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Bi-periodic incomplete Fibonacci sequences

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Abstract

In this paper, we define the bi-periodic incomplete Fibonacci sequences, we study some recurrence relations linked to them, some properties of these numbers and their generating functions. In the case $a = k = b$, we obtain the incomplete k-Fibonacci numbers. If $a = 1 = b$, we have the incomplete Fibonacci numbers.

Keywords: bi-periodic incomplete Fibonacci sequence, bi-periodic Fibonacci sequence, generating function

MSC: 11B39, 11B83, 05A15

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art [10]. The Fibonacci numbers F_n are defined by the recurrence relation

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \; n \geqslant 1.$$

There exist a lot of properties about Fibonacci numbers. In particular, there is a beautiful combinatorial identity

$$F_n = \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} \tag{1.1}$$

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for Fibonacci numbers [10].

In analogy with (1.1), Filipponi [6] introduced the incomplete Fibonacci numbers $F_n(s)$ and the incomplete Lucas numbers $L_n(s)$. They are defined by

$$F_n(s) = \sum_{j=0}^{s} \binom{n-1-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \le s \le \left\lfloor \frac{n-1}{2} \right\rfloor\right),$$

and

$$L_n(s) = \sum_{j=0}^{s} \frac{n}{n-j} \binom{n-j}{j} \quad \left(n = 1, 2, 3, \ldots; 0 \le s \le \left\lfloor \frac{n}{2} \right\rfloor\right).$$

Further in [11], generating functions of the incomplete Fibonacci and Lucas numbers are determined. In [2] Djordević gave the incomplete generalized Fibonacci and Lucas numbers. In [3] Djordević and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal-Lucas numbers. In [15] the authors define the incomplete Fibonacci and Lucas p-numbers. Also the authors define the incomplete bivariate Fibonacci and Lucas p-polynomials in [16]. In [13] we introduce the incomplete k-Fibonacci and k-Lucas numbers and in [12] we study incomplete $h(x)-$Fibonacci and $h(x)-$Lucas polynomials.

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the bi-periodic Fibonacci sequence [4]. For any two nonzero real numbers a and b, the bi-periodic Fibonacci sequence, say $\{q_n\}_{n=0}^{\infty}$, is determined by:

$$q_0 = 0, \quad q_1 = 1, \quad q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \equiv 0 \pmod 2; \\ bq_{n-1} + q_{n-2}, & \text{if } n \equiv 1 \pmod 2; \end{cases} \quad n \geqslant 2. \tag{1.2}$$

These numbers have been studied in several papers; see [1, 4, 5, 8, 9, 17]. In [17], the explicit formula to bi-periodic Fibonacci numbers is

$$q_n = a^{\xi(n-1)} \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-i-1}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}, \tag{1.3}$$

where $\xi(n) = n - 2\lfloor \frac{n}{2} \rfloor$, i.e., $\xi(n) = 0$ when n is even and $\xi(n) = 1$ when n is odd. From equation (1.3) we introduce the bi-periodic incomplete Fibonacci numbers and we obtain new recurrent relations, new identities and generating functions.

2. Bi-Periodic Incomplete Fibonacci Sequence

Definition 2.1. For $n \geqslant 1$, the bi-periodic incomplete Fibonacci numbers are defined as

$$q_n(l) = a^{\xi(n-1)} \sum_{i=0}^{l} \binom{n-i-1}{i} (ab)^{\lfloor \frac{n-1}{2} \rfloor - i}, \quad 0 \le l \le \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{2.1}$$

For $a = b$, $q_n(l) = F^l_{k,n}$, we get incomplete k-Fibonacci numbers [13]. If $a = b = 1$, we obtained incomplete Fibonacci numbers [6]. In Table 1, some values of bi-periodic incomplete k-Fibonacci numbers are provided, with $a = 3$ and $b = 2$.

n/l	0	1	2	3	4	5	6
1	1						
2	3						
3	6	7					
4	18	24					
5	36	54	55				
6	108	180	189				
7	216	396	432	433			
8	648	1296	1476	1488			
9	1296	2808	3348	3408	3409		
10	3888	9072	11340	11700	11715		
11	7776	19440	25488	26748	26838	26839	
12	23328	62208	85536	91584	92214	92232	
13	46656	132192	190512	208656	211176	211302	211303
14	139968	419904	633744	711504	725112	726120	726141
15	279936	886464	1399680	1613520	1658880	1663416	1663584
16	839808	2799360	4618944	5474304	5688144	5715360	5716872

