Orthogonality of the Sheffer system associated to a Levy process

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Abstract

The aim of this paper is to relate some recent results on Lévy processes (see Schoutens and Teugels, 1998. Comm. Statist. Stochastic Models 14, 335–349) to a recent study of the author (1996) on multidimensional natural exponential families. In this way, we consider a natural construction of Sheffer polynomials associated to a d-dimensional Lévy process and we prove that this is the only one that leads to an orthogonal Sheffer system. It is also shown that the orthogonality occurs if and only if the Lévy process wanders throught the class of quadratic natural exponential families. Some interesting martingale properties are reviewed in a multidimensional setting. © 2000 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

Let us recall first what we call a Lévy–Sheffer system associated to a Lévy process. For an up-dated account of the theory of Lévy processes we refer the reader to Bertoin (1996). Let $(X_t)_{t\geqslant 0}$ be a stochastic process on $\mathbb{R}^d$ $(d\geqslant 1)$. We say that $(X_t)_{t\geqslant 0}$ is a *Lévy process* if for every $s,t\geqslant 0$, the increment $X_{t+s} - X_t$ is independent of the process $(X_k)_{0\leqslant k\leqslant t}$ and has the same law as X_s. For all $t\geqslant 0$, we will denote by μ_t the distribution of X_t on $\mathbb{R}^d$. In this way, we obtain what we shall call the *associated distributions* to the Lévy process. The Laplace transform of μ_t will be denoted by

$$L_{\mu_t}(\theta) = \int_{\mathbb{R}^d} \exp(\langle\theta,x\rangle)\mu_t(\mathrm{d}x), \tag{1}$$

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where $x = {}^t(x_1, \ldots, x_d), \theta = {}^t(\theta_1, \ldots, \theta_d)$ and $\langle \theta, x \rangle = \sum_{i=1}^d \theta_i x_i$. The *cumulant function* of μ_t is defined to be

$$k_{\mu_t}(\theta) = \log(L_{\mu_t}(\theta)). \tag{2}$$

We observe that each distribution μ_t is infinitely divisible and we have

$$k_{\mu_t}(\theta) = t k_\mu(\theta), \tag{3}$$

where $\mu = \mu_1$. The following assumptions will be needed throughout the paper and guarantees that the Lévy process has finite exponential moments: It is required that the infinitely divisible measure $\mu = \mu_1$ is not concentrated on an affine hyperplan and Θ_μ, the interior of the domain $\{\theta \in \mathbb{R}^d; k_\mu(\theta) < +\infty\}$, is not empty. We introduce now some multivariate notations. The *order* of $n = {}^t(n_1, \ldots, n_d) \in \mathbb{N}^d$ is, by definition, the integer $|n| = n_1 + \cdots + n_d$. Write

$$x^n = x_1^{n_1} \cdots x_d^{n_d}, \qquad n! = n_1! \cdots n_d!.$$

On $\mathbb{R}^d$, a polynomial of the $|n|$th degree may be written as

$$P_n(x) = \sum_{q \in \mathbb{N}^d, |q| \leqslant |n|} \alpha_q x^q,$$

where at least one of the reals α_q is non-zero when $|q| = |n|$. A family $(P_n)_{n \in \mathbb{N}^d}$ of such polynomials is *a polynomial sequence*.

Definition 1. A polynomial sequence is called a Sheffer system (see Sheffer, 1937) if there exists a neighborhood of $m = 0$, B, and two analytic functions $a : B \to \mathbb{R}^d$ and $b : B \to \mathbb{R}$, such that for all $m \in B$

$$\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_n(x) = \exp(\langle a(m), x \rangle) b(m). \tag{4}$$

For all $t \geqslant 0$, let $(P_{n,t})_{n \in \mathbb{N}^d}$ be a polynomial sequence.

Definition 2. The family of polynomials $(P_{n,t})_{n \in \mathbb{N}^d; t \geqslant 0}$ is called a Lévy–Sheffer system (see Schoutens and Teugels, 1998) if there exists a neighborhood of $m = 0, B$, such that for all $m \in B$ and for all $t \geqslant 0$

$$\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_{n,t}(x) = \exp\{\langle a(m), x \rangle - t k_\mu(a(m))\}, \tag{5}$$

where $a : B \to \mathbb{R}^d$ is analytic with $a(0) = 0$ and μ is an infinitely divisible distribution on $\mathbb{R}^d$.

