Applied Mathematics and Computation 229 (2014) 208–213

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On convolved generalized Fibonacci and Lucas polynomials

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ARTICLE INFO

Keywords:
Convolved $h(x)$-Fibonacci polynomials
$h(x)$-Fibonacci polynomials
$h(x)$-Lucas polynomials
Hessenberg matrices

ABSTRACT

We define the convolved $h(x)$-Fibonacci polynomials as an extension of the classical convolved Fibonacci numbers. Then we give some combinatorial formulas involving the $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. Moreover we obtain the convolved $h(x)$-Fibonacci polynomials from a family of Hessenberg matrices.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every fields of science and art (e.g., see [1]). The Fibonacci numbers F_n are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$, wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci Numbers.

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in\mathbb{N}}$, is defined recurrently by

$$F_{k,0} = 0, \quad F_{k,1} = 1, \quad F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geqslant 1 \tag{1}$$

In [2], k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [2–7].

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by

$$(1 - x - x^2)^{-r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+.$$

If $r = 1$ we have classical Fibonacci numbers.These numbers have been studied in several papers; see [8–10]. Convolved k-Fibonacci numbers have been studied in [11].

Large classes of polynomials can be defined by Fibonacci-like recurrence relation and yield Fibonacci numbers [1]. Such polynomials, called Fibonacci polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x), \quad n \geqslant 1. \tag{2}$$

The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_0(x) = 1, \quad J_1(x) = 1, \quad J_{n+1}(x) = J_n(x) + xJ_{n-1}(x), \quad n \geqslant 1. \tag{3}$$

The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x), \quad n \geqslant 1. \tag{4}$$

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http://dx.doi.org/10.1016/j.amc.2013.12.049

In [12], the authors introduced the $h(x)$-Fibonacci polynomials. That generalize Catalan's Fibonacci polynomials $F_n(x)$ and the k-Fibonacci numbers $F_{k,n}$. In this paper, we introduce the convolved $h(x)$-Fibonacci polynomials and we obtain new identities.

2. Some properties of $h(x)$-Fibonacci polynomials and $h(x)$-Lucas polynomials

Definition 1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$F_{h,0}(x) = 0, \quad F_{h,1}(x) = 1, \quad F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geqslant 1. \tag{5}$$

For $h(x) = x$ we obtain Catalan's Fibonacci polynomials, and for $h(x) = k$ we obtain k-Fibonacci numbers. For $k = 1$ and $k = 2$ we obtain the usual Fibonacci numbers and the Pell numbers.

The characteristic equation associated with the recurrence relation (5) is $v^2 = h(x)v + 1$. The roots of this equation are

$$r_1(x) = \frac{h(x) + \sqrt{h(x)^2 + 4}}{2}, \quad r_2(x) = \frac{h(x) - \sqrt{h(x)^2 + 4}}{2}.$$

Then we have the following basic identities:

$$r_1(x) + r_2(x) = h(x), \quad r_1(x) - r_2(x) = \sqrt{h(x)^2 + 4}, \quad r_1(x)r_2(x) = -1. \tag{6}$$

Some of the properties that the $h(x)$-Fibonacci polynomials verify are summarized bellow (see [12] for the proofs).

- Binet formula: $F_{h,n}(x) = \frac{r_1(x)^n - r_2(x)^n}{r_1(x) - r_2(x)}$.
- Combinatorial formula: $F_{h,n}(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} h^{n-1-2i}(x)$.
- Generating function: $g_f(t) = \frac{t}{1 - h(x)t - t^2}$.

Definition 2. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n\in\mathbb{N}}$ are defined by the recurrence relation

$$L_{h,0}(x) = 2, \quad L_{h,1}(x) = h(x), \quad L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x), \quad n \geqslant 1. \tag{7}$$

For $h(x) = x$ we obtain the Lucas polynomials, and for $h(x) = k$ we have the k-Lucas numbers. For $k = 1$ we obtain the usual Lucas numbers.

Some properties that the $h(x)$-Lucas numbers verify are summarized bellow (see [12] for the proofs).

- Binet formula: $L_{h,n}(x) = r_1(x)^n + r_2(x)^n$.
- Relation with $h(x)$-Fibonacci polynomials: $L_{h,n}(x) = F_{h,n-1}(x) + F_{h,n+1}(x), \ n \geqslant 1$.