Table 1: Numbers $q_n(l)$, for $1 \leqslant n \leqslant 16$, and $a = 3, b = 2$

Some special cases of (2.1) are

$$q_n(0) = a^{\xi(n-1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor};\ (n \geq 1) \tag{2.2}$$

$$q_n(1) = a^{\xi(n-1)}(ab)^{\lfloor \frac{n-1}{2} \rfloor} + a^{\xi(n-1)}(n-2)(ab)^{\lfloor \frac{n-1}{2} \rfloor - 1};\ (n \geq 3) \tag{2.3}$$

$$q_n\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) = q_n;\ (n \geq 1) \tag{2.4}$$

$$q_n\left(\left\lfloor \frac{n-3}{2} \right\rfloor\right) = \begin{cases} q_n - \frac{na}{2}, & \text{if } n \equiv 0 \pmod 2; \\ q_n - 1, & \text{if } n \equiv 1 \pmod 2; \end{cases} \quad n \geq 3. \tag{2.5}$$

2.1. Some recurrence properties of the numbers $q_n(l)$

Proposition 2.2. *The non-linear recurrence relation of the bi-periodic incomplete Fibonacci numbers $q_n(l)$ is*

$$q_{n+2}(l+1) = \begin{cases} aq_{n+1}(l+1) + q_n(l), & \text{if } n \equiv 0 \pmod 2; \\ aq_{n+1}(l+1) + q_n(l), & \text{if } n \equiv 1 \pmod 2; \end{cases} \quad 0 \leq l \leq \frac{n-2}{2}. \tag{2.6}$$

The relation (2.6) *can be transformed into the non-homogeneous recurrence relation*

$$q_{n+2}(l) = \begin{cases} aq_{n+1}(l) + q_n(l) - a\binom{n-l-1}{l}(ab)^{\lfloor \frac{n-1}{2} \rfloor - l}, & \text{if } n \equiv 0 \pmod 2; \\ bq_{n+1}(l) + q_n(l) - \binom{n-l-1}{l}(ab)^{\lfloor \frac{n-1}{2} \rfloor - l}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{2.7}$$

Proof. If n is even, then $\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor + 1$. Use the Definition 2.1 to rewrite the right-hand side of (2.6) as

$$\begin{aligned}
& a\left(a^{\xi(n)}\sum_{i=0}^{l+1}\binom{n-i}{i}(ab)^{\lfloor \frac{n}{2}\rfloor - i}\right) + a^{\xi(n-1)}\sum_{i=0}^{l}\binom{n-i-1}{i}(ab)^{\lfloor \frac{n-1}{2}\rfloor - i} \\
&= a^{\xi(n+1)}\sum_{i=0}^{l+1}\binom{n-i}{i}(ab)^{\lfloor \frac{n}{2}\rfloor - i} + a^{\xi(n+1)}\sum_{i=1}^{l+1}\binom{n-i}{i-1}(ab)^{\lfloor \frac{n-1}{2}\rfloor -(i-1)} \\
&= a^{\xi(n+1)}\left(\sum_{i=0}^{l+1}\left[\binom{n-i}{i}+\binom{n-i}{i-1}\right](ab)^{\lfloor \frac{n}{2}\rfloor - i}\right) - a^{\xi(n+1)}\binom{n}{-1}(ab)^{\lfloor \frac{n+1}{2}\rfloor} \\
&= a^{\xi(n+1)}\sum_{i=0}^{l+1}\binom{n-i+1}{i}(ab)^{\lfloor \frac{n-1}{2}\rfloor - i} - 0 \\
&= q_{n+2}(l+1).
\end{aligned}$$

If n is odd, the proof is analogous. On the other hand, equation (2.7) is clear from (2.6). In fact, if n is even

$$\begin{aligned}
q_{n+2}(l) &= aq_{n+1}(l) + q_n(l-1) = aq_{n+1}(l) + q_n(l) + (q_n(l-1) - q_n(l)) \\
&= aq_{n+1}(l) + q_n(l) - a\binom{n-l-1}{l}(ab)^{\lfloor \frac{n-1}{2}\rfloor - l}.
\end{aligned}$$