One sees via (3) that a Lévy–Sheffer system is connected to a Lévy process with associated distributions $(\mu_t)_{t \geqslant 0}$. We are interested in finding the correspondance between such a Lévy–Sheffer system and the families of associated distributions. At first, in Section 2 we relate the construction of Lévy–Sheffer system with the theory of exponential families. In Section 3 we give a characterization of the pseudo-orthogonal

Lévy–Sheffer sytems. Section 4 is devoted to the study of the orthogonal systems and Section 5 provides a detailed exposition and extension of recent results related to the martingales theory. Sections 3 and 4 are adapted from Pommeret (1996) and we outline the proofs of Theorems 2 and 3, thus making our exposition self-contained.

2. Construction of Lévy–Sheffer systems

We introduce here the notion of natural exponential families (NEFs). The best general references here are Barndorff-Nielsen (1978) and Letac (1992). We denote by ψ_μ the inverse function of k'_μ. The family $F = F(\mu)$ of probabilities

$$P(m, F)(\mathrm{d}x) = f_\mu(x, m)\mu(\mathrm{d}x) \tag{6}$$

$$= \exp\{\langle \psi_\mu(m), x \rangle - k_\mu(\psi_\mu(m))\}\mu(\mathrm{d}x), \tag{7}$$

where $m = {}^{\mathrm{t}}(m_1, \ldots, m_d)$ belongs to the domain $M_F = k'_\mu(\Theta_\mu)$, is called the natural exponential family generated by μ. The variance V_F of $P(m, F)$ is considered as a function of m,

$$V_F(m) = \int_{\mathbb{R}^d} (x - m)^{\mathrm{t}}(x - m) f_\mu(x, m)\mu(\mathrm{d}x).$$

The natural exponential family F is said to be quadratic if the entries of the matrix V_F are second-order polynomials in m, and simple quadratic if the term of second order of the (i, j) entry is of the form $am_i m_j$, where a is a real constant independent of (i, j). The class of simple quadratic NEFs on $\mathbb{R}$ has been described by Morris (1982). Casalis (1996) has extended this classification on $\mathbb{R}^d$ and has split the simple quadratic NEFs into $2d + 4$ types: The $d + 1$ Poisson–Gaussian types (see also Letac, 1989) are composed by the distribution of d independent variables $(X_1, \ldots, X_d)$ where $X_1, \ldots, X_k$ have Poisson distribution and $X_{k+1}, \ldots, X_d$ are Gaussian variables with variance 1. The $d + 1$ negative multinomial gamma types are composed by the distribution of $(X_1, \ldots, X_d)$ where $(X_1, \ldots, X_k)$ has a negative multinomial distribution, the conditional variable $X_{k+1}|(X_1, \ldots, X_k)$ is gamma distributed with shape parameter $\sum_{i=1}^{k} X_i + 1$ and $(X_{k+2}, \ldots, X_d)|(X_1, \ldots, X_{k+1})$ is a Gaussian vector with variance $\mathrm{diag}(X_{k+1}, \ldots, X_{k+1})$. The hyperbolic type is composed by the distribution of $(X_1, \ldots, X_d)$ where $(X_1, \ldots, X_{d-1})$ has a negative multinomial distribution and $X_d|(X_1, \ldots, X_{d-1})$ has an hyperbolic cosine distribution with power convolution parameter $\sum_{i=1}^{d-1} X_i + 1$. The last type of simple quadratic NEFs is the classical multinomial type. Each type is composed by one NEF of the same name and its affinities and convolution powers, i.e. two families $F(\mu)$ and $F(v)$ are said to be of the same type if there exists an affinity $h : x \mapsto Ax + B$ and a positive real number λ such that $\mu = h(v^{*\lambda})$, where $*$ denotes the convolution product. In this case we have

$$F(\mu) = \{h(P(\theta, v^{*\lambda})); \theta \in {}^{\mathrm{t}}A^{-1}\Theta_v\}, \tag{8}$$

$$V_{F(\mu)} = \lambda A V_{F(v)}(A^{-1}(m/\lambda - B)) {}^{\mathrm{t}}A. \tag{9}$$

Remark 1. Among all the simple quadratic NEFs, only the multinomial distributions are not infinitely divisible, its set of convolution powers being $\mathbb{N}^*$.