3. Convolved $h(x)$-Fibonacci polynomials

Definition 3. The convolved $h(x)$-Fibonacci polynomials $F_{h,j}^{(r)}(x)$ are defined by

$$g_h^{(r)}(t) = (1 - h(x)t - t^2)^{-r} = \sum_{j=0}^{\infty} F_{h,j+1}^{(r)}(x)t^j, \quad r \in \mathbb{Z}^+.$$

Note that

$$F_{h,m+1}^{(r)}(x) = \sum_{j_1+j_2+\cdots+j_r=m} F_{h,j_1+1}(x)F_{h,j_2+1}(x)\cdots F_{h,j_r+1}(x). \tag{8}$$

Moreover, using a result of Gould [13, p. 699] on Humbert polynomials (with $n = j, \ m = 2, \ x = h(x)/2, \ y = -1, \ p = -r$ and $C = 1$), we have

$$F_{h,j+1}^{(r)}(x) = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l} \binom{j-l}{l} h(x)^{j-2l}. \tag{9}$$

If $r = 1$ we obtain the combinatorial formula of $h(x)$-Fibonacci polynomials. In Table 1 some polynomials of convolved $h(x)$-Fibonacci polynomials are provided. The purpose of this paper is to investigate the properties of these polynomials.

Table 1
$F_{h,n}^{(r)}(x)$, with $r = 1, 2, 3$.

n	$F_{h,n}^{(1)}(x)$	$F_{h,n}^{(2)}(x)$	$F_{h,n}^{(3)}(x)$
0	0	0	0
1	1	1	1
2	h	2 h	3 h
3	h^2+1	$3h^2+2$	$6h^2+3$
4	h^3+2h	$4h^3+6h$	$10h^3+12h$
5	h^4+3h^2+1	$5h^4+12h^2+3$	$15h^4+30h^2+6$
6	h^5+4h^3+3h	$6h^5+20h^3+12h$	$21h^5+60h^3+30h$
7	$h^6+5h^4+6h^2+1$	$7h^6+30h^4+30h^2+4$	$28h^6+105h^4+90h^2+10$
8	$h^7+6h^5+10h^3+4h$	$8h^7+42h^5+60h^3+20h$	$36h^7+168h^5+210h^3+60h$

Theorem 4. *The following identities hold:*

1. $F_{h,2}^{(r)}(x) = rh(x)$.
2. $F_{h,n}^{(r)}(x) = F_{h,n}^{(r-1)}(x) + h(x)F_{h,n-1}^{(r)}(x) + F_{h,n-2}^{(r)}(x), \; n \geqslant 2.$
3. $nF_{h,n+1}^{(r)}(x) = r(h(x)F_{h,n}^{(r+1)}(x) + 2F_{h,n-1}^{(r+1)}(x)), \; n \geqslant 1.$

Proof.

1. Taking $j = 1$ in (9), we obtain

$$F_{h,2}^{(r)}(x) = \binom{r}{1}\binom{1}{0}h(x) = rh(x).$$

2. This identity is obtained from observing that

$$\sum_{j=0}^{\infty} F_{h,j+1}^{(r)}(x)t^j = (h(x)t + t^2)\sum_{j=0}^{\infty} F_{h,j+1}^{(r)}(x)t^j + \sum_{j=0}^{\infty} F_{h,j+1}^{(r-1)}(x)t^j.$$

3. Taking the first derivative of $g_h^{(r)}(t) = (1 - h(x)t - t^2)^{-r}$, we obtain

$$(g_h^{(r)}(t))' = \sum_{j=1}^{\infty} F_{h,j+1}^{(r)}(x)jt^{j-1} = r\left(\frac{1}{1 - h(x)t - t^2}\right)^{r-1}\left(\frac{h(x) + 2t}{(1 - h(x)t - t^2)^2}\right) = r(h(x) + 2t)g_h^{(r+1)}(t)$$

Therefore the identity is clear. □

In the next theorem we show that the convolved $h(x)$-Fibonacci polynomials can be expressed in terms of $h(x)$-Fibonacci and $h(x)$-Lucas polynomials. This theorem generalizes Theorem 4 of [10] and Theorem 4 of [11].

