

The *classical umbral calculus* consists of the following data:

1. the *alphabet* $A = \{\alpha, \beta, \dots\}$ the of *umbrae*
2. the linear functional $E : R[A] \to R$ *evaluation*, such that
   - $E[1] = 1$
   - $E[\alpha^i \beta^j \cdots \gamma^k] = E[\alpha^i]E[\beta^j]\cdots E[\gamma^k]$ (*uncorrelation* property)
3. two special umbrae ε (*augmentation*) and u (*unity*) such that

$$E[\varepsilon^n] = \delta_{0,n}, \quad \text{for } n = 0, 1, 2, \dots$$

   and

$$E[u^n] = 1, \quad \text{for } n = 0, 1, 2, \dots$$

4. N.B.we assume $R = \mathbb{C}[x]$

Generating functions, umbral equivalence, similarity

1. if $E[\alpha^n] = a_n$ we say α represents the sequence $a = (a_n)_{n\geq 1}$, or a_n is the n-th *moment* of α
2. the *generating function* of α is the exponential formal power series

$$f_\alpha(z) = E[e^{\alpha z}] = 1 + \sum_{n\geq 1} a_n \frac{z^n}{n!},$$

so that

$$E[\alpha^n] = n![z^n]f_\alpha(z)$$

3. umbral equivalence "$\simeq$":

$$\alpha \simeq \gamma \Leftrightarrow E[\alpha] = E[\gamma] \text{ so that } e^{\alpha z} \simeq f_\alpha(z)$$

4. *similarity* "$\equiv$":

$$\alpha \equiv \gamma \Leftrightarrow E[\alpha^n] = E[\gamma^n] \text{ for all } n \geq 0 \Leftrightarrow f_\alpha(z) = f_\gamma(z)$$

Dot operation and composition umbra

1. the *Bell umbra* β and the singleton umbra χ have generating functions
$$f_\beta(t) = e^{e^z - 1} \text{ and } f_\chi(z) = 1 + z,$$
2. $n.\alpha$ $(n \in \mathbb{Z})$ denotes an umbra such that
$$f_{n.\alpha}(z) = f_\alpha(z)^n,$$
3. the dot operation of γ with α is an umbra $\gamma.\alpha$ such that
$$f_{\gamma.\alpha}(z) = f_\gamma[\log f_\alpha(z)],$$
4. composition umbra: dot operation is associative (but noncommutative), so that
$$f_{\gamma.\beta.\alpha}(z) = f_\gamma[f_\alpha(z) - 1].$$

Derivative, compositional inverse and Lagrange involution

1. the compositional inverse of α is an umbra $\alpha^{<-1>}$ such that $\alpha^{<-1>}.\beta.\alpha \equiv \alpha.\beta.\alpha^{<-1>} \equiv \chi$, so that
$$f_{\alpha^{<-1>}}(z) - 1 = [f_\alpha(z) - 1]^{<-1>},$$
2. the derivative of α is an umbra α_D such that ${\alpha_D}^n \simeq \alpha^{n-1}$, that is
$$f_{\alpha_D}(z) = 1 + z f_\alpha(z),$$
3. if $\alpha \simeq 1$ then α_P is defined by
$$\alpha_{DP} \equiv \alpha_{PD} \equiv \alpha,$$
4. we name the umbra $\mathfrak{L}_\alpha \equiv {\alpha_D}^{<-1>}{}_P$ the noncrossing Fourier transform or Lagrange involution of α. Its generating function is
$$f_{\mathfrak{L}_\alpha}(z) = \frac{1}{z}[z f_\alpha(z)]^{<-1>}.$$

Abel polynomials

1. Abel polynomials $A_n(x,a)$ are defined by
$$A_n(x,a) = x(x-na)^{n-1},$$
2. umbral Abel polynomials are obtained by replacing $-na$ with $-n.\alpha$, that is
$$A_n(x,\alpha) = x(x-n.\alpha)^{n-1},$$
3. if $\bar{f}(z) = [ze^{az}]^{<-1>}$ then
$$1 + \sum_{n\geq 1} A_n(x,a)\frac{z^n}{n!} = e^{x\bar{f}(z)},$$
from which
$$[x.\beta.(a.u)_D{}^{<-1>}]^n \simeq A_n(x,a) \qquad (1).$$

Polynomials of binomial type

By replacing a with α in (1) we have

$$(x.\beta.\alpha_D{}^{<-1>})^n \simeq A_n(x,\alpha), \qquad (1u)$$

from which, if $\bar{f}(z) = [zf_\alpha(z)]^{<-1>}$ then

$$1 + \sum_{n\geq 1} p_n(x)\frac{z^n}{n!} = e^{x\bar{f}(z)} \simeq u + \sum_{n\geq 1} A_n(x,\alpha)\frac{z^n}{n!},$$

so that "all polynomials of binomial type are represented by Abel polynomials" (see [RST])

[RST] G.-C. Rota, J. Shen, B.D. Taylor, *All polynomials of binomial type are represented by Abel plynomials*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (1997) **25**, no. 1, 731-738.