Let $(\mu_t)_{t \geqslant 0}$ be the distributions on $\mathbb{R}^d$ associated to a Lévy process $(X_t)_{t \geqslant 0}$ with mean m_t, respectively. From (9) it is easily seen that

$$V_{F(\mu_t)}(m) = tV_{F(\mu)}(m/t). \tag{10}$$

To such a family of probability measures we associate a family of polynomials in d variables constructed from the Taylor expansion of the function $f_{\mu_t}(x,m)$ of the form (7). Let $\mathrm{GL}(\mathbb{R}^d)$ denote the set of invertible $d \times d$ matrices. For all $n \in \mathbb{N}^d$ and for all $A \in \mathrm{GL}(\mathbb{R}^d)$ we define on $\mathbb{R}^d \times M_{F_t}$

$$Q_{A,n,t}(x) = f_{\mu_t}^{(n)}(x, m_t)(Ae_1, \ldots, Ae_d), \tag{11}$$

where $f_{\mu_t}^{(n)}(x, m)(Ae_1, \ldots, Ae_d)$ is the $|n| = n_1 + n_2 + \cdots + n_d$ derivative of $m \mapsto f_{\mu_t}(x, m)$ in the $|n|$ directions Ae_1 (n_1 times),..., Ae_d (n_d times). By induction on $|n|$ it is easy to check that $Q_{A,n,t}$ is a polynomial in x of degree $|n|$ and the $(Q_{A,n})_{n \in \mathbb{N}^d}$ form a basis of $\mathbb{R}[x_1, \ldots, x_d]$. The following theorem expresses this basis of polynomials as a particular construction of Lévy–Sheffer system. We will see in Section 3 that this is the unique one leading to orthogonal Lévy–Sheffer system.

Theorem 1. *The polynomials* $(Q_{tA,n,t})_{n \in \mathbb{N}^d, t \geqslant 0}$ *form a Lévy–Sheffer system.*

Proof. From the analycity of ψ_{μ_t} and k_{μ_t} (see Letac, 1992) it follows that, for all m in an neighborhood of m_t, we have

$$\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} Q_{tA,n,t}(x) = f_{\mu_t}(x, tAm + m_t). \tag{12}$$

Since m_t is the first derivative of k_{μ_t} in $\theta = 0$, we have the following properties:

$$m_t = tm_1, \qquad \psi_{\mu_t}(m) = \psi_\mu(m/t). \tag{13}$$

Hence (5) occurs with $a(m) = \psi_\mu(Am + m_1)$ and $b(m) = \exp\{-k_{\mu_t}(a(m))\}$. $\square$

3. Characterization of Lévy–Sheffer systems

We introduce here two notions of orthogonality.

Definition 3. The polynomials $(P_n)_{n \in \mathbb{N}^d}$ are said to be μ-orthogonal (respectively μ-pseudo-orthogonal) if for all $n, k \in \mathbb{N}^d$ such that $n \neq k$ (respectively $|n| \neq |k|$) we have

$$\int_{\mathbb{R}^d} P_n(x) P_k(x) \mu(\mathrm{d}x) = 0.$$

The Lévy–Sheffer system given by (5) is said to be orthogonal (respectively pseudo-orthogonal) if the polynomials $(P_{n,t})_{n \in \mathbb{N}^d}$ are μ_t-orthogonal (respectively pseudo-ortho-gonal), for all $t \geqslant 0$. Under the assumption of pseudo-orthogonality, the following theorem offers an intrinsic construction of the Lévy–Sheffer systems.