Theorem 5. *Let $j \geqslant 0$ and $r \geqslant 1$. We have*

$$\begin{aligned} F_{h,j+1}^{(r)}(x) = &\sum_{\substack{l=0 \\ r+l\cong 0 \bmod 2}}^{r-1} \binom{r+l-1}{l}\binom{r-l+j-1}{j}\frac{1}{(h(x)^2+4)^{(r+l)/2}}L_{h,r+j-l}^{(r)}(x) \\ &+ \sum_{\substack{l=0 \\ r+l\cong 1 \bmod 2}}^{r-1} \binom{r+l-1}{l}\binom{r-l+j-1}{j}\frac{1}{(h(x)^2+4)^{(r+l-1)/2}}F_{h,r+j-l}^{(r)}(x) \end{aligned} \quad (10)$$

Proof. Given $\alpha, \beta \in \mathbb{C}$, such that $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Then we have the following partial fraction decomposition:

$$(1 - \alpha z)^{-r}(1 - \beta z)^{-r} = \sum_{l=0}^{r-1}\binom{-r}{l}\frac{\alpha^r\beta^l}{(\alpha - \beta)^{r+l}}(1 - \alpha z)^{l-r} + \sum_{l=0}^{r-1}\binom{-r}{l}\frac{\beta^r\alpha^l}{(\beta - \alpha)^{r+l}}(1 - \beta z)^{l-r},$$

where $\binom{t}{0} = 1$ and $\binom{t}{l} = \frac{t(t-1)\cdots(t-l+1)}{l!}$ with $t \in \mathbb{R}$. Using the Taylor expansion

$$(1 - z)^t = \sum_{j=0}^{\infty}(-1)^j\binom{t}{j}z^j$$

Then $(1-\alpha z)^{-r}(1-\beta z)^{-r}=\sum_{j=0}^{\infty}\gamma(j)z^j$, where

$$\gamma(j)=\sum_{l=0}^{r-1}\binom{-r}{l}\frac{\alpha^r\beta^l}{(\alpha-\beta)^{r+l}}(-1)^j\binom{l-r}{j}\alpha^j+\sum_{l=0}^{r-1}\binom{-r}{l}\frac{\beta^r\alpha^l}{(\beta-\alpha)^{r+l}}(-1)^j\binom{l-r}{j}\beta^j$$

Note that $1-h(x)z-z^2=(1-r_1(x)z)(1-r_2(x)z)$. On substituting these values of $\alpha=r_1(x)$ and $\beta=r_2(x)$ and using the identities (6), we obtain

$$F_{h,j+1}^{(r)}(x)=\sum_{l=0}^{r-1}\binom{-r}{l}\frac{(-1)^l\alpha^{r-l}}{(h(x)^2+4)^{(r+l)/2}}(-1)^j\binom{l-r}{j}\alpha^j+\sum_{l=0}^{r-1}\binom{-r}{l}\frac{(-1)^l\beta^{r-l}}{(h(x)^2-4)^{(r+l)/2}}(-1)^j\binom{l-r}{j}\beta^j$$
$$=\sum_{l=0}^{r-1}(-1)^l\binom{-r}{l}(-1)^j\binom{l-r}{j}\frac{1}{(h(x)^2+4)^{(r+l)/2}}\left(\alpha^{r+j-l}+(-1)^{r+l}\beta^{r+j-l}\right)$$

Since that $(-1)^l\binom{-r}{l}=\binom{r+l-1}{l}$ and $(-1)^j\binom{l-r}{j}=\binom{r-l+j-1}{j}$, then

$$F_{h,j+1}^{(r)}(x)=\binom{r+l-1}{l}\binom{r-l+j-1}{j}\frac{1}{(h(x)^2+4)^{(r+l)/2}}\left(\alpha^{r+j-l}+(-1)^{r+l}\beta^{r+j-l}\right)$$

From the above equality and Binet formula, we obtain the Eq. (10). □

4. Hessenberg matrices and convolved $h(x)$-Fibonacci polynomials

An upper Hessenberg matrix, A_n, is an $n\times n$ matrix, where $a_{i,j}=0$ whenever $i>j+1$ and $a_{j+1,j}\neq 0$ for some j. That is, all entries bellow the superdiagonal are 0 but the matrix is not upper triangular:

$$A_n=\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n}\\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n}\\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n}\\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n}\end{pmatrix} \quad (11)$$

We consider a type of upper Hessenberg matrix whose determinants are $h(x)$-Fibonacci numbers. Some results about Fibonacci numbers and Hessenberg can be found in [14]. The following known result about upper Hessenberg matrices will be used.

