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Generalized statistics on S_n and pattern avoidance

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Abstract

Natural q analogues of classical statistics on the symmetric groups S_n are introduced; parameters like: the q-length, the q-inversion number, the q-descent number and the q-major index. Here q is a positive integer. MacMahon's theorem (Combinatory Analysis I–II (1916)) about the equi-distribution of the inversion number and the reverse major index is generalized to all positive integers q. It is also shown that the q-inversion number and the q-reverse major index are equi-distributed over subsets of permutations avoiding certain patterns. Natural q analogues of the Bell and the Stirling numbers are related to these q statistics—through the counting of the above pattern-avoiding permutations.

1. Introduction

MacMahon's celebrated theorem about the equi-distribution of the *length* (or the *inversion-number*) and the *major index* statistics on the symmetric group S_n [10]—has received far-reaching refinements and generalizations through the last three decades. For a brief review on these refinements—see [12]. In [12] we extended the various classical S_n statistics, in a natural way, to the alternating group A_{n+1}. This was done via the canonical presentations of the elements of these groups, and by a certain covering map $f\colon A_{n+1} \to S_n$.

Further refinements of MacMahon's theorem were obtained in [12] by the introduction of the '*delent*' statistics on these groups. Then these equi-distribution theorems for S_n were 'lifted' back, via $f\colon A_{n+1} \to S_n$, thus yielding equi-distribution theorems for A_{n+1}.

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This paper continues [12] and might be considered as its *q-analogue*. Note that here q is a positive integer; the generalization to an arbitrary q is still open. We introduce the *q-analogues* of the classical statistics on the symmetric groups: the *q-length*, the *q-inversion number*, the *q-descent number*, the *q-major index* and the *q-reverse-major index* of a permutation. The *q-delent* statistics are also introduced. We then extend classical properties to these q-analogues. For example, it is proved that the q-length equals the q-inversion number of a permutation; furthermore, it is proved that the q-inversion number and the q-reverse major index are equi-distributed on S_{n+q-1}. See below.

It is realized that the above map $f: A_{n+1} \to S_n$ is the restriction to A_{n+1} of a covering map $f_2: S_{n+1} \to S_n$. More generally, we have similar covering maps $f_q: S_{n+q-1} \to S_n$ for all positive integers q. These maps are defined via the canonical presentations of the elements in S_{n+q-1}. It is proved that the map f_q sends the q-statistics on S_{n+q-1} to the corresponding classical statistics on S_n, see Proposition 8.6 below. For example, if $\pi \in S_{n+q-1}$, it is proved there that the q-inversion number of π equals the inversion number of $f_q(\pi)$.

Dashed patterns in permutations were introduced by Babson and Steingrimsson [2]. For example, a permutation σ contains the pattern $(1-32)$ if $\sigma = [\ldots, a, \ldots, c, b, \ldots]$ for some $a < b < c$; if no such a, b, c exist then σ is said to avoid $(1-32)$. Connections between the number of permutations avoiding $(1-32)$—and various combinatorial objects, like the Bell and the Stirling numbers, as well as the number of left-to-right-minima in permutations were proved by Claesson [3]. Via the various q-statistics we obtain q-analogues for these connections and results.

For a permutation $\pi \in S_{n+q-1}$ it is proved that the *q-descent* and the *q-delent* numbers of π are equal exactly when π avoids a certain collection of dashed patterns, and that the number of these permutations is $(q-1)! \sum_k q^k S(n,k)$, where $S(n,k)$ are the Stirling numbers of the second kind, see Corollary 2.8. Also, the number of permutations in S_{n+q-1} for which the *q-delent* number equals $k-1$ is $(q-1)! q^k c(n,k)$, where $c(n,k)$ are the Stirling numbers of the first kind; see Proposition 2.9.

Equi-distribution of q-statistics is studied in Section 11. A *q-analogue* of MacMahon's classical equi-distribution theorem is given, see Theorem 2.5 below. Multivariate refinements of MacMahon's theorem, due to Foata–Schützenberger and others [7, 12, 14], also have corresponding q-analogues. These analogues are described in Section 11.1, see Theorem 11.5 and its consequences.

An intensive study of equi-distribution over subsets of permutations avoiding patterns has been carried out recently, cf. [1, 5, 6, 13]. In Section 11.2 it is shown that certain q-statistics are equi-distributed on the above subsets of dashed-patterns-avoiding permutations. See Theorems 2.6 and 11.7 below.

2. The main results

Throughout the paper q is a positive integer. Recall the unique canonical presentation of a permutation in S_n as a product of shortest coset representatives along the principal flag, see Section 3.1 below. The *q-length* of a permutation $\pi \in S_n$, $\ell_q(\pi)$, is the number of Coxeter generators in the canonical presentation of π, where the generators $s_1, \ldots, s_{q-1}$ are not counted.

$$\mathrm{inv}_q(\pi) := \sum_{i=q+1}^{n} m_q(i),$$

where

$$m_q(i) := \min\{i - q, \#\{j < i \mid \pi(j) > \pi(i)\}\}.$$

Also $\mathrm{inv}_q(\pi) := 0$ if $n \le q$. Thus $\ell_1(\pi) = \ell(\pi)$ and $\mathrm{inv}_1(\pi) = \mathrm{inv}(\pi)$.

As in the (classical) case $q = 1$, we have

Proposition 2.1 (See Proposition 4.2). *For every $\sigma \in S_n$*

$$\ell_q(\sigma) = \mathrm{inv}_q(\sigma).$$

Proposition 2.2 (See Proposition 6.1). *For every $\pi \in A_n$, $\ell_2(\pi)$ is the length with respect to the set of generators $\{a_1, \ldots, a_{n-2}\} \subset A_n$, where $a_i := s_1 s_{i+1}$.*

Define the *q-delent number*, $\mathrm{del}_q(\pi)$, to be the number of times s_q appears in the canonical presentation of π.