If n is odd, the proof is analogous. □

Proposition 2.3. *One has*

$$\sum_{i=0}^{s}\binom{s}{i}q_{n+i}(l+i)a^{\lfloor \frac{i+\xi(n+1)}{2}\rfloor}b^{\lfloor \frac{i+\xi(n)}{2}\rfloor} = q_{n+2s}(l+s), \quad 0 \le l \le \frac{n-s-1}{2}. \tag{2.8}$$

Proof. (By induction on s.) The sum (2.8) clearly holds for $s = 0$ and $s = 1$ (see (2.6)). Now suppose that the result is true for all $j < s+1$, we prove it for $s+1$. If n is even, then

$$\begin{aligned}
& \sum_{i=0}^{s+1}\binom{s+1}{i}q_{n+i}(l+i)a^{\lfloor \frac{i+1}{2}\rfloor}b^{\lfloor \frac{i}{2}\rfloor} \\
&= \sum_{i=0}^{s+1}\left[\binom{s}{i}+\binom{s}{i-1}\right]q_{n+i}(l+i)a^{\lfloor \frac{i+1}{2}\rfloor}b^{\lfloor \frac{i}{2}\rfloor} \\
&= \sum_{i=0}^{s+1}\binom{s}{i}q_{n+i}(l+i)a^{\lfloor \frac{i+1}{2}\rfloor}b^{\lfloor \frac{i}{2}\rfloor} + \sum_{i=0}^{s+1}\binom{s}{i-1}q_{n+i}(l+i)a^{\lfloor \frac{i+1}{2}\rfloor}b^{\lfloor \frac{i}{2}\rfloor} \\
&= q_{n+2s}(l+s) + \binom{s}{s+1}q_{n+s+1}(l+s+1)a^{\lfloor \frac{s+2}{2}\rfloor}b^{\lfloor \frac{s+1}{2}\rfloor}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=-1}^{s}\binom{s}{i}q_{n+i+1}(l+i+1)a^{\lfloor\frac{i+2}{2}\rfloor}b^{\lfloor\frac{i+1}{2}\rfloor}\\
&=q_{n+2s}(l+s)+0+a\sum_{i=0}^{s}\binom{s}{i}q_{n+i+1}(l+i+1)a^{\lfloor\frac{i}{2}\rfloor}b^{\lfloor\frac{i+1}{2}\rfloor}+\binom{s}{-1}q_{n}(l)a^{\lfloor\frac{1}{2}\rfloor}b^{0}\\
&=q_{n+2s}(l+s)+a\sum_{i=0}^{s}\binom{s}{i}q_{n+i+1}(l+i+1)a^{\lfloor\frac{i}{2}\rfloor}b^{\lfloor\frac{i+1}{2}\rfloor}+0\\
&=q_{n+2s}(l+s)+aq_{n+2s+1}(l+s+1)\\
&=q_{n+2s+2}(l+s+1).
\end{aligned}
$$

If n is odd, the proof is analogous. □

Proposition 2.4. *For $n \geq 2l+2$,*

$$
\sum_{i=0}^{s-1}a^{\lfloor\frac{s-\xi(n+1)}{2}\rfloor-\lfloor\frac{i+\xi(n)}{2}\rfloor}b^{\lfloor\frac{s-\xi(n)}{2}\rfloor-\lfloor\frac{i+\xi(n+1)}{2}\rfloor}q_{n+i}(l)
= q_{n+s+1}(l+1)-a^{\lfloor\frac{s+\xi(n+1)}{2}\rfloor}b^{\lfloor\frac{s+\xi(n)}{2}\rfloor}q_{n+1}(l+1). \quad (2.9)
$$

Proof. (By induction on s.) Sum (2.9) clearly holds for $s=1$ (see (2.6)). Now suppose that the result is true for all $i<s$. We prove it for s. If n is even, then

