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Research Article

Incomplete k-Fibonacci and k-Lucas Numbers

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We define the incomplete k-Fibonacci and k-Lucas numbers; we study the recurrence relations and some properties of these numbers.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art (e.g., see [1]). Fibonacci numbers F_n are defined by the recurrence relation

$$F_0 = 0, \qquad F_1 = 1, \qquad F_{n+1} = F_n + F_{n-1}, \quad n \geqslant 1. \tag{1}$$

There exist a lot of properties about Fibonacci numbers. In particular, there is a beautiful combinatorial identity to Fibonacci numbers [1]

$$F_n = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i}. \tag{2}$$

From (2), Filipponi [2] introduced the incomplete Fibonacci numbers $F_n(s)$ and the incomplete Lucas numbers $L_n(s)$. They are defined by

$$F_n(s) = \sum_{j=0}^{s} \binom{n-1-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \le s \le \left\lfloor \frac{n-1}{2} \right\rfloor\right),$$
$$L_n(s) = \sum_{j=0}^{s} \frac{n}{n-j} \binom{n-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \le s \le \left\lfloor \frac{n}{2} \right\rfloor\right). \tag{3}$$

Further in [3], generating functions of the incomplete Fibonacci and Lucas numbers are determined. In [4], Djordjević gave the incomplete generalized Fibonacci and Lucas numbers. In [5], Djordjević and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. In [6], the authors define the incomplete Fibonacci and Lucas p-numbers. Also the authors define the incomplete bivariate Fibonacci and Lucas p-polynomials in [7].

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci Numbers.

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \qquad F_{k,1} = 1, \qquad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geqslant 1. \tag{4}$$

In [8], k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [8–14].

For any positive real number k, the k-Lucas sequence, say $\{L_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$L_{k,0} = 2, \qquad L_{k,1} = k, \qquad L_{k,n+1} = kL_{k,n} + L_{k,n-1}. \tag{5}$$

If $k = 1$, we have the classical Lucas numbers. Moreover, $L_{k,n} = F_{k,n-1} + F_{k,n+1}$, $n \geqslant 1$; see [15].

In [12], the explicit formula to k-Fibonacci numbers is

$$F_{k,n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i} k^{n-2i-1}, \tag{6}$$

and the explicit formula of k-Lucas numbers is

$$L_{k,n}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} k^{n-2i}. \tag{7}$$

From (6) and (7), we introduce the incomplete k-Fibonacci and k-Lucas numbers and we obtain new recurrent relations, new identities, and their generating functions.

2. The Incomplete k-Fibonacci Numbers

Definition 1. The incomplete k-Fibonacci numbers are defined by

$$F_{k,n}^{l} = \sum_{i=0}^{l} \binom{n-1-i}{i} k^{n-2i-1}, \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{8}$$

In Table 1, some values of incomplete k-Fibonacci numbers are provided.

We note that

$$F_{1,n}^{\lfloor (n-1)/2 \rfloor} = F_n. \tag{9}$$

For $k = 1$, we get incomplete Fibonacci numbers [2].

Some special cases of (8) are

$$\begin{gathered} F_{k,n}^{0} = k^{n-1}; \quad (n \ge 1), \\ F_{k,n}^{1} = k^{n-1} + (n-2) k^{n-3}; \quad (n \ge 3) \\ F_{k,n}^{2} = k^{n-1} + (n-2) k^{n-3} + \frac{(n-4)(n-3)}{2} k^{n-5}; \quad (n \ge 5) \\ F_{k,n}^{\lfloor (n-1)/2 \rfloor} = F_{k,n}; \quad (n \ge 1) \\ F_{k,n}^{\lfloor (n-3)/2 \rfloor} = \begin{cases} F_{k,n} - \dfrac{nk}{2} & (n \text{ even}) \\ F_{k,n} - 1 & (n \text{ odd}) \end{cases} (n \ge 3). \end{gathered} \tag{10}$$

2.1. Some Recurrence Properties of the Numbers $F_{k,n}^{l}$

Proposition 2. *The recurrence relation of the incomplete k-Fibonacci numbers $F_{k,n}^{l}$ is*

$$F_{k,n+2}^{l+1} = k F_{k,n+1}^{l+1} + F_{k,n}^{l}, \quad 0 \le l \le \frac{n-2}{2}. \tag{11}$$