Theorem 2. *For all* $t \geqslant 0$, *let the polynomial sequence* $(P_{n,t})_{n \in \mathbb{N}^d}$ *be a* μ_t-*pseudo-orthogonal basis of* $\mathbb{R}[x_1, \ldots, x_d]$. *Then the two following assertions are equivalent*:
(i) *The polynomials* $(P_{n,t})$ *form a Sheffer system.*
(ii) *There exists* $A \in \mathrm{GL}(\mathbb{R}^d)$ *such that* $P_{n,t} = Q_{tA,n,t}$, *for all* $(n,t) \in \mathbb{N}^d \times \mathbb{R}^+$.

Proof. The proof is similar to Pommeret (1996). We only give the main ideas. Theorem 1 shows that (ii) implies (i). Suppose that (i) occurs. From Pommeret (1996, Lemma 2.5.), there exists a neighborhood of 0, B, such that for all $m \in B$

$$\int \left(\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_{n,t}(x) \right) \mu_t(\mathrm{d}x) = \sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} \int_{\mathbb{R}^d} P_{n,t}(x) \mu_t(\mathrm{d}x)$$

$$= \int P_0(x) \mu_t(\mathrm{d}x) = 1.$$

In addition, from (4) we obtain

$$\int \left(\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_{n,t}(x) \right) \mu_t(\mathrm{d}x) = \int \exp(\langle a(m), x \rangle) b(m) \mu_t(\mathrm{d}x)$$

$$= \exp\{k_{\mu_t}(a(m))\} b(m). \tag{14}$$

It follows that $b(z) = \exp\{-k_{\mu_t}(a(m))\}$. Letting $\mathscr{P}_1(x) = {}^{\mathrm{t}}(P_{e_1,t}(x), \ldots, P_{e_d,t}(x))$, and proceeding in a similar manner to that above, we obtain the following vectorial equality:

$$\int \left(\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_{n,t}(x) \right) \mathscr{P}_1(x) \mu_t(\mathrm{d}x) = \left(\int \mathscr{P}_1(x) {}^{\mathrm{t}}\mathscr{P}_1(x) \mu_t(\mathrm{d}x) \right) m. \tag{15}$$

Since $\mathscr{P}_1(x)$ is a vector of polynomials of degree 1, there exists $\tilde{A} \in \mathrm{GL}(\mathbb{R}^d)$ and $\beta \in \mathbb{R}^d$ such that

$$\mathscr{P}_1(x) = \tilde{A}x + \beta. \tag{16}$$

Since

$$\int \mathscr{P}_1(x) \mu_t(\mathrm{d}x) = \int \mathscr{P}_1(x) P_0(x) \mu_t(\mathrm{d}x) = 0$$

we have that $\beta = -\tilde{A}m_t$ and

$$\int \mathscr{P}_1(x) {}^{\mathrm{t}}\mathscr{P}_1(x) \mu_t(\mathrm{d}x) = \int \tilde{A}(x - m_t) {}^{\mathrm{t}}(x - m_t) {}^{\mathrm{t}}\tilde{A} \mu_t(\mathrm{d}x) \tag{17}$$

$$= \tilde{A} V_{F_t}(m_t) {}^{\mathrm{t}}\tilde{A}, \tag{18}$$

where $F_t = F(\mu_t)$. Combining (15) with (16) and (18) yields

$$\tilde{A} V_{F_t}(m_t) {}^{\mathrm{t}}\tilde{A}m = \tilde{A}(k'_{\mu_t}(a(m)) - m_t).$$

It follows that

$$a(m) = \psi_{\mu_t}(V_{F_t}(m_t) {}^{\mathrm{t}}\tilde{A}m + m_t)$$

$$= \psi_\mu(V_{F_1}(m_1) {}^{\mathrm{t}}\tilde{A}m + m_1).$$

From this we can rewrite (5) as

$$\sum_{n \in \mathbb{N}^d} \frac{m^n}{n!} P_{n,t}(x) = f_{\mu_t}(x, tV_{F_1}(m_1)^{\mathrm{t}}\tilde{A}m + tm_1).$$