$$
\begin{aligned}
&\sum_{i=0}^{s}a^{\lfloor\frac{s}{2}\rfloor-\lfloor\frac{i}{2}\rfloor}b^{\lfloor\frac{s+1}{2}\rfloor-\lfloor\frac{i+1}{2}\rfloor}q_{n+i}(l)\\
&=\sum_{i=0}^{s-1}a^{\lfloor\frac{s}{2}\rfloor-\lfloor\frac{i}{2}\rfloor}b^{\lfloor\frac{s+1}{2}\rfloor-\lfloor\frac{i+1}{2}\rfloor}q_{n+i}(l)+q_{n+s}(l)\\
&=a^{\xi(s+1)}b^{\xi(s)}\sum_{i=0}^{s-1}a^{\lfloor\frac{s-1}{2}\rfloor-\lfloor\frac{i}{2}\rfloor}b^{\lfloor\frac{s}{2}\rfloor-\lfloor\frac{i+1}{2}\rfloor}q_{n+i}(l)+q_{n+s}(l)\\
&=a^{\xi(s+1)}b^{\xi(s)}\Big(q_{n+s+1}(l+1)-a^{\lfloor\frac{s+1}{2}\rfloor}b^{\lfloor\frac{s}{2}\rfloor}q_{n+1}(l+1)\Big)+q_{n+s}(l)\\
&=\Big(a^{\xi(s+1)}b^{\xi(s)}q_{n+s+1}(l+1)+q_{n+s}(l)\Big)-a^{\xi(s+1)+\lfloor\frac{s+1}{2}\rfloor}b^{\xi(s)+\lfloor\frac{s}{2}\rfloor}q_{n+1}(l+1)\\
&=\Big(a^{\xi(s+1)}b^{\xi(s)}q_{n+s+1}(l+1)+q_{n+s}(l)\Big)-a^{\lfloor\frac{s+2}{2}\rfloor}b^{\lfloor\frac{s+1}{2}\rfloor}q_{n+1}(l+1)\\
&=q_{n+s+2}(l+1)-a^{\lfloor\frac{s+2}{2}\rfloor}b^{\lfloor\frac{s+1}{2}\rfloor}q_{n+1}(l+1).
\end{aligned}
$$

If n is odd, the proof is analogous. □

Following proposition shows the sum of the nth row of the array in Table 1.

Proposition 2.5. *One has*

$$
\sum_{l=0}^{\lfloor\frac{n-1}{2}\rfloor}q_{n}(l)=(l+1)q_{n}(l)-a^{\xi(n-1)}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}i\binom{n-i-1}{i}(ab)^{\lfloor\frac{n-1}{2}\rfloor-i}. \quad (2.10)
$$

Proof. Let $h = \left\lfloor \frac{n-1}{2} \right\rfloor$, then

$$\begin{aligned}
\sum_{l=0}^{h} q_n(l) &= q_n(0) + q_n(1) + \cdots + q_n(h) \\
&= a^{\xi(n-1)} \binom{n-1-0}{0} (ab)^h \\
&+ a^{\xi(n-1)} \left[\binom{n-1-0}{0} (ab)^h + \binom{n-1-1}{1} (ab)^{h-1} \right] + \cdots \\
&+ a^{\xi(n-1)} \left[\binom{n-1-0}{0} (ab)^h + \cdots + \binom{n-1-h}{h} (ab)^{h-h} \right] \\
&= a^{\xi(n-1)} \left[(h+1) \binom{n-1-0}{0} (ab)^h + h \binom{n-1-1}{1} (ab)^{h-1} + \right. \\
&\left. \cdots + \binom{n-1-h}{h} (ab)^{h-h} \right] \\
&= a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (h+1-i) \binom{n-1-i}{i} (ab)^{h-i} \\
&= a^{\xi(n-1)} (h+1) \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-1-i}{i} (ab)^{h-i} \\
&- a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i \binom{n-1-i}{i} (ab)^{h-i} \\
&= (h+1)\, q_n(l) - a^{\xi(n-1)} \sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} i \binom{n-1-i}{i} (ab)^{h-i}. \qquad \square
\end{aligned}$$

3. Generating function of the bi-periodic incomplete Fibonacci numbers

In this section, we give the generating functions of bi-periodic incomplete Fibonacci numbers.

Lemma 3.1. *Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following non-homogeneous and non-linear recurrence relation:*

$$s_n = \begin{cases} as_{n-1} + s_{n-2} + ar_n, & \text{if } n \equiv 1 \pmod 2; \\ bs_{n-1} + s_{n-2} + s_{n-1}, & \text{if } n \equiv 0 \pmod 2; \end{cases} \quad (n > 1), \tag{3.1}$$

where a and b are complex numbers and $\{r_n\}_{n=0}^{\infty}$ is a given complex sequence. Then *the generating function $U(t)$ of the sequence $\{s_n\}_{n=0}^{\infty}$ is*