The relation (11) *can be transformed into the nonhomogeneous recurrence relation*

$$F_{k,n+2}^{l} = k F_{k,n+1}^{l} + F_{k,n}^{l} - \binom{n-1-l}{l} k^{n-1-2l}. \tag{12}$$

Proof. Use Definition 1 to rewrite the right-hand side of (11) as

$$\begin{aligned} & k \sum_{i=0}^{l+1} \binom{n-i}{i} k^{n-2i} + \sum_{i=0}^{l} \binom{n-i-1}{i} k^{n-2i-1} \\ & = \sum_{i=0}^{l+1} \binom{n-i}{i} k^{n-2i+1} + \sum_{i=1}^{l+1} \binom{n-i}{i-1} k^{n-2i+1} \\ & = k^{n-2i+1} \left(\sum_{i=0}^{l+1} \left[\binom{n-i}{i} + \binom{n-i}{i-1} \right] \right) - k^{n+1} \binom{n}{-1} \\ & = \sum_{i=0}^{l+1} \binom{n-i+1}{i} k^{n-2i+1} - 0 \\ & = F_{k,n+2}^{l}. \end{aligned} \tag{13}$$

□

Proposition 3. *One has*

$$\sum_{i=0}^{s} \binom{s}{i} F_{k,n+i}^{l+i} k^{i} = F_{k,n+2s}^{l+s}, \quad 0 \le l \le \frac{n-s-1}{2}. \tag{14}$$

Proof (by induction on s). Sum (14) clearly holds for $s = 0$ and $s = 1$ (see (11)). Now suppose that the result is true for all $j < s + 1$; we prove it for $s + 1$:

$$\begin{aligned} & \sum_{i=0}^{s+1} \binom{s+1}{i} F_{k,n+i}^{l+i} k^{i} \\ & = \sum_{i=0}^{s+1} \left[\binom{s}{i} + \binom{s}{i-1} \right] F_{k,n+i}^{l+i} k^{i} \\ & = \sum_{i=0}^{s+1} \binom{s}{i} F_{k,n+i}^{l+i} k^{i} + \sum_{i=0}^{s+1} \binom{s}{i-1} F_{k,n+i}^{l+i} k^{i} \\ & = F_{k,n+2s}^{l+s} + \binom{s}{s+1} F_{k,n+s+1}^{l+s+1} k^{s+1} + \sum_{i=-1}^{s} \binom{s}{i} F_{k,n+i+1}^{l+i+1} k^{i+1} \\ & = F_{k,n+2s}^{l+s} + 0 + \sum_{i=0}^{s} \binom{s}{i} F_{k,n+i+1}^{l+i+1} k^{i+1} + \binom{s}{-1} F_{k,n}^{l} \\ & = F_{k,n+2s}^{l+s} + k \sum_{i=0}^{s} \binom{s}{i} F_{k,n+i+1}^{l+i+1} k^{i} + 0 \\ & = F_{k,n+2s}^{l+s} + k F_{k,n+2s+1}^{l+s+1} \\ & = F_{k,n+2s+2}^{l+s+1}. \end{aligned} \tag{15}$$

□

Proposition 4. *For $n \ge 2l + 2$,*

$$\sum_{i=0}^{s-1} F_{k,n+i}^{l} k^{s-1-i} = F_{k,n+s+1}^{l+1} - k^{s} F_{k,n+1}^{l+1}. \tag{16}$$

Table 1: The numbers $F^l_{k,n}$, for $1 \leqslant n \leqslant 10$.

$n \backslash l$	0	1	2	3	4
1	1				
2	k				
3	k^2	k^2+1			
4	k^3	k^3+2k			
5	k^4	k^4+3k^2	k^4+3k^2+1		
6	k^5	k^5+4k^3	k^5+4k^3+3k		
7	k^6	k^6+5k^4	$k^6+5k^4+6k^2$	$k^6+5k^4+6k^2+1$	
8	k^7	k^7+6k^5	$k^7+6k^5+10k^3$	$k^7+6k^5+10k^3+4k$	
9	k^8	k^8+7k^6	$k^8+7k^6+15k^4$	$k^8+7k^6+15k^4+10k^2$	$k^8+7k^6+15k^4+10k^2+1$
10	k^9	k^9+8k^7	$k^9+8k^7+21k^5$	$k^9+8k^7+21k^5+20k^3$	$k^9+8k^7+21k^5+20k^3+5k$

