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THE CLASSICAL UMBRAL CALCULUS: READING BLISSARD WITH THE KEY GIVEN BY G. C. ROTA AND B. D. TAYLOR

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Abstract

A study of Blissard's first paper is presented using the foundations of umbral calculus given by Rota and Taylor.

1. Introduction

In 1994, G. C. Rota and B. D. Taylor, in SIAM J. Math. Anal. [19] published a rigorous presentation of the umbral calculus, as formerly applied heuristically by Blissard, Bell, Riordan and others. We were at that time studying operators that commute with differentiation (and with translation) and came in contact with the umbral calculus, something totally unknown to us, by chance; trying to understand the meaning of it, we soon saw that the question was not easy, by any means. Looking in the literature, we found a vast amount of papers related to the subject but they did not help much, in the beginning, to clarify the meaning of that esoteric calculus but; precisely, this initial difficulty put us on the track to "discovery" and, little by little, some light appeared; this light was, mainly, the work, previously mentioned, by Rota and Taylor. At first, the

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necessity of introducing the strange definition given by them seemed to us incomprehensible but we decided to follow the advice of the authors and reread the classics with this key in hand and see the result; to decide who were the classics was not an easy matter either but after looking through the references we thought the best candidate, to start with, was the Rev. John Blissard. It seems that he is today acknowledged as the founder of the umbral calculus (he called it *representative notation* or *symbolic method*) even though, for some time, as far as we know (we are following E. T. Bell here [2]) the credit was given to E. Lucas [13,14,15]. Blissard wrote between the years 1862 and 1868 several papers, all published in the same journal, in fact the Quart. J. Pure Appl. Math (as Rota and Taylor say "the editors of the journal must be given credit for publishing papers that thoroughly lacked in rigor but relied on a suggestive and often powerful notation"); by the way, it would be interesting to know what would happen today in a situation like this.

In this article, therefore, we go through the first of the papers published by Blissard, precisely the one titled "Theory of generic equations" [4], examine it carefully with the foundations of umbral calculus given in [19] and see the outcome. What follows is an exposition of this rewarding task with the difficulties encountered and the clarifications due to Rota and Taylor.

2. Theory of Generic Equations: Definitions and First Principles

The basic concepts, "generic equation" and "quantities of two sorts, actual and representative", are introduced in paragraphs 1 and 4 of Chapter I of [4]; a summary of it, following the original notation and taking out the paragraphs not relevant, is reproduced here to make easier the reading of this article.

1. Let $U_1, U_2, \ldots, U_n$ form a class of *quantities* which are either determinable from, or connected by, a general law of relation, then the equation which expresses that general law is called a *Generic Equation*.

2. If this equation is expressed in finite terms, the values of $U_1, U_2, \ldots, U_n$, & c. can be determined from it.

Example 1. $B_1, B_2, \ldots, B_n$, & *c.* are the numbers of Bernoulli obtained from the generic equation

$$nB_{n-1} + \frac{n(n-1)}{1.2} B_{n-2}, \,\&\, c. + nB_1 + 1 = 0, \quad n > 1.$$

Example 2. Consider the numbers $A_1, A_2, \ldots, A_n$ & *c.* given by the generic equation

$$2A_n + nA_{n-1} + \frac{n(n-1)}{1.2} A_{n-2}, \,\&\, c. + nA_1 + 1 = 0$$

that are connected with Bernoulli's numbers by

$$-A_n = \frac{2^{n+2} - 2}{n+1} B_{n+1}.$$

3. If $U_1, U_2, \ldots, U_n$ form a class of *functions* which are connected by a known general law, the equation which expresses that law is of the nature of a Generic Equation, and can be operated upon by the method adopted in this theory.

4. According to this method, quantities are considered as divided into two sorts, actual and representative. A representative quantity, indicated by the use of a capital letter without a subindex, as $A, B, \ldots, P, Q, \ldots, U, V, \ldots$ is such that U^n is conventionally held to be equivalent to, and may be replaced by U_n. All quantities not denoted by capital letters are actual quantities.