$$U(t) = \frac{aG(t) + s_0 - r_0 + (s_1 - as_0 - ar_1)t + (b-a)tf(t) + (1-a)R(t)}{1 - at - t^2}, \quad (3.2)$$

where $G(t)$ denotes the generating function of $\{r_n\}_{n=0}^{\infty}$, $f(t)$ denotes the generating function of $\{s_{2n+1}\}_{n=0}^{\infty}$ and $R(t)$ denotes the generating function of $\{r_{2n}\}_{n=0}^{\infty}$. Moreover,

$$f(t) = \frac{atR(t) + a(1-t^2)R'(t) + (s_1 - a(r_1 + r_0))t + (a(s_0 + r_1) - s_1)t^3}{1 - (ab+2)t^2 + t^4}, \quad (3.3)$$

where $R'(t)$ denotes the generating function of $\{r_{2n-1}\}_{n=1}^{\infty}$.

Proof. We begin with the formal power series representation of the generating function for $\{s_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$,

$$U(t) = s_0 + s_1 t + s_2 t^2 + \cdots + s_k t^k + \cdots,$$
$$G(t) = r_0 + r_1 t + r_2 t^2 + \cdots + r_k t^k + \cdots.$$

Note that,

$$atU(t) = as_0 t + as_1 t^2 + as_2 t^3 + \cdots + as_k t^{k+1} + \cdots,$$
$$t^2 U(t) = s_0 t^2 + s_1 t^3 + s_2 t^4 + \cdots + s_k t^{k+1} + \cdots,$$

and,

$$aG(t) = ar_0 + ar_1 t + ar_2 t^2 + \cdots + ar_k t^k + \cdots.$$

Since $s_{2k+1} = as_{2k} + s_{2k-1} + ar_{2k+1}$, we get

$$\begin{aligned}&(1 - at - t^2)U(t) - aG(t)\\ &= (s_0 - ar_0) + (s_1 - a(s_0 + r_1))t + \sum_{m=1}^{\infty} (s_{2m} - as_{2m-1} - s_{2m-2} - ar_{2m})\, t^{2m}.\end{aligned}$$

Since $s_{2k} = bs_{2k-1} + s_{2k-2} + r_{2k}$, we get

$$\begin{aligned}&(1 - at - t^2)U(t) - aG(t)\\ &= (s_0 - ar_0) + (s_1 - a(s_0 + r_1))t + \sum_{m=1}^{\infty} ((b-a)s_{2m-1} + (1-a)r_{2m})\, t^{2m}\\ &= (s_0 - ar_0) + (s_1 - a(s_0 + r_1))t + (b-a)t \sum_{m=1}^{\infty} s_{2m-1} t^{2m-1} + (1-a) \sum_{m=1}^{\infty} r_{2m} t^{2m}\\ &= (s_0 - ar_0) + (s_1 - a(s_0 + r_1))t + (b-a)tf(t) + (1-a)R(t) - (1-a)r_0\\ &= (s_0 - r_0) + (s_1 - a(s_0 + r_1))t + (b-a)tf(t) + (1-a)R(t).\end{aligned}$$

Then equation (3.2) is clear.

On the other hand,

$$\begin{aligned}
s_{2m-1} &= as_{2m-2} + s_{2m-3} + ar_{2m-1} \\
&= a(bs_{2m-3} + s_{2m-4} + r_{2m-2}) + s_{2m-3} + ar_{2m-1} \\
&= (ab+1)s_{2m-3} + as_{2m-4} + a(r_{2m-2} + r_{2m-1}) \\
&= (ab+1)s_{2m-3} + s_{2m-3} - s_{2m-5} - ar_{2m-3} + a(r_{2m-2} + r_{2m-1}) \\
&= (ab+2)s_{2m-3} - s_{2m-5} + a(-r_{2m-3} + r_{2m-2} + r_{2m-1}).
\end{aligned}$$

Then

$$\begin{aligned}
&(1-(ab+2)t^2+t^4)f(t) - atR(t) + a(t^2-1)R'(t) \\
&= (s_1 - a(r_0+r_1))t + (s_3 - (ab+2)s_1 - ar_2 + a(r_1 - r_3))t^3 \\
&+ \sum_{m=3}^{\infty}(s_{2m-1} - (ab+2)s_{2m-3} + s_{2m-5} - ar_{2m-2} \\
&+ a(r_{2m-3} - r_{2m-1}))t^{2m-1} \\
&= (s_1 - a(r_0+r_1))t + (s_3 - (ab+2)s_1 - ar_2 + a(r_1 - r_3))t^3 \\
&= (s_1 - a(r_0+r_1))t + (a(s_0+r_1) - s_1)t^3.
\end{aligned}$$