Theorem 6. *Let $a_1, p_{i,j}\ (i\leqslant j)$ be arbitrary elements of a commutative ring R, and let the sequence $a_1, a_2, \ldots$ be defined by:*

$$a_{n+1}=\sum_{i=1}^{n}p_{i,n}a_i,\quad (n=1,2,\ldots).$$

If

$$A_n=\begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n}\\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n}\\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n}\\ 0 & 0 & 0 & \cdots & -1 & a_{n,n}\end{pmatrix}$$

then

$$a_{n+1}=a_1\ \det\ A_n. \quad (12)$$

In particular, if

$$F_n^{(h)}=\begin{pmatrix} h(x) & 1 & 0 & \cdots & 0 & 0\\ -1 & h(x) & 1 & \cdots & 0 & 0\\ 0 & -1 & h(x) & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & h(x) & 1\\ 0 & 0 & 0 & \cdots & -1 & h(x)\end{pmatrix} \quad (13)$$

then from Theorem 6 we have that

$$\det F_n^{(h)} = F_{h,n+1}(x), \quad (n = 1, 2, \ldots). \tag{14}$$

It is clear that the principal minor $M^{(h)}(i)$ of $F_n^{(h)}$ is equal to $F_{h,i}(x)F_{h,n-i+1}(x)$. It follows that the principal minor $M^{(h)}(i_1, i_2, \ldots, i_l)$ of the matrix $F_n^{(h)}$ is obtained by deleting rows and columns with indices $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$:

$$M^{(h)}(i_1, i_2, \ldots, i_l) = F_{h,i_1}(x)F_{h,i_2-i_1}(x) \cdots F_{h,i_l-i_{l-1}}(x)F_{h,n-i_l+1}(x). \tag{15}$$

Then we have the following theorem.

Theorem 7. *Let $S_{n-l}^{(h)}$, $(l = 0, 1, 2, \ldots, n-1)$ be the sum of all principal minors of $F_n^{(h)}$ or order $n - l$. Then*

$$S_{n-l}^{(h)} = \sum_{j_1+j_2+\cdots+j_{l+1}=n-l} F_{h,j_1+1}(x)F_{h,j_2+1}(x) \cdots F_{h,j_{l+1}+1}(x) = F_{h,n-l+1}^{(l+1)}(x). \tag{16}$$

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

Corollary 8. *The convolved $h(x)$-Fibonacci polynomials $F_{h,n-l+1}^{(l+1)}(x)$ is equal, up to the sign, to the coefficient of t^l in the characteristic polynomial $p_n(t)$ of $F_n^{(h)}$.*

Corollary 9. *The following identity holds:*

$$F_{h,n-l+1}^{(l+1)}(x) = \sum_{i=0}^{\lfloor (n-l)/2 \rfloor} \binom{n-i}{i} \binom{n-2i}{l} h(x)^{n-2i-l}.$$

Proof. The characteristic matrix of $F_n^{(h)}$ has the form

$$\begin{pmatrix} t-h(x) & 1 & 0 & \cdots & 0 & 0 \\ -1 & t-h(x) & 1 & \cdots & 0 & 0 \\ 0 & -1 & t-h(x) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t-h(x) & 1 \\ 0 & 0 & 0 & \cdots & -1 & t-h(x) \end{pmatrix} \tag{17}$$

Then $p_n(t) = F_{n+1}(t - h(x))$, where $F_{n+1}(t)$ is a Fibonacci polynomial. Then from Corollary 8 and the following identity for Fibonacci polynomial [5]:

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i},$$

we obtain that

$$F_{n+1}(t - h(x)) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \sum_{l=0}^{n-2i} \binom{n-2i}{l} (-1)^{n-l} h(x)^{n-2i-l} t^l.$$

Therefore the corollary is obtained. □

Acknowledgments

The author would like to thank the anonymous referees for their helpful comments. The author was partially supported by Universidad Sergio Arboleda under Grant No. USA-II-2012-14.

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