5. By this notation, the Generic Equation for Bernoulli's numbers, viz.

$$nB_{n-1} + \frac{n(n-1)}{1.2} B_{n-2}, \,\&\, c. + nB_1 + 1 = 0, \quad n > 1,$$

is expressed by

$$(1 + B)^n - B^n = 0, \quad n > 1,$$

and the Generic Equation for the A numbers, viz.,

$$2A_n + nA_{n-1} + \frac{n(n-1)}{1.2} A_{n-2}, \,\&\, c. + nA_1 + 1 = 0$$

by

$$(A+1)^n + A^n = 0, \quad n > 0.$$

6. It is here of great importance to remark that an equation that involves a representative quantity is from the definition, and, as the nature of the case, not , susceptible of any algebraic operation, by which the indices of that quantity would be affected. Thus $\sin(B\theta) = -\frac{1}{2}\theta$, but it does not follow that $\sin^2(B\theta) = \frac{1}{4}\theta^2$. In fact, $\sin^2(B\theta) = \frac{1-\cos 2B\theta}{2}$ will be shown to be $\frac{1-\theta\cot\theta}{2}$. Such equations therefore will only admit of transposition of terms, or development of functions, or the affection of all the terms by coefficients not involving the representative quantity, or the processes of differentiation and integration with respect to some actual and not representative quantity. The only difficulty in the use of the present notation, consists in a careful observance of this caution.

7. A leading object of the present theory is to obtain Generic Equations in a form adapted to the application of Taylor's theorem. This is the case with the Generic Equations of Art. 6. Thus it is easy to see that from

$$(A+1)^n + A^n = 0, \quad n > 0,$$

we obtain

$$f\{x+(A+1)\theta\} + f\{x+A\theta\} = 2f(x)$$

and from

$$(1+B)^n - B^n = 0, \quad n > 1,$$

we obtain

$$f\{x+(B+1)\theta\} - f\{x+B\theta\} = \theta\frac{df(x)}{dx}.$$

3. Reading Blissard with the Outlook of Rota and Taylor

It is not easy to understand the meaning of the previous paragraph; mainly the definition given in 4, the formulas of 5 and the caution given in 6 give the sensation of something weird going on. The peculiar thing is that the calculations obtained in Chapter II of [4] are all valid and easily deduced as the author claims. Let us review paragraphs 4, 5, 6 and 7 with the aid of the foundations given by Rota and Taylor in hand [19]; (note that the theory of Bernoulli numbers is developed in umbral calculus in [19]).

What are the representative quantities? This is clear; precisely the elements of an alphabet, the set A [19], whose elements are called *letters* or *umbrae*. The actual variables take values on C, (in [19], a commutative integral domain D, whose quotient field is of characteristic zero is considered).

The convention adopted in paragraph 4 can be explained (with caution) using the linear functional **eval**, "evaluation", defined on the polynomial ring $D[A]$ and taking values in D. The example given in 4, that is,

$$e^{U\theta} = 1 + U\theta + \frac{U^2\theta^2}{1.2} + \cdots + \frac{U^n\theta^n}{1.2...n} \ \&\ c.$$

that becomes

$$1 + U_1\theta + \frac{U_2\theta^2}{1.2} + \cdots + \frac{U_n\theta^n}{1.2...n} \ \&\ c.$$

can be considered as a linear functional acting on $D[A]$ if we understand the previous substitution for every n, that is,

$$\mathbf{eval}\left(\sum_{k=0}^{n} \frac{1}{k!} U^k\theta^k\right) = \sum_{k=0}^{n} \frac{1}{k!} U_k\theta^k, \quad \text{for all } n$$

and U is the *umbra* that represents the sequence $U_1, U_2, \ldots, U_n$, (or, (U_i) is umbrally represented by the umbra U) because

$$\mathbf{eval}(U^n) = U_n, \quad n = 0, 1, 2\ldots.$$

Necessarily $\mathbf{eval}(U^0) = U_0 = 1$, for any umbra U.