For $0 \le k \le n-1$ define the kth *almost left-to-right-minima* in a permutation $\pi \in S_n$ (denoted a^k.l.t.r.min) as the set of indices

$$\mathrm{Del}_{k+1}(\pi) := \{i \mid k+2 \le i \le n, \#\{j < i \mid \pi(j) < \pi(i)\} \le k\}.$$

Thus $\mathrm{Del}_q(\pi)$ is the set of a^{q-1}.l.t.r.min in π. See Example 5.10 below.

Proposition 2.3 (See Proposition 5.2). *The number of occurrences of s_{k+1} in the canonical presentation of $\pi \in S_n$, $\mathrm{del}_{k+1}(w)$, equals the number of a^k.l.t.r.min in π.*

The second delent statistics del_2 on even permutations in A_{n+1} and the first delent statistics del_1 on S_n have analogous interpretations. See, for example, Proposition 6.1.

The *q-descent set* of $\pi \in S_{n+q-1}$ is defined as

$$\mathrm{Des}_q(\pi) := \{i \mid i \text{ is a } q\text{-descent in } \pi\},$$

and the *q-descent number* is defined as

$$\mathrm{des}_q(\pi) := \#\mathrm{Des}_q(\pi).$$

For $\pi \in S_{n+q-1}$ define the *q-major index*

$$\mathrm{maj}_q(\pi) := \sum_{i \in \mathrm{Des}_q(\pi)} i$$

and the *q-reverse major index*

$$\mathrm{rmaj}_{q,m}(\pi) := \sum_{i \in \mathrm{Des}_q(\pi)} (m - i),$$

where $m = n + q - 1$.

Thus Des_1 is the standard descent set of a permutation in S_n. The definition of the q-descent set is justified by the following phenomena:

(1) Des_2 is the descent set on the alternating group A_n with respect to the distinguished set of generators $\{a_1, \ldots, a_{n-2}\}$, where $a_i := s_1 s_{i+1}$, see Proposition 6.1.
(2) The q-descent set, Des_q, is strongly related with pattern avoiding permutations, see Proposition 9.3.
(3) Des_q is involved in the definition of the q-(reverse) major index, and thus in the q-analogue of MacMahon's equi-distribution theorem (Theorem 11.2).

Given q, denote by

$$\mathrm{Pat}(q) = \{(\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q+2), (q+1)) \mid \sigma \in S_q\}$$

the set with these $q!$ dashed patterns. For example, $\mathrm{Pat}(1) = \{(1-32)\}$ $\mathrm{Pat}(2) = \{(1-2-43), (2-1-43)\}$.

Denote by $\mathrm{Avoid}_q(n+q-1)$ the set of permutations in S_{n+q-1} avoiding all the $q!$ patterns in $\mathrm{Pat}(q)$.

Proposition 2.4 (See Proposition 9.3). *A permutation* $\pi \in S_{n+q-1}$ *avoids* $\mathrm{Pat}(q)$ *exactly when* $\mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)$:

$$\mathrm{Avoid}_q(n+q-1) = \{\pi \in S_{n+q-1} \mid \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}.$$

The following is a q-analogue of MacMahon's equi-distribution theorem.

Theorem 2.5 (See Theorem 11.2).

$$\sum_{\pi \in S_{n+q-1}} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)} = \sum_{\pi \in S_{n+q-1}} t^{\mathrm{inv}_q(\pi)}$$
$$= q!(1+tq)(1+t+t^2q)\cdots(1+t+\cdots+t^{n-2}+t^{n-1}q).$$

Far reaching multivariate refinements of MacMahon's theorem, which imply equi-distribution on subsets of permutations, were given by Foata and Schützenberger and others, cf. [7, 8, 12, 14]. In Section 11.1 we describe some q-analogues of these refinements, see Theorem 11.4 and Corollary 11.6 below.

The above q-statistics are equi-distributed on permutations avoiding $\mathrm{Pat}(q)$.

Theorem 2.6 (See Corollary 11.8).

$$\sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t_1^{\mathrm{rmaj}_{q,n+q-1}(\pi)} t_2^{\mathrm{des}_q(\pi)} = \sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t_1^{\mathrm{inv}_q(\pi)} t_2^{\mathrm{des}_q(\pi)}.$$

For example, for $q = 1$

$$\sum_{\pi^{-1} \in \mathrm{Avoid}(1-32)} t_1^{\mathrm{rmaj}_n(\pi)} t_2^{\mathrm{des}(\pi)} = \sum_{\pi^{-1} \in \mathrm{Avoid}(1-32)} t_1^{\mathrm{inv}(\pi)} t_2^{\mathrm{des}(\pi)}.$$

For $q = 2$

$$\sum_{\pi^{-1} \in \mathrm{Avoid}(1-2-43,2-1-43)} t_1^{\mathrm{rmaj}_{2,n+1}(\pi)} t_2^{\mathrm{des}_2(\pi)}$$
$$= \sum_{\pi^{-1} \in \mathrm{Avoid}(1-2-43,2-1-43)} t_1^{\mathrm{inv}_2(\pi)} t_2^{\mathrm{des}_2(\pi)}.$$

Bell and Stirling numbers (of both kinds) appear naturally in the enumeration of permutations with respect to their q-statistics.

Let $c(n,k)$ be the kth Stirling number of the first kind and $S(n,k)$ be the kth Stirling number of the second kind. Let the nth q-Bell number be $b_q(n) := \sum_k q^k S(n,k)$. Let $B_q(x) := \sum_{n=0}^{\infty} b_q(n)\frac{x^n}{n!}$ denote the exponential generating function of $\{b_q(n)\}$. Then

$$B_q(x) = \exp(q\mathrm{e}^x - q).$$

The classical formula $b_1(n) = \frac{1}{\mathrm{e}}\sum_{r=0}^{\infty}\frac{r^n}{r!}$ [4] (see also [15, (1.6.10)]) generalizes as follows:

$$b_q(n) = \frac{1}{\mathrm{e}^q}\sum_{r=0}^{\infty}\frac{q^r r^n}{r!},$$

see Remark 10.4.