Proof (by induction on s). Sum (16) clearly holds for $s = 1$ (see (11)). Now suppose that the result is true for all $j < s$. We prove it for s:

$$\begin{aligned}\sum_{i=0}^{s} F^l_{k,n+i} k^{s-i} &= k \sum_{i=0}^{s-1} F^l_{k,n+i} k^{s-i-1} + F^l_{k,n+s} \\ &= k\left(F^{l+1}_{k,n+s+1} - k^s F^{l+1}_{k,n+1}\right) + F^l_{k,n+s} \\ &= \left(kF^{l+1}_{k,n+s+1} + F^l_{k,n+s}\right) - k^{s+1} F^{l+1}_{k,n+1} \\ &= F^{l+1}_{k,n+s+2} - k^{s+1} F^{l+1}_{k,n+1}.\end{aligned} \tag{17}$$

□

Note that if k, in (4), is a real variable, then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0 \\ x & \text{if } n = 1 \\ xF_n(x) + F_{n-1}(x) & \text{if } n > 1. \end{cases} \tag{18}$$

Lemma 5. *One has*

$$\begin{aligned} F'_n(x) &= \frac{nF_{n+1}(x) - xF_n(x) + nF_{n-1}(x)}{x^2+4} \\ &= \frac{nL_n(x) - xF_n(x)}{x^2+4}. \end{aligned} \tag{19}$$

See Proposition 13 of [12].

Lemma 6. *One has*

$$\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} i \binom{n-1-i}{i} k^{n-1-2i} = \frac{\left(\left(k^2+4\right)n - 4\right)F_{k,n} - nkL_{k,n}}{2\left(k^2+4\right)}. \tag{20}$$

Proof. From (6) we have that

$$kF_{k,n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} k^{n-2i}. \tag{21}$$

By deriving into the previous equation (respect to k), it is obtained

$$\begin{aligned} F_{k,n} + kF'_{k,n} &= \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (n-2i) \binom{n-1-i}{i} k^{n-2i-1} \\ &= nF_{k,n} - 2\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} i \binom{n-1-i}{i} k^{n-2i-1}. \end{aligned} \tag{22}$$

From Lemma 5,

$$\begin{aligned} F_{k,n} + k\left(\frac{nL_{k,n} - kF_{k,n}}{k^2+4}\right) \\ = nF_{k,n} - 2\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} i \binom{n-1-i}{i} k^{n-2i-1}. \end{aligned} \tag{23}$$

From where, after some algebra (20) is obtained. □

Proposition 7. *One has*

$$\sum_{l=0}^{\lfloor (n-1)/2 \rfloor} F^l_{k,n} = \begin{cases} \dfrac{4F_{k,n} + nkL_{k,n}}{2\left(k^2+4\right)} & (n \textit{ even}) \\[2ex] \dfrac{\left(k^2+8\right)F_{k,n} + nkL_{k,n}}{2\left(k^2+4\right)} & (n \textit{ odd}). \end{cases} \tag{24}$$

Table 2: The numbers $L^l_{k,n}$, for $1 \leqslant n \leqslant 9$.

$n\backslash l$	0	1	2	3	4
1	k				
2	k^2	k^2+2			
3	k^3	k^3+3k			
4	k^4	k^4+4k^2	k^4+4k^2+2		
5	k^5	k^5+5k^3	k^5+5k^3+5k		
6	k^6	k^6+6k^4	$k^6+6k^4+9k^2$	$k^6+6k^4+9k^2+2$	
7	k^7	k^7+7k^5	$k^7+7k^5+14k^3$	$k^7+7k^5+14k^3+7k$	
8	k^8	k^8+8k^6	$k^8+8k^6+20k^4$	$k^8+8k^6+20k^4+16k^2$	$k^8+8k^6+20k^4+16k^2+2$
9	k^9	k^9+9k^7	$k^9+9k^7+27k^5$	$k^9+9k^7+27k^5+30k^3$	$k^9+9k^7+27k^5+30k^3+9k$