$p = \sum_{k=0}^{n} \frac{1}{n!} U^k \theta^k$ is an element of $D[A]$, umbral polynomial (following [19]) with support equal to $\{U\}$. Besides $p = \sum_{k=0}^{n} \frac{1}{n!} U^k \theta^k$ is umbrally equivalent to $\sum_{k=0}^{n} \frac{1}{k!} U_k \theta^k$.

Consider, now, the formulas given in 5. The one concerning the Bernoulli numbers, that is,

$$(1 + B)^n - B^n = 0, \quad n > 1,$$

means

$$\mathbf{eval}(1 + B)^n = \mathbf{eval}\, B^n, \quad n > 1$$

but $\mathbf{eval}(1 + B)^n$ has to be computed carefully if the Generic Equation for Bernoulli's numbers is to be found; precisely, how does the linear functional **eval** work in products? Is $\mathbf{eval}\, B^n = \mathbf{eval}(B^j)\mathbf{eval}(B^k)$, $j + k = n$? The answer is no as is, explicitly, indicated in [19]; in fact, if $\alpha, \beta, \ldots, \gamma$ are distinct umbrae and if $i, j, \ldots, k$ are nonnegative integers, then

$$\mathbf{eval}(\alpha^i \beta^j \ldots \gamma^k) = \mathbf{eval}(\alpha^i)\mathbf{eval}(\beta^j)\ldots\mathbf{eval}(\gamma^k).$$

The same remark applies to

$$(A + 1)^n + A^n = 0, \quad n > 0.$$

Using the relation $\simeq$ (umbrally equivalent, see [19]) the previous formulas read

$$(1 + B)^n \simeq B^n, \quad n > 1,$$

and

$$(A + 1)^n \simeq -A^n = 0, \quad n > 0$$

while with the definition of exchangeability, in symbols $\equiv$ (see [19]), we have

$$1 + B \equiv B$$

and

$$A + 1 \equiv -A.$$

Now the caution given in paragraph 6 makes sense; the representative quantities can be managed with the restriction imposed by the properties of the linear functional **eval**. It is interesting, though, to compute the formula

$$\sin^2(B\theta) = \frac{1 - \theta \cot \theta}{2}$$

which will be done later on.

Concerning paragraph 7, that is to obtain Generic Equations "suitable" to the application of Taylor's theorem, the first thing that one is uncomfortable about is the introduction of series without any notion of convergence and the application of algebraic formulas to processes that seem to be, at first, infinite ones. Looking at it more carefully we see that the formulas

$$(1 + B)^n \simeq B^n, \quad n > 1 \quad \text{and} \quad (A + 1)^n \simeq -A^n = 0, \quad n > 0$$

can be generalized to

$$f\{x + (B + 1)\theta\} - f(x + B\theta) \simeq \theta \frac{df(x)}{dx}$$

and

$$f\{x + (A + 1)\theta\} - f(x + A\theta) \simeq 2f(x)$$

meaning for the first one (with $\theta = 0$) the following algebraic equations (assuming that f is a C^∞ function in the "actual variable x")

$$f(x) - f(x) = 0$$

$$(B + 1)\frac{df(x)}{dx}\theta - B\frac{df(x)}{dx}\theta = \theta\frac{df(x)}{dx} \quad (B + 1 - B = 1)$$

$$(B+1)^2 \frac{d^2 f(x)}{dx^2}\frac{\theta^2}{2!} - B^2 \frac{d^2 f(x)}{dx^2}\frac{\theta^2}{2!} = 0 \quad ((B+1)^2 \simeq B^2)$$

....

$$(B+1)^n \frac{d^n f(x)}{dx^n}\frac{\theta^n}{n!} - B^n \frac{d^n f(x)}{dx^n}\frac{\theta^n}{n!} = 0 \quad ((B+1)^n \simeq B^n) \ \text{ for all } n > 1.$$

A similar set of equations for the second formula concerning the numbers represented by A can be written.