Proposition 2.7 (See Proposition 10.8).

$$\#\{\sigma \in S_{n+q-1} \mid \mathrm{Del}_q(\sigma) - 1 = \mathrm{Des}_q(\sigma) \textit{ and } \mathrm{del}_q(\sigma) = k-1\}$$
$$= (q-1)!q^k S(n,k).$$

Corollary 2.8 (See Propositions 9.3 and 10.5).

$$(q-1)!b_q(n) = \#\{\pi \in S_{n+q-1} \mid \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\} = \mathrm{Avoid}_q(n+q-1).$$

Proposition 2.9 (See Proposition 10.10).

$$\#\{\pi \in S_{n+q-1} \mid \mathrm{del}_q(\pi) = k-1\} = c_q(n,k),$$

where $c_q(n,k) = q^k(q-1)!c(n,k)$.

3. Preliminaries

3.1. The S_n canonical presentation

A basic tool, both in [12] and in this paper, is the canonical presentation of a permutation, which we now describe.

Recall that the transpositions $s_i = (i, i+1)$, $1 \le i < n-1$, are the Coxeter generators of the symmetric group S_n. For each $1 \le j \le n-1$ define

$$R_j^S = \{1, s_j, s_j s_{j-1}, \ldots, s_j s_{j-1} \cdots s_1\} \tag{1}$$

and note that $R_1^S, \ldots, R_{n-1}^S \subseteq S_n$.

The following is a classical theorem; see for example [9, pp. 61–62]. See also [12, Theorem 3.1].

Theorem 3.1. *Let $w \in S_n$, then there exist unique elements $w_j \in R_j^S$, $1 \le j \le n-1$, such that $w = w_1 \cdots w_{n-1}$. Thus, the presentation $w = w_1 \cdots w_{n-1}$ is unique; it is called the* **canonical presentation** *of w.*

Note that R_j^S is the complete list of representatives of minimal length of right cosets of S_j in S_{j+1}. Thus, the canonical presentation of $w \in S_n$ is the unique presentation of w as a product of shortest coset representatives along the principal flag

$$\{e\} = S_1 < S_2 < \cdots < S_n.$$

We remark that a similar canonical presentation for the alternating groups A_n is given in [12], see Section 3.2 below.

The descent set $\mathrm{Des}(\pi)$ of a permutation $\pi \in S_n$ is a classical notion. In [12] the '*delent*' statistic was introduced: $\mathrm{Del}(\pi)$ is the set of indices i which are left-to-right-minima of π, and $\mathrm{del}(\pi) = \#\mathrm{Del}(\pi)$. By Proposition 7.2 of [12], $\mathrm{del}(\pi)$ equals the number of times that $s_1 = (1, 2)$ appears in the canonical presentation of π.

Theorem 9.1 is the main theorem of [12] and we now state its part about S_n (it also has a similar part about A_n).

Theorem 3.2. *For every subset $D_1 \subseteq [n-1]$ and $D_2 \subseteq [n-1]$*

$$\sum_{\{\pi \in S_n \mid \mathrm{Des}_S(\pi^{-1}) \subseteq D_1, \mathrm{Del}_S(\pi^{-1}) \subseteq D_2\}} q^{\mathrm{rmaj}_{S_n}(\pi)} = \sum_{\{\pi \in S_n \mid \mathrm{Des}_S(\pi^{-1}) \subseteq D_1, \mathrm{Del}_S(\pi^{-1}) \subseteq D_2\}} q^{\ell_S(\pi)}.$$

In the following case, a simple explicit generating function is given.

Theorem 3.3 ([12, Theorem 6.1]).

$$\sum_{\sigma \in S_n} q^{\ell_S(\sigma)} t^{\mathrm{del}_S(\sigma)} = \sum_{\sigma \in S_n} q^{\mathrm{rmaj}_{S_n}(\sigma)} t^{\mathrm{del}_S(\sigma)} = (1+qt)(1+q+q^2t)\cdots(1+q+\cdots+q^{n-1}t).$$

3.2. *The alternating group*

The alternating group serves as a motivating example. Here are some results from [12], which are applied in Section 6 and Appendix and in the formulation and proof of Proposition 8.5. The reader who is not interested in this motivating example may skip this subsection.

Let

$$a_i := s_1 s_{i+1} \qquad (1 \le i \le n-1) \quad \text{and} \quad A := \{a_i \mid 1 \le i \le n-1\}.$$

The set A generates the alternating group on $n+1$ letters A_{n+1}. This generating set and its following properties appear in [11].

Proposition 3.4 ([11, Proposition 2.5]). *The defining relations of A are*

$$(a_i a_j)^2 = 1 \qquad (|i-j| > 1); \qquad (a_i a_{i+1})^3 = 1 \qquad (1 \le i < n-1);$$
$$a_1^3 = 1 \qquad \textit{and} \qquad a_i^2 = 1 \qquad (1 < i \le n-1).$$

For each $1 \le j \le n-1$ define

$$R_j^A = \{1, a_j, a_j a_{j-1}, \ldots, a_j \cdots a_2, a_j \cdots a_2 a_1, a_j \cdots a_2 a_1^{-1}\} \tag{2}$$

and note that $R_1^A, \ldots, R_{n-1}^A \subseteq A_{n+1}$.

Theorem 3.5. *Let $v \in A_{n+1}$, then there exist unique elements $v_j \in R_j^A$, $1 \le j \le n-1$, such that $v = v_1 \cdots v_{n-1}$, and this presentation is unique.*

This presentation is called the A canonical presentation of v.

For $\sigma \in A_{n+1}$ let $\ell_A(\sigma)$ be the length of the A canonical presentation of σ. Let

$$\mathrm{Des}_A(\sigma) := \{i \mid \ell_A(\sigma) \le \ell_A(\sigma a_i)\}$$

and $\mathrm{des}_A(\sigma) := \#\mathrm{Des}_A(\sigma)$, define $\mathrm{maj}_A(\sigma) := \sum_{i \in \mathrm{Des}_a(\sigma)} i$, and $\mathrm{rmaj}_{A_{n+1}}(\sigma) := \sum_{i \in \mathrm{Des}_a(\sigma)} (n-i)$. Let $\mathrm{del}_A(\sigma)$ be the number of appearances of $a_1^{\pm 1}$ in its A canonical presentation. It is proved in [12] that this number equals the number of almost-left-to-right-minima in σ.