Proof

$$\begin{aligned}
&\sum_{l=0}^{\lfloor (n-1)/2\rfloor} F^l_{k,n} \\
&= F^0_{k,n} + F^1_{k,n} + \cdots + F^{\lfloor (n-1)/2\rfloor}_{k,n} \\
&= \binom{n-1-0}{0} k^{n-1} \\
&\quad + \left[\binom{n-1-0}{0} k^{n-1} + \binom{n-1-1}{1} k^{n-3}\right] \\
&\quad + \cdots + \left[\binom{n-1-0}{0} k^{n-1} + \binom{n-1-1}{1} k^{n-3}\right. \\
&\qquad \left. + \cdots + \binom{n-1-\lfloor \frac{n-1}{2}\rfloor}{\lfloor \frac{n-1}{2}\rfloor} k^{n-1-2\lfloor (n-1)/2\rfloor}\right] \\
&= \left(\left\lfloor \frac{n-1}{2}\right\rfloor + 1\right)\binom{n-1-0}{0} k^{n-1} \\
&\quad + \left\lfloor \frac{n-1}{2}\right\rfloor \binom{n-1-1}{1} k^{n-3} \\
&\quad + \cdots + \binom{n-1-\lfloor \frac{n-1}{2}\rfloor}{\lfloor \frac{n-1}{2}\rfloor} k^{n-1-2\lfloor (n-1)/2\rfloor} \\
&= \sum_{i=0}^{\lfloor (n-1)/2\rfloor} \left(\left\lfloor \frac{n-1}{2}\right\rfloor + 1 - i\right)\binom{n-1-i}{i} k^{n-1-2i} \\
&= \sum_{i=0}^{\lfloor (n-1)/2\rfloor} \left(\left\lfloor \frac{n-1}{2}\right\rfloor + 1\right)\binom{n-1-i}{i} k^{n-1-2i} \\
&\quad - \sum_{i=0}^{\lfloor (n-1)/2\rfloor} i \binom{n-1-i}{i} k^{n-1-2i} \\
&= \left(\left\lfloor \frac{n-1}{2}\right\rfloor + 1\right) F_{k,n} - \sum_{i=0}^{\lfloor (n-1)/2\rfloor} i \binom{n-1-i}{i} k^{n-1-2i}.
\end{aligned} \tag{25}$$

From Lemma 6, (24) is obtained. □

3. The Incomplete k-Lucas Numbers

Definition 8. The incomplete k-Lucas numbers are defined by

$$L^l_{k,n} = \sum_{i=0}^{l} \frac{n}{n-i}\binom{n-i}{i} k^{n-2i}, \quad 0 \le l \le \left\lfloor \frac{n}{2}\right\rfloor. \tag{26}$$

In Table 2, some numbers of incomplete k-Lucas numbers are provided.

We note that

$$L^{\lfloor n/2\rfloor}_{1,n} = L_n. \tag{27}$$

Some special cases of (26) are

$$\begin{aligned}
L^0_{k,n} &= k^n; \quad (n \ge 1), \\
L^1_{k,n} &= k^n + nk^{n-2}; \quad (n \ge 2), \\
L^2_{k,n} &= k^n + nk^{n-2} + \frac{n(n-3)}{2} k^{n-4}; \quad (n \ge 4), \\
L^{\lfloor n/2\rfloor}_{k,n} &= L_{k,n}; \quad (n \ge 1), \\
L^{\lfloor (n-2)/2\rfloor}_{k,n} &= \begin{cases} L_{k,n} - 2 & (n \text{ even}) \\ L_{k,n} - nk & (n \text{ odd}) \end{cases} \quad (n \ge 2).
\end{aligned} \tag{28}$$

3.1. Some Recurrence Properties of the Numbers $L_{k,n}^{l}$

Proposition 9. *One has*

$$L_{k,n}^{l} = F_{k,n-1}^{l-1} + F_{k,n+1}^{l}, \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{29}$$

Proof. By (8), rewrite the right-hand side of (29) as

$$\begin{aligned}
&\sum_{i=0}^{l-1} \binom{n-2-i}{i} k^{n-2-2i} + \sum_{i=0}^{l} \binom{n-i}{i} k^{n-2i} \\
&= \sum_{i=1}^{l} \binom{n-1-i}{i-1} k^{n-2i} + \sum_{i=0}^{l} \binom{n-i}{i} k^{n-2i} \\
&= \sum_{i=0}^{l} \left[\binom{n-1-i}{i-1} + \binom{n-i}{i} \right] k^{n-2i} - \binom{n-1}{-1} \\
&= \sum_{i=0}^{l} \frac{n}{n-i} \binom{n-i}{i} k^{n-2i} + 0 \\
&= L_{k,n}^{l}.
\end{aligned} \tag{30}$$

□

Proposition 10. *The recurrence relation of the incomplete k-Lucas numbers $L_{k,n}^{l}$ is*

$$L_{k,n+2}^{l+1} = kL_{k,n+1}^{l+1} + L_{k,n}^{l}, \quad 0 \le l \le \left\lfloor \frac{n}{2} \right\rfloor. \tag{31}$$