Remark 3.1. Note formula (2) of [19] using the augmentation umbra ε that verifies

$$\mathbf{eval}(\varepsilon^n) = \delta_{n,0},$$

where δ is the Kronecker delta. Using this umbra ε two umbrae α and β are said to be *inverse* [19] (concept used but not explicitly mentioned in [4]) when $\alpha + \beta \equiv \varepsilon$, that is,

$$\mathbf{eval}(\alpha+\beta)^n = \mathbf{eval}(\varepsilon^n)$$

but how do α and β behave when the computation is done ? For $n = 1$, we have

$$\mathbf{eval}(\beta) = -\mathbf{eval}(\alpha)$$

but for $n = 2$

$$\mathbf{eval}(\alpha+\beta)^n = \mathbf{eval}(\alpha^2 + \beta^2 + 2\alpha\beta) \neq \alpha_2 + \beta_2 - 2\alpha_1\beta_1,$$

where α_n and β_n are the sequences represented by the umbrae α and β (easily checked taking α the umbra representing the Bernoulli numbers). Therefore

$$\mathbf{eval}(2\alpha\beta) = -2\alpha_2$$

or, generally

$$\mathbf{eval}(\alpha^n\beta^m) = (-1)^m \alpha_{n+m}.$$

4. On Developments

In the Chapter II of his work Blissard treat principally on the properties of numbers derived from Generic Equations and their application in the expansion of various functions. The A and B numbers are considered first and then some others, called by the *author* U and V, which are not necessary for our purposes. We are mainly concerned with the introduction in [19] of the "saturated umbral calculi"; reading the mentioned paper the need of that esoteric definition seemed to us quite mysterious; of course we thought that there must have been an explanation for the introduction of it but we could not find it. Again the advice given in [19] was sound; reading Blissard, the explanation looked for was found as we will shown later after making some computations to show how the method's work and it makes "complete sense in retrospect" [19].

Take $f(x) = e^x$ in $f\{x + (B+1)\theta\} - f(x + B\theta) \simeq \theta \dfrac{df(x)}{dx}$; then we have

$$e^{\{x+(B+1)\theta\}} - e^{x+B\theta} \simeq \theta e^x$$

which implies (for $x = 0$)

$$e^{(B+1)\theta} - e^{B\theta} \simeq \theta$$

but, as e^θ behaves as an "actual variable", we get

$$(e^\theta - 1)e^{B\theta} \simeq \theta$$

which means the following algebraic equations

$$\sum_{j+k=n} \frac{1}{(j+1)!}\frac{1}{k!}B_k = 0, \quad n \neq 1$$

$$= 1, \quad n = 1$$

expressed by the author (Rota and Taylor use the symbol $\simeq$)

$$e^{B\theta} = \frac{\theta}{e^\theta - 1}.$$

Changing θ by $-\theta$ in $(e^{\theta}-1)e^{B\theta} \simeq \theta$ results

$$(e^{-\theta}-1)e^{-B\theta} \simeq -\theta,$$

that is,

$$\sum_{j+k=n} \frac{(-1)^{j+1}}{(j+1)!}\frac{(-1)^k}{k!} B_k = 0, \quad n \neq 1$$

$$= -1, \quad n = 1$$

or

$$e^{-B\theta} = \frac{\theta e^{\theta}}{e^{\theta}-1}$$

finding

$$B_1 = -\frac{1}{2}, \quad B_3 = 0 = B_5 = \ldots B_{2n+1}.$$

Hence

$$\frac{\theta}{e^{\theta}-1} = 1 + B_1\theta + B_2\frac{\theta^2}{1.2} + B_4\frac{\theta^4}{1.2.3.4} + \cdots + B_{2n}\frac{\theta^{2n}}{1.2...2n} + \cdots$$

showing the methods' beauty.