Theorems 3.1 and 3.5 allow us to introduce in [12] the following covering map:

Definition 3.6. Define $f: A_{n+1} \to S_n$ as follows.

$$f(a_1) = f(a_1^{-1}) = s_1 \qquad \text{and} \qquad f(a_i) = s_i, \qquad 2 \le i \le n-1.$$

Now extend $f: R_j^A \to R_j^S$ via

$$f(a_j a_{j-1} \cdots a_\ell) = s_j s_{j-1} \cdots s_\ell, \qquad f(a_j \cdots a_1) = f(a_j \cdots a_1^{-1}) = s_j \cdots s_1.$$

Finally, let $v \in A_{n+1}$, $v = v_1 \cdots v_{n-1}$ its unique A canonical presentation, then

$$f(v) = f(v_1) \cdots f(v_{n-1})$$

which is clearly the S canonical presentation of $f(v)$.

Proposition 3.7 ([12, Propositions 5.3–5.4]). *For every $\pi \in A_{n+1}$,*

$$\ell_A(\pi) = \ell_S(f(\pi)), \qquad \mathrm{Des}_A(\pi) = \mathrm{Des}_S(f(\pi)), \qquad \mathrm{Del}_A(\pi) = \mathrm{Del}_S(f(\pi))$$

Thus $\mathrm{des}_A(\pi) = \mathrm{des}_S(f(\pi))$, $\mathrm{maj}_A(\pi) = \mathrm{maj}_S(f(\pi))$, $\mathrm{rmaj}_{A_{n+1}}(\pi) = \mathrm{rmaj}_{S_n}(f(\pi))$ *and* $\mathrm{del}_A(\pi) = \mathrm{del}_S(f(\pi))$.

4. Basic concepts I

Let $\pi \in S_n$. Recall that its length $\ell(\pi)$ equals the number of the Coxeter generators $s_1, \ldots, s_{n-1}$ in its canonical presentation. It is well known that $\ell(\pi)$ also equals $\mathrm{inv}(\pi)$, the number of inversions of π. Also, it is easily seen that $\mathrm{inv}(\pi)$ can be written as

$$\mathrm{inv}(\pi) = \sum_{i=2}^{n} m(i),$$

where

$$m(i) = \min\{i - 1, \#\{j < i \mid \pi(j) > \pi(i)\}\}.$$

Thus, the following definition is a natural q-analogue of these two classical statistics.

Definition 4.1. Let $\pi \in S_n$.

1. $(\ell_q)\ell_q(\pi)$ as follows:
   $\ell_q(\pi) :=$ the number of Coxeter generators in the canonical presentation of π, where $s_1, \ldots, s_{q-1}$ are not counted (thus, for example, $\ell_2(s_1) = 0$ and $\ell_2(s_1 s_2 s_1 s_3 s_2 s_1) = 3$).
2. (inv_q) begining of Section 2.

Thus $\ell_1(\pi) = \ell(\pi)$ and $\mathrm{inv}_1(\pi) = \mathrm{inv}(\pi)$.
As in the (classical) case $q = 1$, we have

Proposition 4.2. *For every $\sigma \in S_n$*

$$\ell_q(\sigma) = \mathrm{inv}_q(\sigma).$$

Proof. We may assume that $q < n$. Let $\sigma = w_1 \cdots w_{n-1}$ with $w_j \in R_j$ be the canonical presentation of σ, and denote $\pi = w_1 \cdots w_{n-2}$, then $\pi \in S_{n-1}$, hence $\pi = [b_1, \ldots, b_{n-1}, n]$. If $w_{n-1} = 1$ then $\sigma \in S_{n-1}$ and we are done by induction. Hence assume $w_{n-1} \neq 1$, so that $w_{n-1} = s_{n-1} \cdots s_k$ for some $1 \leq k \leq n - 1$, and therefore $\sigma = [b_1, \ldots, b_{k-1}, n, b_k, \ldots, b_{n-1}]$.

Case 1. $1 \leq k \leq q$, in which case

$$\ell_q(w_{n-1}) = n - q \quad \text{and} \quad \sigma = [b_1, \ldots, b_{k-1}, n, b_k, \ldots, b_q, \ldots, b_{n-1}].$$

Then for $q \leq i \leq n - 1$,

$$\#\{j < i + 1 \mid \sigma(j) > \sigma(i + 1)\} = \#\{j < i \mid b_j > b_i\} + 1$$

(the '+1' comes from $n > b_i$). Therefore $m_q(i + 1, \sigma) = m_q(i, \pi) + 1$, since

$$\begin{aligned} m_q(i + 1, \sigma) &= \min\{i + 1 - q; \#\{j < i + 1 \mid \sigma(j) > \sigma(i + 1)\}\} \\ &= \min\{i + 1 - q; \#\{j < i \mid b_j > b_i\} + 1\} \\ &= \min\{i - q; \#\{j < i \mid b_j > b_i\}\} + 1 = m_q(i, \pi) + 1. \end{aligned}$$

Thus

$$\mathrm{inv}_q(\sigma) = \sum_{i=q+1}^{n} m_q(i, \sigma) = \sum_{i=q}^{n-1} m_q(i + 1, \sigma) = \sum_{i=q}^{n-1} m_q(i, \pi) + (n - q)$$

(by induction)

$$= \ell_q(\pi) + n - q = \ell_q(\pi) + \ell_q(w_{n-1}) = \ell_q(\sigma).$$

Case 2. $q + 1 \leq k$, hence $\ell_q(w_{n-1}) = \ell_1(w_{n-1}) = n - k$, $\sigma = [b_1, \ldots, b_q, \ldots, b_{k-1}, n, b_k, \ldots, b_{n-1}]$. Here

1. $m_q(i,\sigma) = m_q(i,\pi)$ if $q+1 \le i \le k-1$,
2. $m_q(k,\sigma) = 0$ $(i = k)$, and, as in Case 1,
3. $m_q(i+1,\sigma) = m_q(i,\pi) + 1$ if $k \le i \le n-1$.