Then, setting $A = V_{F_1}(m_1)^{\mathrm{t}}\tilde{A}$, we have

$$P_{n,t}(x) = t^{|n|} f_{\mu_t}^{(n)}(x, m_t)(Ae_1, \ldots, Ae_d)$$

$$= Q_{tA,n,t}(x). \quad \square$$

4. Orthogonal Lévy–Sheffer systems

We are now interested in finding for which measures μ_t on $\mathbb{R}^d$ the orthogonality of the Lévy–Sheffer polynomials occurs. Since we restrict our attention to infinitely divisible measures, we omit the multinomial distributions from the simple quadratic natural exponential families (see Remark 1). However, a similar result of Theorem 3(ii) occurs in the multinomial case. The following result is an extension of Feinsilver's (1986).

Theorem 3. *Let* $(P_{n,t})_{n \in \mathbb{N}^d, t \geq 0}$ *be the Lévy–Sheffer system defined by* (5). *Then we have the following two assertions*:

(i) *The Lévy–Sheffer system is pseudo-orthogonal if and only if* $F(\mu)$ *is quadratic and there exists* $(A, b) \in \mathrm{GL}(\mathbb{R}^d) \times \mathbb{R}^d$ *such that* $a(m) = \psi_\mu(Am + b)$.

(ii) *Assume that the NEF* $F(\mu)$ *is not the product of two NEFs on* $\mathbb{R}^{d-k} \times \mathbb{R}^k$. *Then the Lévy–Sheffer system is orthogonal if and only if* $F(\mu)$ *is simple quadratic and there exists* $(A, b) \in \mathrm{GL}(\mathbb{R}^d) \times \mathbb{R}^d$ *such that* $a(m) = \psi_\mu(Am + b)$ *and* $A^{-1}V_{F(\mu)}(m_1)^{\mathrm{t}}A^{-1}$ *is diagonal.*

Proof. We content ourselves only with showing part (ii) since part (i) follows this proof. Assume the polynomials $P_{n,t}$ are μ_t-orthogonal. From Theorem 2 there exist $A = V_{F_1}(m_1)\tilde{A}$, where $\tilde{A}$ has the form (18), such that $P_{n,t} = Q_{tA,n,t}$ and $a(m) = \psi_\mu(Am + m_1)$. We have

$$A^{-1}V_{F_1}(m_1)^{\mathrm{t}}A^{-1} = (\tilde{A}V_{F_1}(m_1)^{\mathrm{t}}\tilde{A})^{-1}$$

$$= \left(\int \mathscr{P}_1(x)^{\mathrm{t}}\mathscr{P}_1(x)\mu_1(\mathrm{d}x) \right)^{-1}$$

and from the μ-orthogonality of the polynomials $(P_{n,1})_{n \in \mathbb{N}^d}$, the above positive definite matrix is diagonal. To show that F is simple quadratic, we may suppose $A = I$ (the identity), (9) giving the general case. Write $P_n = Q_{I,n,t}$ (for short). Writting $k'_\mu(\theta) = m_1$, from (14) we have

$$\exp\{k_\mu(\psi_\mu(m) + \psi_\mu(z)) - k_\mu(\psi_\mu(m)) - k_\mu(\psi_\mu(z))\}$$

$$= \sum_{n \in \mathbb{N}^d} \frac{(m - m_1)^n (z - m_1)^n}{(n!)^2} \int P_{n,t}^2(x)\mu(\mathrm{d}x). \tag{19}$$

Differentiating two times (19) with respect to m and taking $m = m_1$ yields F simple quadratic.

Conversely, note first that from (10) if F_1 is simple quadratic then F_t is simple quadratic for all $t \geqslant 0$. Thus we content to show that the polynomials $P_n = P_{n,1}$ are μ-orthogonal. Write $\theta = \psi_\mu(m_1)$. We have

$$\exp\{\langle \theta, x \rangle\} = \sum_{n \in \mathbb{N}^d} \frac{(k'_\mu(\theta) - m_1)^n}{n!} P_n(x) \exp\{k_\mu(\theta)\}.$$