Abel polynomials and Lagrange inversion formula: I

1. by setting $x = \chi$ in (1) we recover

$$(-na)^{n-1} = n![z^n][ze^{az}]^{<-1>},$$

2. $x = \chi$ in (1u) gives

$$(-n.\alpha)^{n-1} \simeq n![z^n][ze^{\alpha z}]^{<-1>},$$

3. if we intend $[ze^{\alpha z}]^{<-1>} \simeq [zf_\alpha(z)]^{<-1>}$, then we have the Lagrange inversion formula

$$\frac{1}{n}[z^{n-1}]\left(\frac{1}{f_\alpha(z)}\right)^n = [z^n][zf_\alpha(z))]^{<-1>}.$$

Abel polynomials and Lagrange inversion formula: II

1. by replacing x with another umbra γ we have

$$(\gamma.\beta.\alpha_D{}^{<-1>})^n \simeq A_n(\gamma, \alpha) \qquad (\star)$$

2. since

$$(\gamma.\beta.\alpha_D{}^{<-1>})^n \simeq n![z^n]f_\gamma\left([zf_\alpha(z)]^{<-1>}\right)$$

and

$$A_n(\gamma, \alpha) \simeq \sum_{i=0}^{n-1} \binom{n-1}{i} \gamma^{i+1}(-n.\alpha)^{n-1-i} \simeq (n-1)![z^{n-1}]f'_\gamma(z)\left(\frac{1}{f_\alpha(z)}\right)^n,$$

then we recover a more general version of Lagrange inversion

$$[z^n]f_\gamma\left([zf_\gamma(z)]^{<-1>}\right) = \frac{1}{n}[z^{n-1}]f'_\gamma(z)\left(\frac{1}{f_\alpha(z)}\right)^n$$

Generalized umbral Abel polynomials

We define generalized umbral Abel polynomials

$$A_n^{(k)}(x, \alpha) = x(x + k.\alpha)^{n-1}.$$

We set $A_n^{(k)}(\alpha) = A_n^{(k)}(\alpha, \alpha)$. A combinatorial treatment of $k = n$ is given in [PS].

Theorem (First Abel Inversion Theorem)

$$\gamma^n \simeq A_n^{(k)}(\alpha), \text{ for } n = 1, 2, \ldots$$

if and only if

$$\alpha^n \simeq A_n^{(-k)}(\gamma, \alpha) \text{ for } n = 1, 2, \ldots.$$

[PS] P. Petrullo, D. Senato, *An instance of umbral methods in representation theory: the parking function module*, arXiv: 0807.4840v2.

Abel polynomials and Lagrange inversion formula: III

Theorem (Abel form of LIF)

$$A_n^{(k)}(\mathfrak{L}_\alpha) \simeq A_n^{(n+k)}(-1.\alpha) \simeq -A_n^{(-(n+k+2))}(\alpha)$$

Proof.

$$k \neq -1 \Rightarrow [z^n]\{[zf(\alpha,z)]^{<-1>}\}^{k+1} = \frac{k+1}{n}[z^{n-k-1}]\left(\frac{1}{f(\alpha,z)}\right)^n \qquad (2)$$

$$k = -1 \Rightarrow [z^n]\log\left(\frac{1}{z}[zf(\alpha,z)]^{<-1>}\right) = \frac{1}{n}[z^n]\left(\frac{1}{f(\alpha,z)}\right)^n \qquad (3)$$

□

Second inversion rule

Theorem (Second Abel Inversion Theorem)

$$\gamma^n \simeq A_n^{(n+k)}(\alpha), \text{ for } n = 1, 2, \ldots$$

if and only if

$$\alpha^n \simeq A_n^{(-(n+k))}(\gamma, \mathfrak{L}_{-1.\alpha}), \text{ for } n = 1, 2, \ldots.$$

Cumulant umbrae

If $a = (a_n)_{n \geq 1}$, let $\alpha^n \simeq a_n = n! a'_n$ and $a' = (a'_n)_{n \geq 1}$. We define κ_α, η_α and $\mathfrak{K}_\alpha$ to be such that

$$\begin{aligned} \alpha &\equiv \beta . \kappa_\alpha, \\ \alpha &\equiv \bar{u} . \beta . \eta_\alpha, \\ \alpha &\equiv \mathfrak{L}_{-1 . \mathfrak{K}_\alpha}, \end{aligned}$$

where $\bar{u} \equiv -1 . - \chi$. In this way

$$\begin{aligned} {\kappa_\alpha}^n &\simeq k_n(a), \\ {\eta_\alpha}^n &\simeq n! s_n(a'), \\ {\mathfrak{K}_\alpha}^n &\simeq n! r_n(a'). \end{aligned}$$