It follows that

$$\begin{aligned}\mathrm{inv}_q(\sigma) &= \sum_{i=q+1}^{n} m_q(i,\sigma) = \sum_{i=q+1}^{k-1} m_q(i,\pi) + \sum_{i=k}^{n-1} m_q(i,\pi) + n - k \\ &= \sum_{i=q}^{n-1} m_q(i,\pi) + (n-k) \text{ (by induction)} \\ &= \ell_q(\pi) + n - k = \ell_q(\pi) + \ell_q(w_{n-1}) = \ell_q(\sigma). \quad \square\end{aligned}$$

The following lemma was proved in [12].

Lemma 4.3 ([12, Lemma 3.7]). *Let $w = s_{i_1} \cdots s_{i_p}$ be the canonical presentation of $w \in S_n$. Then the canonical presentation of w^{-1} is obtained from the presentation $w^{-1} = s_{i_p} \cdots s_{i_1}$ by commuting moves only—without any braid moves. Similarly for $v, v^{-1} \in A_{n+1}$.*

Proposition 4.4. *For every $\sigma \in S_n$,*

$$\ell_q(\sigma^{-1}) = \ell_q(\sigma), \qquad \textit{hence also } \mathrm{inv}_q(\sigma^{-1}) = \mathrm{inv}_q(\sigma).$$

Proof. Lemma 4.3 easily implies that $\ell_q(\sigma^{-1}) = \ell_q(\sigma)$, while this, together with Proposition 4.2 implies the equality $\mathrm{inv}_q(\sigma^{-1}) = \mathrm{inv}_q(\sigma)$. $\square$

5. Basic concepts II

A natural q-analogue of the del statistics from [12] is introduced in this section. This allows us to introduce below a (less intuitive) q-analogue of the descent statistics.

5.1. The del statistics

Recall the definitions of Del and del (of types S and A) from [12]: given a permutation w in S_n, $\mathrm{Del}_S(w)$ is the set of indices which are *left-to-right-minima* (l.t.r.min) in w, and $\mathrm{Del}_A(w)$ is the set of indices which are *almost left-to-right-minima* (a.l.t.r.min) in w. Let $s_i = (i, i+1), i = 1, \ldots, n-1$, denote the Coxeter generators of S_n. The following classical fact is of fundamental importance in this paper.

Let $R_j = \{1, s_j, s_j s_{j-1}, \ldots, s_j s_{j-1} \cdots s_1\}$ and let $w \in S_n$, then there exist unique elements $w_j \in R_j$, $1 \le j \le n-1$, such that $w = w_1 \cdots w_{n-1}$; this is the (unique) canonical presentation of w, see Theorem 3.1 in [12].

Similarly $a_i = s_1 s_{i+1}, i = 1, \ldots, n-1$, are the corresponding generators for the alternating group A_{n+1}, and there is a corresponding unique canonical presentation for the elements of A_{n+1}, see Section 3 in [12]. The following was observed in [12]:

1. The number of times s_1 appears in the canonical presentation of w (i.e. $\mathrm{del}_S(w)$) equals the number of l.t.r.min in w (hence $\#\mathrm{Del}_S(w) = \mathrm{del}_S(w)$), see [12, Proposition 7.2].
2. The number of times s_2 appears in w equals the number of a.l.t.r.min in w. Moreover, if $w \in A_{n+1}$, that number equals the number of times $a_1^{\pm 1}$ appears in the A-canonical presentation of w, which by definition is $\mathrm{del}_A(w)$, and $\mathrm{del}_A(w) = \#\mathrm{Del}_A(w)$, see [12, Proposition 7.6].

In this paper, 'sub S' is replaced by 'sub 1': $\mathrm{Del}_S = \mathrm{Del}_1$ and $\mathrm{del}_S = \mathrm{del}_1$, etc. Similarly (in A_n) 'sub A' is replaced by 'sub 2'. We shall also encounter 'sub q' for every positive integer q.

Definition 5.1. Let $\pi \in S_n$ and let $1 \le q \le n-1$.

1. Define $\mathrm{del}_q(\pi)$ to be the number of times s_q appears in the canonical presentation of π.
2. For $0 \le k \le n-1$ define the kth *almost-left-to-right-minima* in a permutation $\pi \in S_n$ (denoted a^k.l.t.r.min) as the set of indices

$$\mathrm{Del}_{k+1}(\pi) := \{i \mid k+2 \le i \le n, \#\{j < i \mid \pi(j) < \pi(i)\} \le k\}.$$

Thus $\mathrm{Del}_q(\pi)$ is the set of a^{q-1}.l.t.r.min in π.

See Example 5.10 below.

Note that if $i \le k+1$ then, trivially, $\#\{j < i \mid \pi(j) < \pi(i)\} \le k$, however these indices are not counted as a^k.l.t.r.min. Also note that a^0.l.t.r.min is simply l.t.r.min.

Proposition 5.2. *Let $w \in S_n$. Then for every nonnegative integer k, the number of occurrences of s_{k+1} in the canonical presentation of w, $\mathrm{del}_{k+1}(w)$, equals the number of a^k.l.t.r.min in w. Writing $k+1 = q$ we have*

$$\#\mathrm{Del}_q(w) = \mathrm{del}_q(w).$$

Proof. (Generalizes the Proof of Proposition 7.6 in [12]). We first need the following two lemmas.

Lemma 5.3. *Let $1 \le k+1 \le n$, let $w \in S_n$ and let $\pi \in S_{k+1}$. Also let $i \le n$. Then i is a^k.l.t.r.min of w if and only if i is a^k.l.t.r.min of πw. In particular, the number of a^k.l.t.r.min of w equals the number of a^k.l.t.r.min of πw.*

Proof. Denote $w = [b_1, \ldots, b_n]$ (namely $w(r) = b_r$), and compare w with πw: π permutes only the b_r's in $\{1, \ldots, k+1\}$. If $b_i \in \{1, \ldots, k+1\}$, the total number of b_j's smaller than b_i is $\le k$; in particular such i is a^k.l.t.r.min in both w and πw, provided $i \ge k+2$. If on the other hand $b_i \notin \{1, \ldots, k+1\}$ then b_i is greater than all the elements in that subset; thus such i is a^k.l.t.r.min of w if and only if i is a^k.l.t.r.min of πw. This implies the proof. □