By differentiating this equality with respect to θ and by identification we obtain for all $i = 1, \ldots, d$,

$$x_i P_n(x) = \sum_{|n|-1 \leqslant |q| \leqslant |n|+1} \alpha_{q,i} P_q(x), \tag{20}$$

where $(\alpha_{q,i})_{q \in \mathbb{N}^d} \in \mathbb{R}$. By induction on $|n|$, (7) and (20) imply the following three results (see Pommeret, 1996):

1. $\forall n \in \mathbb{N}^d \setminus \{0\}$, $\int P_n(x) \mu(\mathrm{d}x) = 0$.
2. There exists $(\beta^n_{l,q}) \in \mathbb{R}$ such that for all $q \in \mathbb{N}^d$, if $|q| < |n|$, then

$$x^q P_n(x) = \sum_{|n|-|q| \leqslant |l| \leqslant |n|+|q|} \beta^n_{l,q} P_l(x).$$

3. Write $V_{F_1}(m_1) = \mathrm{diag}(v_1, \ldots, v_d)$. There exists $(\varepsilon_{q,n}) \in \mathbb{R}$ such that

$$P_n(x) = \varepsilon_{n,n} x^n + \sum_{q; |q| < |n|} \varepsilon_{n,q} x^q,$$

where $\varepsilon_{n,n} = (v_1^{n_1} \cdots v_d^{n_d})^{-1}$.

From these three above equalities, if $|n| \neq |q|$, then $\int P_n(x) P_q(x) \mu(\mathrm{d}x) = 0$. $\quad \square$

5. Applications and examples

The characterization of orthogonal Sheffer system on $\mathbb{R}$ is due to Meixner (1934) (see Morris (1982) for a treatment via the exponential families) and for a recent account of the theory of orthogonal Lévy–Sheffer system on $\mathbb{R}$ we refer the reader to Schoutens and Teugels (1998). As for prerequisites, the reader is expected to be familiar with some classical polynomials on $\mathbb{R}$ which are described in the Koekoek and Swarttouw (1994) report. We use the notation of Koekoek and Swarttouw. Following Schoutens and Teugels, we relate our results to the theory of martingales associated with Lévy–Sheffer processes. For this purpose, we recall here an important property of the orthogonal Lévy–Sheffer system $(P_{n,t})_{n \in \mathbb{N}^d, t \geqslant 0}$ (see Schoutens and Teugels):

$$\forall t > s \geqslant 0, \quad \mathbb{E}(P_{n,t}(X_t) | X_s) = P_{n,s}(X_s), \tag{21}$$

where $(X_t)_{t \geqslant 0}$ is the Lévy process with the associated distributions $(\mu_t)_{t \geqslant 0}$. In the remainder of this section we have compiled a few examples of orthogonal Lévy–Sheffer

system on $\mathbb{R}^d$ (see Pommeret (1996) for the case $d=2$, the other ones are in a submitted paper). For each one we give the form of the variance function of the simple quadratic natural exponential family (see Casalis, 1996 for more details) and we consider some particular values of m_1 to apply (21). From now on, $X_t = (X_t^{(1)}, \ldots, X_t^{(d)})$ denotes the random vector with distribution μ_t and we will denote by $m_t = (m_t^{(1)}, \ldots, m_t^{(d)})$ its mean. It is a simple matter to apply (21) and we only describe the polynomials $P_{n,t}$. The Pochhammer symbol is defined by

$$(a)_0 = 1 \quad \text{and} \quad (a)_k = a(a+1)(a+2)\cdots(a+k-1).$$

The family of negative binomial distributions on $\mathbb{R}^2$: Let $F = F(\mu)$ be the negative multinomial family on $\mathbb{R}^2$ with variance function

$$V_F(m) = am^t m + \begin{pmatrix} m^{(1)} & 0 \\ 0 & m^{(2)} \end{pmatrix},$$

where $a > 0$. If $m_1^{(1)} = 1/a$ and $m_1^{(2)} = (\sqrt{5}-1)/a$, then the orthogonal Lévy–Sheffer polynomials are products of Meixner polynomials:

$$P_{n,t}(x) = (x^{(1)}+p)_{n_1}(n_1+p)_{n_2}(c_1)^{n_1/2}(-1/\sqrt{2p})^{|n|}M_{n_1}^{c_1,\beta_1}(x^{(2)})M_{n_2}^{c_2,\beta_2}(x^{(1)}),$$

where $p = t/a$; $2c_1 = 3 - \sqrt{5}$; $\beta_1 = x^{(1)}$; $c_2 = \frac{1}{2}$ and $\beta_2 = n_1 + p$.