Abel parametrization for classical cumulants

1. we have
$$\kappa_\alpha{}^n \simeq \alpha(\alpha - 1.\alpha)^{n-1} = A_n^{(-1)}(\alpha),$$
2. by applying First Abel Inversion Theorem
$$\alpha^n \simeq \kappa_\alpha(\kappa_\alpha + \alpha)^{n-1} = A_n^{(1)}(\kappa_\alpha, \alpha), \qquad (4)$$
3. identity (4) is a result of Rota-Shen [RS], the umbra κ_α has been deeply studied by Di Nardo-Senato [DNS]

[RS] G.-C. Rota, J. Shen, *On the combinatorics of cumulants*, J. Combin. Theory Ser. A (2000) **91**, 283-304.
[DNS] E. Di Nardo, D. Senato, *An umbral setting for cumulants and factorial moments*, European J. Combin. (2006) **27**, 394-413.

Abel parametrization for free and boolean cumulants

1. we have
$$\eta_\alpha{}^n \simeq \alpha(\alpha - 2.\alpha)^{n-1} = A_n^{(-2)}(\alpha)$$
and
$$\mathfrak{K}_\alpha{}^n \simeq \alpha(\alpha - n.\alpha)^{n-1} = A_n^{(-n)}(\alpha),$$
2. by using First Abel Inversion Theorem and Abel form of LIF we obtain
$$\alpha^n \simeq \eta_\alpha(\eta_\alpha + 2.\alpha)^{n-1} = A_n^{(2)}(\eta_\alpha, \alpha)$$
and
$$\alpha^n \simeq \mathfrak{K}_\alpha(\mathfrak{K}_\alpha + n.\mathfrak{K}_\alpha)^{n-1} = A_n^{(n)}(\mathfrak{K}_\alpha).$$

Mixed parametrization

Second Abel inversion Theorem gives

Theorem (Mixed Abel parametrization of cumulants)

$$\kappa_\alpha{}^n \simeq A_n^{(1)}(\eta_\alpha, \alpha) \simeq A_n^{(n-1)}(\mathfrak{K}_\alpha),$$

$$\eta_\alpha{}^n \simeq A_n^{(-1)}(\kappa_\alpha, \alpha) \simeq A_n^{(n-2)}(\mathfrak{K}_\alpha),$$

$$\mathfrak{K}_\alpha{}^n \simeq A_n^{(1-n)}(\kappa_\alpha, \alpha) \simeq A_n^{(2-n)}(\eta_\alpha, \alpha).$$

From the parametrization to the formulae: an example

If $\alpha^n \simeq a_n$ then we have

$$\alpha(\alpha + \gamma . \alpha)^n \simeq \sum_{\mu \vdash n} \mathrm{d}_\mu (\gamma)_{\ell(\mu)-1} a_\mu,$$

where $\mu = (\mu_1, \mu_2, \ldots) = [1^{m_1} 2^{m_2} \ldots]$, $a_\mu = a_{\mu_1} a_{\mu_2} \ldots$, $\ell(\mu) = m_1 + m_2 + \cdots$, and

$$\mathrm{d}_\mu = \frac{n!}{\mu_1! \mu_2! \cdots m_1! m_2! \cdots}.$$

Since ${\eta_\alpha}^n \simeq n! s_n$ and ${\mathfrak{K}_\alpha}^n \simeq n! r_n$, from

$${\eta_\alpha}^n \simeq \mathfrak{K}_\alpha(\mathfrak{K}_\alpha + (n-2) . \mathfrak{K}_\alpha)^{n-1},$$

we obtain

$$s_n = \sum_{\mu \vdash n} \frac{(n-2)_{\ell(\mu)-1}}{m_1! m_2! \cdots} r_\mu.$$

Convolution umbrae

1. the disjoint sum of α and γ is an umbra $\alpha \dot{+} \gamma$ such that
$$(\alpha \dot{+} \gamma)^n \simeq \alpha^n + \gamma^n,$$
2. if $\alpha^n \simeq a_n = n! a'_n$ and $\gamma^n \simeq b_n = n! b'_n$, then we define $\alpha \star \gamma$, $\alpha \uplus \gamma$ and $\alpha \boxplus \gamma$ to be umbrae such that
$$\begin{aligned} \kappa_{\alpha \star \gamma} &\equiv \kappa_\alpha \dot{+} \kappa_\gamma, \\ \eta_{\alpha \uplus \gamma} &\equiv \eta_\alpha \dot{+} \eta_\gamma, \\ \mathfrak{K}_{\alpha \boxplus \gamma} &\equiv \mathfrak{K}_\alpha \dot{+} \mathfrak{K}_\gamma. \end{aligned}$$
3. in this way
$$\begin{aligned} (\alpha \star \gamma)^n &\simeq (a \star b)_n, \\ (\alpha \uplus \gamma)^n &\simeq n!(a' \uplus b')_n, \\ (\alpha \boxplus \gamma)^n &\simeq n!(a' \boxplus b')_n. \end{aligned}$$