Lemma 5.4. *Let $1 \le k \le n-1$ and denote $s_{[k,n-1]} = s_k s_{k+1} \cdots s_{n-1}$. Let $\sigma \in S_{n-1}$ and write $\sigma = [b_1, \ldots, b_{n-1}, n]$. Then $s_{[k,n-1]}\sigma = [c_1, \ldots, c_{n-1}, k]$, and the two tuples $(b_1, \ldots, b_{n-1})$ and $(c_1, \ldots, c_{n-1})$ are order-isomorphic, namely for all i, j, $b_i < b_j$ if and only if $c_i < c_j$.*

Proof. Comparing σ with $s_{[k,n-1]}\sigma$, we see that

1. the (position with) n in σ is replaced in $s_{[k,n-1]}\sigma$ by k;
2. each j in σ, $k \le j \le n-1$, is replaced by $j+1$ in $s_{[k,n-1]}\sigma$;
3. each j, $1 \le j \le k-1$ is unchanged.

This implies the proof. □

The Proof of Proposition 5.2 is by induction on n. If $n \le k+1$, the number of a^k.l.t.r.min of any permutation in S_n is zero, and also $s_{k+1} \notin S_n$, hence 5.2 holds in that case.

Next assume 5.2 holds for $n-1$ and prove for n. Let $w = w_1 \cdots w_{n-1}$ be the canonical presentation of $w \in S_n$ and denote $\sigma = w_1 \cdots w_{n-2}$, then $\sigma \in S_{n-1}$. If $w_{n-1} = 1$ then $w \in S_{n-1}$ and the proof follows by induction. So let $w_{n-1} \ne 1$, then we can write $w_{n-1} = s_{n-1}s_{n-2}\cdots s_d v$, where $d \ge k+1$ and $v \in \{1, s_k, s_k s_{k-1}, \ldots, s_k s_{k-1}\cdots s_1\}$ hence $v \in S_{k+1}$. If $d \ge k+2$ then necessarily $v = 1$ and in that case the number of times s_{k+1} appears in w and in σ is the same. If $d = k+1$, that number in w is one more than in σ. We show that the same holds for the number of a^k.l.t.r.min for these two permutations σ and w.

By Lemma 3.4 of [12], it suffices to prove that statement for the inverse permutations w^{-1} and σ^{-1}. Now, $w^{-1} = \pi s_{[d,n-1]}\sigma^{-1}$, where $\pi = v^{-1} \in S_{k+1}$, hence by Lemma 5.3 it suffices to compare the number of a^k.l.t.r.min in σ^{-1} with that in $s_{[d,n-1]}\sigma^{-1}$. By Lemma 5.4 $\sigma^{-1} = [b_1, \ldots, b_{n-1}, n]$ and $s_{[d,n-1]}\sigma^{-1} = [c_1, \ldots, c_{n-1}, d]$ where the b's and the c's are order isomorphic.

The case $d \ge k+2$. Here the two last positions—n in σ^{-1} and d in $s_{[d,n-1]}\sigma^{-1}$—are not a^k.l.t.r.min, and the above order isomorphism implies the proof in that case.

The case $d = k+1$. By a similar argument, now the last position in $s_{[d,n-1]}\sigma^{-1}$ (which is $k+1$) is one additional a^k.l.t.r.min.

The proof now follows. □

Proposition 5.5. *For every positive integer q and every permutation $\pi \in S_{n+q-1}$*

$$\mathrm{del}_q(\pi) = \mathrm{del}_q(\pi^{-1}).$$

Proof. This is a straightforward consequence of Lemma 3.7 of [12], which says the following: let $\pi \in S_n$ and let $\pi = s_{i_1}\cdots s_{i_r}$ be its canonical presentation. Then the canonical presentation of π^{-1} is obtained from the equation $\pi^{-1} = s_{i_r}\cdots s_{i_1}$ by commuting moves only, without any braid moves. Thus, the number of times a particular s_j appears in π and in π^{-1} is the same. This clearly implies the proof. □

Corollary 5.6. *For every positive integer q and every permutation $\pi \in S_{n+q-1}$ the number of a^{q-1}.l.t.r.min in π equals the number of a^{q-1}.l.t.r.min in π^{-1}.*

Proof. Combining Proposition 5.2 with Proposition 5.5. □

Remark 5.7. Setting $q = k+1$ in Lemma 5.3, deduce that for any two permutations σ and η in S_{n+q-1}, if σ and η belong to the same right coset of S_q, i.e. $\eta \in S_q\sigma$, then

$$\mathrm{Del}_q(\eta) = \mathrm{Del}_q(\sigma) \qquad (\text{and therefore } \mathrm{del}_q(\eta) = \mathrm{del}_q(\sigma)).$$

The same is also true for the left cosets: let $\eta \in \sigma S_q$ then again

$$\mathrm{Del}_q(\eta) = \mathrm{Del}_q(\sigma) \qquad \text{(and therefore } \mathrm{del}_q(\eta) = \mathrm{del}_q(\sigma)\text{)}.$$

This easily follows from Definition 5.1, since if $\sigma = [b_1, \ldots, b_q, \ldots, b_n]$, $\tau \in S_q$ and $\eta = \sigma\tau$, then $\eta = [b_{\tau(1)}, \ldots, b_{\tau(q)}, b_{q+1}, \ldots, b_n]$.

Let now σ and η belong to the same left coset or right coset of S_q, then by the same reasoning, for any $q \le d$, $\mathrm{del}_d(\eta) = \mathrm{del}_d(\sigma)$ since $S_q \subseteq S_d$. Since

$$\ell_q(\eta) = \sum_{d=q}^{n-1} \mathrm{del}_d(\eta), \qquad \text{and} \qquad \ell_q(\sigma) = \sum_{d=q}^{n-1} \mathrm{del}_d(\sigma),$$

deduce that in that case $\ell_q(\eta) = \ell_q(\sigma)$.

5.2. The q-descent set

Recall that i is a *descent* of π if $\pi(i) > \pi(i+1)$, and let $\mathrm{Des}(\pi)$ denote the ('classical') descent-set of π. The following definition seems to be the appropriate q-analogue for descents.