A few words concerning the result stated in paragraph 6 of [4] seem now to be in order; precisely the formula

$$\sin^2 B\theta = \frac{1-\theta\cot\theta}{2}.$$

It is easily deduce (and fundamentally right with the observations made above) and done in paragraph 15 of [4] that $\sin B\theta = -\frac{1}{2}\theta$ and $\cos B\theta = \frac{1}{2}\theta\cot\frac{1}{2}\theta$. In fact

$$\cos(B+1)\theta - \cos B\theta = 0$$

and similarly

$$\sin(B+1)\theta - \sin B\theta = \theta$$

from the generic equation

$$(B+1)^n - B^n = 0, \quad n > 1.$$

Therefore $\cos 2B\theta = \theta \cot \theta$ as the "actual" variable θ is substituted by 2θ; from the formula

$$\sin^2 B\theta = \frac{1 - \cos 2B\theta}{2}$$

(true because needs only the linearity of the evaluation operator; it is enough to use the Taylor's series of $\sin^2 x$ and $\cos x$, the trigonometric formula

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

the generic equation for the Bernoulli numbers and the substitution B^n by B_n) it follows the result. On the other hand we can not say that

$$\sin^2 B\theta = \sin B\theta. \sin B\theta = \frac{1}{4}\theta^2$$

as this operation involves the multiplication properties of the evaluation operator.

The introduction of the "saturated umbral calculi" is absolutely natural looking at Example 6 of paragraph 21 of [4] but before doing that let us go through Example 4 of the same paragraph as explanation of certain "strange manipulations" that will be needed, namely the changing of an "actual variable" for a "representative variable".

From

$$f\{x + (A+1)\theta\} + f\{x + A\theta\} = 2f(x)$$

we have for $\theta = 1$

$$f\{x + (A+1)\} + f\{x + A\} = 2f(x)$$

and for $f(x) = x^n$

$$(x + A + 1)^n + (x + A)^n \doteq 2x^n$$

and so

$$(-B + A + 1)^n + (-B + A)^n = 2(-B)^n$$

as

$$\sum_{k=0}^{n}\binom{n}{k}x^k(A+1)^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k(A)^{n-k} = 2x^n$$

valid for $x = -B_1$, $x^2 = B_2$, $x^3 = -B_3$... which amounts to a double use of the linear operator **eval** (change of subscripts by powers) applied to the umbrae A and B and therefore (by the usual procedure)

$$f\{x + (-B + A + 1)\theta\} + f\{x + (-B + A)\theta\} = 2f(x - B\theta)$$

for which $\theta = 1$ writes

$$f\{x + (-B + A + 1)\} + f\{x + (-B + A)\} = 2f(x - B)$$

and can be looked upon as a substitution of x for $x - B$, (observe that the umbra $-B$ denotes an inverse umbra to B and so **eval**$(-B)^n = (-1)^n B_n$, $n > 1$).

Consider now the formula of Example 6

$$f\{x + (mB + 1)\theta\} + f\{x + (mB + 2)\theta\} + \cdots + f\{x + (mB + m)\theta\}$$
$$= mf\{x + (B + 1)\theta\}$$

deduced in paragraph 19 of Blissard's article; letting θ take the value 1 it follows

$$f\{x + (mB + 1)\} + f\{x + (mB + 2)\} + \cdots + f\{x + (mB + m)\} = mf\{x + (B + 1)\}$$

and substituting, as it was shown before, x for $x + mA$ we get

$$f(2mB + 1 + x) + f(2mB + 2 + x) + \cdots + f(2mB + m + x)$$
$$= mf(mA + B + 1 + x)$$

remembering that $(A + B)^n = (2B)^n$. Therefore putting successively the quantities $x + 2A,\ x + 2^2 A,\ x + 2^3 A, \ldots, x + 2^{n-1} A$ in the previous formula we obtain

$$f(m.2^n B + 1 + x) + f(m.2^n B + 2 + x) + \cdots + f(m.2^n B + m + x)$$

$$= mf\{(A + 2A + \cdots + 2^{n-1} A) + 1 + B + x\},$$

where the representative quantities $A, 2A, 2^2 A, \ldots, 2^{n-1} A$, all receive, in words of the author, a separate development. What does it mean, a separate development? To illustrate the situation Blissard gives the following example; take $f(x) = x^r$, $m = 2$, $n = 2$, $x = -1$. Then

$$(8B)^r + (8B + 1)^r = 2\{B + 2(A + 2A)\}^r$$

and if $r = 2$ it follows

$$21 - 8 + 1 = 24\frac{1}{2^2} + 32\frac{1}{2^2}$$

which is true but if we sum the variables we get

$$128\frac{1}{6} + 16\left(-\frac{1}{2}\right) + 1 = 2\left(\frac{1}{6} + 12\frac{1}{4}\right)$$

which is not true.