The relation (31) *can be transformed into the nonhomogeneous recurrence relation*

$$L_{k,n+2}^{l} = kL_{k,n+1}^{l} + L_{k,n}^{l} - \frac{n}{n-l} \binom{n-l}{l} k^{n-2l}. \tag{32}$$

Proof. Using (29) and (11), we write

$$\begin{aligned}
L_{k,n+2}^{l+1} &= F_{k,n+1}^{l} + F_{k,n+3}^{l+1} \\
&= kF_{k,n}^{l} + F_{k,n-1}^{l-1} + kF_{k,n+2}^{l+1} + F_{k,n+1}^{l} \\
&= k\left(F_{k,n}^{l} + F_{k,n+2}^{l+1}\right) + F_{k,n-1}^{l-1} + F_{k,n+1}^{l} \\
&= kL_{k,n+1}^{l+1} + L_{k,n}^{l}.
\end{aligned} \tag{33}$$

□

Proposition 11. *One has*

$$kL_{k,n}^{l} = F_{k,n+2}^{l} - F_{k,n-2}^{l-2}, \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{34}$$

Proof. By (29),

$$F_{k,n+2}^{l} = L_{k,n+1}^{l} - F_{k,n}^{l-1}, \qquad F_{k,n-2}^{l-2} = L_{k,n-1}^{l-1} - F_{k,n}^{l-1}, \tag{35}$$

whence, from (31),

$$F_{k,n+2}^{l} - F_{k,n-2}^{l-2} = L_{k,n+1}^{l} - L_{k,n-1}^{l-1} = kL_{k,n}^{l}. \tag{36}$$

□

Proposition 12. *One has*

$$\sum_{i=0}^{s} \binom{s}{i} L_{k,n+i}^{l+i} k^{i} = L_{k,n+2s}^{l+s}, \quad 0 \le l \le \frac{n-s}{2}. \tag{37}$$

Proof. Using (29) and (14), we write

$$\begin{aligned}
\sum_{i=0}^{s} \binom{s}{i} L_{k,n+i}^{l+i} k^{i} &= \sum_{i=0}^{s} \binom{s}{i} \left[F_{k,n+i-1}^{l+i-1} + F_{k,n+i+1}^{l+i} \right] k^{i} \\
&= \sum_{i=0}^{s} \binom{s}{i} F_{k,n+i-1}^{l+i-1} k^{i} + \sum_{i=0}^{s} \binom{s}{i} F_{k,n+i+1}^{l+i} k^{i} \\
&= F_{k,n-1+2s}^{l-1+s} + F_{k,n+1+2s}^{l+s} = L_{k,n+2s}^{l+s}.
\end{aligned} \tag{38}$$

□

Proposition 13. *For $n \ge 2l+1$,*

$$\sum_{i=0}^{s-1} L_{k,n+i}^{l} k^{s-1-i} = L_{k,n+s+1}^{l+1} - k^{s} L_{k,n+1}^{l+1}. \tag{39}$$

The proof can be done by using (31) and induction on s.

Lemma 14. *One has*

$$\sum_{i=0}^{\lfloor n/2 \rfloor} i \frac{n}{n-i} \binom{n-i}{i} k^{n-2i} = \frac{n}{2} \left[L_{k,n} - kF_{k,n} \right]. \tag{40}$$

The proof is similar to Lemma 6.

Proposition 15. *One has*

$$\sum_{l=0}^{\lfloor n/2 \rfloor} L_{k,n}^{l} = \begin{cases} L_{k,n} + \dfrac{nk}{2} F_{k,n} & (n \text{ even}) \\ \dfrac{1}{2} \left(L_{k,n} + nkF_{k,n} \right) & (n \text{ odd}). \end{cases} \tag{41}$$

Proof. An argument analogous to that of the proof of Proposition 7 yields

$$\sum_{l=0}^{\lfloor n/2 \rfloor} L_{k,n}^{l} = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) L_{k,n} - \sum_{i=0}^{\lfloor n/2 \rfloor} i \frac{n}{n-i} \binom{n-i}{i} k^{n-2i}. \tag{42}$$

From Lemma 14, (41) is obtained. □

4. Generating Functions of the Incomplete k-Fibonacci and k-Lucas Number

In this section, we give the generating functions of incomplete k-Fibonacci and k-Lucas numbers.