Definition 5.8. i is a *q-descent* in $\pi \in S_{n+q-1}$ if $i \ge q$ and at least one of the following two conditions holds:

(1) $i \in \mathrm{Des}(\pi)$;
(2) $i + 1$ is an a^{q-1}.l.t.r.min in π.

Thus $\mathrm{Des}_q(\pi) = (\mathrm{Des}(\pi) \cap \{q, q+1, \ldots, n-1\}) \cup (\mathrm{Del}_q(\pi) - 1)$, hence for all q, $\mathrm{Del}_q(\pi) - 1 \subseteq \mathrm{Des}_q(\pi)$ where $\mathrm{Del}_q(\pi) - 1 = \{i - 1 \mid i \in \mathrm{Del}_q(\pi)\}$.

Note that when $q = 1$, condition (2) says that $i + 1$ is l.t.r.min, which implies that i is a descent. Thus, a 1-descent is just a descent in the classical sense.

Definition 5.9. 1. The *q-descent set* of $\pi \in S_{n+q-1}$ is defined as

$$\mathrm{Des}_q(\pi) := \{i \mid i \text{ is a } q\text{-descent in } \pi\}.$$

2. The *q-descent number* of π is defined as $\mathrm{des}_q(\pi) := \#\mathrm{Des}_q(\pi)$.
3. The *q-major index* and the *q-reverse major index* of $\pi \in S_{n+q-1}$ are defined as

$$\mathrm{maj}_q(\pi) := \sum_{i \in \mathrm{Des}_q(\pi)} i \qquad \text{and} \qquad \mathrm{rmaj}_{q,m}(\pi) := \sum_{i \in \mathrm{Des}_q(\pi)} (m - i),$$

where $m = n + q - 1$.

Example 5.10. Let $\sigma = [7, 8, 6, 5, 2, 9, 4, 1, 3]$.

When $q = 2$, $\mathrm{Del}_2(\sigma) = \{3, 4, 5, 7, 8\}$ and $\mathrm{Des}_2(\sigma) = \mathrm{Del}_2(\sigma) - 1 = \{2, 3, 4, 6, 7\}$.

When $q = 3$, $\mathrm{Del}_3(\sigma) = \{4, 5, 7, 8, 9\}$, hence $\mathrm{Des}_3(\sigma) = \{3, 4, 6, 7\} \cup \{3, 4, 6, 7, 8\} = \{3, 4, 6, 7, 8\}$.

Also, $\mathrm{Des}_4(\sigma) = \{4, 6, 7, 8\}$, etc.

6. Motivating examples

When $q = 1$, the corresponding statistics are classical. By definition, for every $\pi \in S_n$, $\ell_1(\pi) = \ell_S(\pi)$, $\mathrm{Des}_1(\pi) = \mathrm{Des}_S(\pi)$, and $\mathrm{Del}_1(\pi) = \mathrm{Del}_S(\pi)$. It follows that for every $\pi \in S_n$, $\mathrm{des}_1(\pi) = \mathrm{des}_S(\pi)$, $\mathrm{maj}_1(\pi) = \mathrm{maj}_S(\pi)$, $\mathrm{ramj}_{1,n}(\pi) = \mathrm{rmaj}_{S_n}(\pi)$, and $\mathrm{del}_1(\pi) = \mathrm{del}_S(\pi)$. The delent statistics, del_S, were introduced in [12].

The corresponding A-statistics were also studied in [12]; these A-statistics correspond to the case $q = 2$ and are restricted to the alternating groups. This is the following proposition.

Proposition 6.1. *For every even permutation* $\pi \in S_{n+1}$

(1) $\ell_2(\pi) = \ell_A(\pi)$,
(2) $\mathrm{Des}_2(\pi) = \mathrm{Des}_A(\pi)$, *and*
(3) $\mathrm{Del}_2(\pi) = \mathrm{Del}_A(\pi)$.

Proof. (1) follows from [12, Proposition 4.5]. (2) follows from Lemma A.1 in the Appendix. For (3) see [12, Proposition 7.5]. □

An alternative and more conceptual proof is given below (see Remark 8.9).

Corollary 6.2. *For every even permutation* $\pi \in S_{n+1}$, $\mathrm{des}_2(\pi) = \mathrm{des}_A(\pi)$, $\mathrm{maj}_2(\pi) = \mathrm{maj}_A(\pi)$, $\mathrm{ramj}_{2,n}(\pi) = \mathrm{rmaj}_{A_n}(\pi)$, *and* $\mathrm{del}_2(\pi) = \mathrm{del}_A(\pi)$.

7. The double cosets of $S_q \subseteq S_{n+q-1}$

Let S_q be the subgroup of S_{n+q-1} generated by $\{s_1, \ldots, s_{q-1}\}$. It is shown here that the previous q-statistics are invariant on the double cosets of S_q in S_{n+q-1}.

Proposition 7.1. *For any two permutations π and σ in S_{n+q-1}, if π and σ belong to the same double coset of S_q (namely, $\pi \in S_q \sigma S_q$), then*

(1) $\mathrm{Del}_q(\pi) = \mathrm{Del}_q(\sigma)$, *hence* $\mathrm{del}_q(\pi) = \mathrm{del}_q$;
(2) $\mathrm{Des}_q(\pi) = \mathrm{Des}_q(\sigma)$, *hence* $\mathrm{des}_q(\pi) = \mathrm{des}_q$;
(3) $\mathrm{inv}_q(\pi) = \mathrm{inv}_q(\sigma) = \ell_q(\pi) = \ell_q(\sigma)$.

Proof. It suffices to prove that if there exists $\tau \in S_q$, such that $\pi = \tau\sigma$ or $\pi = \sigma\tau$, then equalities 1–3 hold.

(1) Part 1 was proved in Remark 5.7.