The family of negative binomial hyperbolic distributions on $\mathbb{R}^2$: Let $F = F(\mu)$ be the negative binomial hyperbolic family with variance function

$$am^t m - \begin{pmatrix} m^{(1)} & m^{(2)} \\ m^{(2)} & -m^{(1)} \end{pmatrix},$$

where $a > 0$. If $m_1 = 1/a(2,0)$ then the orthogonal Lévy–Sheffer system is a mixing of Meixner and Meixner–Pollaczek polynomials:

$$P_{n,t}(x) = (x^{(1)})_{n_2}(p+n_2)_{n_1}(-1)^{n_1}(1/\sqrt{2p})^{|n|}P_{n_2}^{\lambda}(x^{(2)}/2; \pi/2)M_{n_1}^{c_1,\beta_1}(x^{(1)}-p),$$

where $p = t/a$; $\lambda = x^{(1)}/2$; $c_1 = 1/2$; $\beta_1 = n_2 + p$; and P_{n_2}, M_{n_1} denote Pollaczek and Meixner polynomials, respectively.

The Poisson–Gaussian family on $\mathbb{R}^d$: Let $F = F(\mu)$ be the family of Poisson–Gaussian distributions with variance function

$$V_F(m) = \text{diag}(m^{(1)}, \ldots, m^{(k)}, 1, \ldots, 1).$$

If $m_1^{(j)} = 1$ for $j = 1, \ldots, k$ and if $m_1^{(j)} = 0$ for $l = k+1, \ldots, d$, then the orthogonal Lévy–Sheffer system is given by

$$P_{n,t}(x) = t^{|n|} \prod_{i=1}^{k}(-1)^{n_i}C_{n_i}^{1}(x^{(i)}) \prod_{j=k+1}^{d}(1/\sqrt{2})^{n_j}H_{n_j}(\sqrt{2}x^{(j)}),$$

where C and H denote Charlier and Hermite polynomials, respectively.

The gamma Gaussian family on $\mathbb{R}^d$: Let $F = F(\mu)$ be the gamma Gaussian family with variance function

$$V_F(m) = am^t m + \operatorname{diag}(0, m^{(1)}, \ldots, m^{(d)}),$$

where $a > 0$. Let $m_1 = (1/a, 0, \ldots, 0)$, then the orthogonal Lévy–Sheffer system is

$$P_{n,t}(x) = (-a/t)^{n_1}(n_1)! L_{n_1}^{\alpha_t}(x^{(1)}) \prod_{i=2}^{d} \left(\frac{a\sqrt{x^{(1)}}}{t\sqrt{2}} \right)^{n_i} H_{n_i} \left(\frac{x^{(i)}}{\sqrt{2x^{(1)}}} \right),$$

where $\alpha_t = n_2 + \cdots + n_d - 1 + t/a$; and L, H denote Laguerre and Hermite polynomials.

The negative multinomial gamma family on $\mathbb{R}^d$: Let $F = F(\mu)$ be the negative multinomial gamma family with variance function

$$V_F(m) = am^t m + \operatorname{diag}(m^{(1)}, \ldots, m^{(k)}, 0, m^{(k+1)}, \ldots, m^{(k+1)})$$

with $a > 0$. If $m_1 = (1/a, \ldots, 1/a)$, then the orthogonal Lévy–Sheffer system is given by a product of Laguerre and Charlier polynomials:

$$P_{n,t}(x) = (-ax^{(d)}/t)^{|n|} L_{n_d}^{\alpha_t}(x^{(d)}) \prod_{i=1}^{d-1} C_{n_i}^{x^{(d)}}(x^{(i)}),$$

where $\alpha_t = n_1 + \cdots + n_{d-1} - 1 + t/a$.

Acknowledgements

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