Boolean convolution vs free convolutions

Theorem

$$\mathfrak{L}_{\alpha \boxplus \gamma} \equiv \mathfrak{L}_\alpha \uplus \mathfrak{L}_\gamma \text{ and } \mathfrak{L}_{\alpha \uplus \gamma} \equiv \mathfrak{L}_\alpha \boxplus \mathfrak{L}_\gamma$$

Proof.

From $\alpha \equiv \bar{u}.\beta.\eta_\alpha$ we have ${\eta_\alpha}^n \simeq -(-1.\alpha)^n$. In this way

$$-1.(\alpha \uplus \gamma) \equiv (-1.\alpha) \dot{+} (-1.\gamma).$$

From $\alpha \equiv \mathfrak{L}_{-1.\mathfrak{K}_\alpha}$ we have $\mathfrak{K}_\alpha \equiv -1.\mathfrak{L}_\alpha$, so that

$$\mathfrak{L}_{\alpha \boxplus \gamma} \equiv -1.\mathfrak{K}_{\alpha \boxplus \gamma} \equiv -1.(\mathfrak{K}_\alpha \dot{+} \mathfrak{K}_\alpha),$$

that is $\mathfrak{L}_{\alpha \boxplus \gamma} \equiv \mathfrak{L}_\alpha \uplus \mathfrak{L}_\gamma$. Second similarity is analogous. □

Abel-type convolutions

We call Abel-type convolution of α and γ every umbra $\alpha_{(k)}\gamma$ such that

$$A_n^{(k)}(\alpha_{(k)}\gamma) = A_n^{(k)}(\alpha) + A_n^{(k)}(\gamma).$$

Then, if $k = -1$ then

$$\alpha_{(-1)}\gamma \equiv \alpha + \gamma,$$

otherwise

$$\alpha_{(k)}\gamma \equiv \frac{1}{1+k}.\left[(1+k).\alpha \dot{+} (1+k).\gamma\right],$$

where, in general $(\alpha \dot{+} \gamma)^n \simeq \alpha^n + \gamma^n$

Convolution umbrae via Abel-type convolutions

Theorem

$$\alpha_{(-1)}\gamma \equiv \alpha \star \gamma,$$

$$\alpha_{(-2)}\gamma \equiv \alpha \uplus \gamma,$$

$$(\alpha_{(-n)}\gamma)^n \simeq (\alpha \boxplus \gamma)^n.$$

Proof.

Via Abel parametrization. □

Alphabets and umbrae

1. given a formal power series $f(z)$, Lascoux [L] consider the alphabet $\mathbb{A}$ such that

$$f(z) = H_z(\mathbb{A}) \text{ and } f(z)^k = H_z(k\mathbb{A}),$$

where $H_z(\mathbb{A}) = 1 + \sum_{n \geq 1} h_n(\mathbb{A}) z^n$,

2. if $e^{\alpha z} \simeq f(z) = H_z(\mathbb{A})$ and $e^{\gamma z} \simeq g(z) = H_z(\mathbb{B})$ then we have

$$E[\alpha^n] = h_n(\mathbb{A}),$$

$$E[(k{\bullet}\alpha)^n] = h_n(k\mathbb{A})$$

and

$$E[(\alpha + \gamma)^n] = h_n(\mathbb{A} + \mathbb{B}).$$

[L] A. Lascoux, *Alphabet splitting*, in: Algebraic combinatorics and computer science, Springer Verlag, Italia, (2001), 431-444.

Summary

1. polynomials $A_n^{(k)}(x, \alpha) = x(x + k.\alpha)^{n-1}$ encode Lagrange inversion formula, for instance
$$A_n^{(k)}(\mathfrak{L}_\alpha) \simeq A_n^{(n+k)}(-1.\alpha),$$
2. cumulants are represented by $\alpha(\alpha - k.\alpha)^{n-1}$, with $k = 1, 2, n$,
3. convolutions are represented by umbrae $\alpha_{(k)}\gamma$ such that
$$A_n^{(k)}(\alpha_{(k)}\gamma) = A_n^{(k)}(\alpha) + A_n^{(k)}(\gamma).$$
4. umbrae encode the alphabet splitting.

Thanks

Thank you