Therefore equation (3.3) is obtained. □

Theorem 3.2. *The generating function of the bi-periodic incomplete Fibonacci numbers $q_n(l)$ is given by*

$$Q_l(t) = \sum_{i=0}^{\infty} q_i(l)t^i \tag{3.4}$$

$$= \frac{aG(t) + q_{2l+1} + (q_{2l+2} - aq_{2l+1})t + (b-a)tf(t) + (1-a)R(t)}{1-at-t^2}, \tag{3.5}$$

where

$$G(t) = -\frac{1}{2}\left(\frac{t^2}{(1-(ab)^{1/2}t)^{l+1}}(1+(ab)^{-1/2}) + \frac{t^2}{(1+(ab)^{1/2}t)^{l+1}}(1-(ab)^{-1/2})\right), \tag{3.6}$$

$$f(t) = \frac{q_{2l+2}t + (aq_{2l+1} - q_{2l+2})t^3 + atR(t) + a(1-t^2)R'(t)}{1-(ab+2)t^2+t^4} \tag{3.7}$$

and

$$R(t) = -\frac{1}{2}\left(\frac{t^2}{(1-(ab)^{1/2}t)^{l+1}} + \frac{t^2}{(1+(ab)^{1/2}t)^{l+1}}\right), \tag{3.8}$$

$$R'(t) = -\frac{1}{2(ab)^{1/2}}\left(\frac{t^2}{(1-(ab)^{1/2}t)^{l+1}} - \frac{t^2}{(1+(ab)^{1/2}t)^{l+1}}\right). \tag{3.9}$$

Proof. Let l be a fixed positive integer. From (2.1) and (2.7), $q_n(l) = 0$ for $0 \leq n < 2l+1$, $q_{2l+1}(l) = q_{2l+1}$, and $q_{2l+2}(l) = q_{2l+2}$, and

$$q_n(l) = \begin{cases} aq_{n-1}(l) + q_{n-2}(l) - a\binom{n-l-3}{l}(ab)^{\lfloor\frac{n-3}{2}\rfloor - l}, & \text{if } n \equiv 0 \pmod 2; \\ bq_{n-1}(l) + q_{n-2}(l) - \binom{n-l-3}{l}(ab)^{\lfloor\frac{n-3}{2}\rfloor - l}, & \text{if } n \equiv 1 \pmod 2. \end{cases} \tag{3.10}$$

Now let

$$s_0 = q_{2l+1}(l) = q_{2l+1}, \quad s_1 = q_{2l+2}(l) = q_{2l+1}, \text{ and}$$

$$s_n = q_{n+2l+1}(l).$$

Also let

$$r_0 = r_1 = 0 \text{ and } r_n = \binom{n+l-2}{n-2}(ab)^{\lfloor\frac{n}{2}\rfloor - 1}.$$

The generating function of the sequence $\{-r_n\}$ is

$$G(t) = -\frac{1}{2}\left(\frac{t^2}{(1-(ab)^{1/2}t)^{l+1}}(1+(ab)^{-1/2}) + \frac{t^2}{(1+(ab)^{1/2}t)^{l+1}}(1-(ab)^{-1/2})\right)$$

See [14, p. 355] and bisection generating functions [7]. Thus, from Lemma 3.1, we get the generating function $Q_l(t)$ of sequence $\{q_n(l)\}_{n=0}^{\infty}$. □

4. Conclusion

In this paper, we introduce the notion of bi-periodic incomplete Fibonacci numbers, and we obtain new identities. An open question is to evaluate the right sum in Proposition 2.5. On the other hand, in [9], authors introduced the bi-periodic Lucas numbers. They are defined by the recurrence relation

$$p_0 = 2, \quad p_1 = 1, \quad p_n = \begin{cases} ap_{n-1} + p_{n-2}, & \text{if } n \equiv 0 \pmod 2; \\ bp_{n-1} + p_{n-2}, & \text{if } n \equiv 1 \pmod 2; \end{cases} \quad n \geqslant 2. \tag{4.1}$$

It would be interesting to study the bi-periodic incomplete Lucas numbers and research their properties.

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