Lemma 16 (see [3, page 592]). *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$s_n = as_{n-1} + bs_{n-2} + r_n \quad (n > 1), \tag{43}$$

where a and b are complex numbers and $\{r_n\}$ *is a given complex sequence. Then, the generating function* $U(t)$ *of the sequence* $\{s_n\}$ *is*

$$U(t) = \frac{G(t) + s_0 - r_0 + (s_1 - s_0 a - r_1)t}{1 - at - bt^2}, \quad (44)$$

where $G(t)$ *denotes the generating function of* $\{r_n\}$.

Theorem 17. *The generating function of the incomplete k-Fibonacci numbers* $F^l_{k,n}$ *is given by*

$$\begin{aligned} R_{k,l}(x) &= \sum_{i=0}^{\infty} F^l_{k,i} t^i \\ &= t^{2l+1}\left[F_{k,2l+1} + (F_{k,2l+2} - kF_{k,2l+1})t - \frac{t^2}{(1-kt)^{l+1}}\right] \\ &\quad \times \left[1 - kt - t^2\right]^{-1}. \end{aligned} \quad (45)$$

Proof. Let l be a fixed positive integer. From (8) and (12), $F^l_{k,n} = 0$ for $0 \le n < 2l+1$, $F^l_{k,2l+1} = F_{k,2l+1}$, and $F^l_{k,2l+2} = F_{k,2l+2}$, and

$$F^l_{k,n} = kF^l_{k,n-1} + F^l_{k,n-2} - \binom{n-3-l}{l} k^{n-3-2l}. \quad (46)$$

Now let

$$s_0 = F^l_{k,2l+1}, \qquad s_1 = F^l_{k,2l+2}, \qquad s_n = F^l_{k,n+2l+1}. \quad (47)$$

Also let

$$r_0 = r_1 = 0, \qquad r_n = \binom{n+l-2}{n-2} k^{n-2}. \quad (48)$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2/(1-kt)^{l+1}$ (see [16, page 355]). Thus, from Lemma 16, we get the generating function $R_{k,l}(x)$ of sequence $\{s_n\}$. □

Theorem 18. *The generating function of the incomplete k-Lucas numbers* $L^l_{k,n}$ *is given by*

$$\begin{aligned} S_{k,l}(x) &= \sum_{i=0}^{\infty} L^l_{k,i} t^i \\ &= t^{2l}\left[L_{k,2l} + (L_{k,2l+1} - kL_{k,2l})t - \frac{t^2(2-t)}{(1-kt)^{l+1}}\right] \\ &\quad \times \left[1 - kt - t^2\right]^{-1}. \end{aligned} \quad (49)$$

Proof. The proof of this theorem is similar to the proof of Theorem 17. Let l be a fixed positive integer. From (26) and (32), $L^l_{k,n} = 0$ for $0 \le n < 2l$, $L^l_{k,2l} = L_{k,2l}$, and $L^l_{k,2l+1} = L_{k,2l+1}$, and

$$L^l_{k,n} = kL^l_{k,n-1} + L^l_{k,n-2} - \frac{n-2}{n-2-l}\binom{n-2-l}{n-2-2l} k^{n-2-2l}. \quad (50)$$

Now let

$$s_0 = L^l_{k,2l}, \qquad s_1 = L^l_{k,2l+1}, \qquad s_n = L^l_{k,n+2l}. \quad (51)$$

Also let

$$r_0 = r_1 = 0, \qquad r_n = \binom{n+2l-2}{n+l-2} k^{n+2l-2}. \quad (52)$$

The generating function of the sequence $\{r_n\}$ is $G(t) = t^2(2-t)/(1-kt)^{l+1}$ (see [16, page 355]). Thus, from Lemma 16, we get the generating function $S_{k,l}(x)$ of sequence $\{s_n\}$. □

5. Conclusion

In this paper, we introduce incomplete k-Fibonacci and k-Lucas numbers, and we obtain new identities. In [17], the authors introduced the $h(x)$-Fibonacci polynomials. That generalizes Catalan's Fibonacci polynomials and the k-Fibonacci numbers. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$\begin{gathered} F_{h,0}(x) = 0, \qquad F_{h,1}(x) = 1, \\ F_{h,n+1}(x) = h(x) F_{h,n}(x) + F_{h,n-1}(x), \quad n \geqslant 1. \end{gathered} \quad (53)$$

It would be interesting to study a definition of incomplete $h(x)$-Fibonacci polynomials and research their properties.

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