The answer is, of course, that they "behave as different variables" and that, we suppose, took Rota and Taylor to the definition of $n.\alpha$, for every umbra α and for every positive integer n (called an *auxiliary umbra*, that is exchangeable with the sum

$$\alpha' + \alpha'' + \cdots + \alpha''',$$

where $\alpha', \alpha'', \ldots, \alpha'''$ are a set of n distinct umbrae, each of which is exchangeable with the given umbra α) (in an analogous way they introduce $-n.\alpha$ and $0.\alpha$) and the definition of saturated umbral calculi (3.1 of [19]). The properties of this "scalar multiplication" $n.\alpha$ are given in Proposition 3.2 of [19].

As stated in Definition 3.1 there are infinitely many umbrae; in Example 7 of [4] the computation of a particular umbra related to integration is determined, proving once more the method's possibilities. Consider $\frac{1}{A+1}, \frac{1}{(A+1)^2}, \ldots, \frac{1}{(A+1)^n} \ldots$; the point is to find out the sequence represented by this umbra, that is, $\mathbf{eval}\left(\frac{1}{(A+1)^n}\right)$, but why the quotient notation is used? At first this notation does not make sense and the obvious explanation, that is, $\mathbf{eval}\left(\frac{1}{(A+1)^n}\right) = \frac{1}{\mathbf{eval}[(A+1)^n]}$ is not the answer so the author is defining a new umbra suitable for his purpose. Let us follow what is done: the formula

$$x^A = \frac{2}{x+1}$$

is exactly

$$e^{A\theta} = \frac{2}{e^\theta + 1} \quad \text{or} \quad e^{(A+1)\theta} = \frac{2e^\theta}{e^\theta + 1},$$

that is,

$$\sum_{n=0}^{\infty} \frac{1}{n!}(A+1)^n \theta^n = \frac{2e^\theta}{e^\theta + 1}$$

and now integration with respect to θ is possible finding (consider the operator **eval**)

$$\theta + (A+1)\frac{\theta^2}{2!} + (A+1)^2 \frac{\theta^3}{3!} + \cdots = 2\log(e^\theta + 1) + C$$

and so $C = 2\log 2$ taking $\theta = 0$. Defining $\frac{(A+1)^n}{A+1} = (A+1)^{n-1}$, that is, they have the same **eval**, it can be written

$$\frac{1}{A+1}\left(1 + (A+1)\theta + (A+1)^2 \frac{\theta^2}{2!} + (A+1)^3 \frac{\theta^3}{3!} + \ldots\right)$$

$$= 2\log(e^\theta + 1) - 2\log 2 + \frac{1}{A+1}$$

or symbolically

$$\frac{e^{(A+1)\theta}}{A+1} = 2\log(e^{\theta}+1)$$

and so

$$\mathbf{eval}\left(\frac{1}{A+1}\right) = 2\log 2.$$

Proceeding in an analogous way it follows

$$\mathbf{eval}\left(\frac{1}{(A+1)^n}\right) = \left(2 - \frac{1}{2^{n-2}}\right)S_n, \text{ where } S_n = \sum_{k=1}^{\infty}\frac{1}{k^n}, \quad n \geq 2.$$

Remark 4.1. Note that the author integrates the formula

$$x^A = \frac{2}{x+1}$$

getting

$$\frac{x^{A+1}}{A+1} = 2\log(1+x) + C$$

takes $x = 0$ and $\frac{1}{A+1}$ follows (we suppose that something similar to what we have done was in his mind).

To finish these comments about the symbolic calculus let us note our admiration that Blissard could manage his way through all these difficulties and arrive safely to harbour; fortunately for us the foundations of Rota and Taylor have cleared the path.

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