(2) Denote $\sigma = [b_1, \ldots, b_{n+q-1}]$ and $\pi = [b'_1, \ldots, b'_{n+q-1}]$. Since $\mathrm{Des}_q(\pi) = (\mathrm{Des}(\pi) \cap \{q, q+1, \ldots, n\}) \cup (\mathrm{Del}_q(\pi) - 1)$, and the same for $\mathrm{Des}_q(\sigma)$, it suffices to prove the following: let $i \geq q$ and $i \in \mathrm{Des}(\sigma)$, then either $i \in \mathrm{Des}(\pi)$ or $i + 1 \in \mathrm{Del}_q(\pi)$.

We prove first the case of the right cosets: $\pi = \tau\sigma$. It is given that $b_i > b_{i+1}$.

Case 1. $b_i, b_{i+1} \notin \{1, \ldots, q\}$. Then $b_i = b'_i$ and $b_{i+1} = b'_{i+1}$ and we are done.

Case 2. $b_i \notin \{1, \ldots, q\}$ and $b_{i+1} \in \{1, \ldots, q\}$. Then $b_i = b'_i > q$ while $b'_{i+1} \in \{1, \ldots, q\}$ and we are done.

Case 3. $b_i, b_{i+1} \in \{1, \ldots, q\}$. Then at most $q - 1$ b_js in σ are left and smaller than b_{i+1}. Thus (by 1) $i + 1 \in \mathrm{Del}_q(\sigma) = \mathrm{Del}_q(\pi)$.

We prove next the case of the left cosets: $\pi = \sigma\tau$.

By the argument in Remark 5.7, the claim holds if $i > q$. Therefore examine the case $i = q$. If $q \in \mathrm{Des}(\pi)$, then we are done. Recall that $b_q > b_{q+1}$ and assume $q \notin \mathrm{Des}(\pi)$ (i.e. $b_{\tau(q)} < b_{q+1}$). It follows that

$$\#\{j < q + 1 \mid b_{\tau(j)} < b_{q+1}\} < q,$$

hence $q + 1 \in \mathrm{Del}_q(\pi)$, which completes the proof of part 2.

(3) This follows from Remark 5.7 and from Proposition 4.2, since $\mathrm{inv}_q(\pi) = \ell_q(\pi)$ and similarly for σ. □

8. The covering map f_q

Motivated by Proposition 8.5 below, we introduce the map f_q from S_{n+q-1} onto S_n, which sends all the elements in the same double coset of S_q to the same element in S_n. The function f_q is applied later to 'pull-back' the equi-distribution results from the (classical) case $q = 1$ to the general q-case.

Definition 8.1. Let $\pi \in S_{n+q-1}$ and let $\pi = s_{i_1} \cdots s_{i_r}$ be its canonical presentation, then define $f_q : S_{n+q-1} \to S_n$ as follows:

$$f_q(\pi) = f_q(s_{i_1}) \cdots f_q(s_{i_r}),$$

where $f_q(s_1) = \cdots = f_q(s_{q-1}) = 1$, and $f_q(s_j) = s_{j-q+1}$ if $j \geq q$.

Remark 8.2. It is easy to verify that for any q_1, q_2, $f_{q_1} \circ f_{q_2} = f_{q_1+q_2-1}$. Thus, for every natural q, $f_q = f_2^{q-1}$.

Proposition 8.3. *The map f_q is invariant on the double cosets of S_q: Let $\sigma \in S_{n+q-1}$ and $\pi \in S_q\sigma S_q$, then $f_q(\sigma) = f_q(\pi)$.*

Proof. It suffices to prove that if $\sigma \in S_{n+q-1}$ and $\tau \in S_q$ then $f_q(\sigma\tau) = f_q(\tau\sigma) = f_q(\sigma)$. By Remark 8.2, it suffices to prove when $q = 2$ and hence when $\tau = s_1$. As usual, let $\sigma = w_1 \cdots w_n \in S_{n+1}$ be the canonical presentation of σ. By analysing the two cases $w_1 = 1$ and $w_1 = s_1$, it easily follows that $f_2(s_1\sigma) = f_2(\sigma)$.

We now show that $f_2(\sigma s_1) = f_2(\sigma)$. The proof in that case follows from the definition of f_2 and by induction on n, by analysing the following cases:

$w_n = 1$;
$w_n = s_n s_{n-1} \cdots s_k$ with $k \geq 3$;
$w_n = s_n s_{n-1} \cdots s_2$, and
$w_n = s_n s_{n-1} \cdots s_2 s_1$.

We verify, for example, the case $k \geq 3$. Denote $\pi = w_1 \cdots w_{n-1}$, so $\sigma = \pi w_n$. Now $f_2(\sigma s_1) = f_2(\pi s_1 \cdot w_n) = f_2(\pi s_1) f_2(w_n) =$ (by induction) $= f_2(\pi) f_2(w_n) = f_2(\sigma)$.

The proof in the last two cases follows similarly, and from the fact that $f_2(s_n s_{n-1} \cdots s_2) = f_2(s_n s_{n-1} \cdots s_2 s_1) = s_{n-1} \cdots s_2 s_1$. □

Note that f_q is not a group homomorphism. For example, let $q = 2$, $g = s_2$ and $h = s_1 s_2$. Then $f_2(g) = f_2(h) = s_1$ so $f_2(g) f_2(h) = 1$, but $gh = s_1 s_2 s_1$, hence $f_2(gh) = s_1$. Nevertheless we do have the following

Proposition 8.4. *For any permutation π, $f_q(\pi^{-1}) = (f_q(\pi))^{-1}$.*

Proof. Again by Remark 8.2, it suffices to prove for $q = 2$. The proof is based on Lemma 4.3. Denote $s_0 := 1$, then note that if $s_i s_j = s_j s_i$ then also $s_{i-1} s_{j-1} = s_{j-1} s_{i-1}$ (the converse is false, as $s_1 s_2 \neq s_2 s_1$).

Let $\pi = s_{i_1} \cdots s_{i_r}$ be the canonical presentation of π. By commuting moves, $\pi^{-1} = s_{i_r} \cdots s_{i_1} = \cdots = s_{p_1} \cdots s_{p_r}$ where the right hand side is the canonical presentation of π^{-1}. By definition, $f_2(\pi^{-1}) = s_{p_1 -1} \cdots s_{p_r -1}$. Now by the same commuting moves $s_{i_r -1} \cdots s_{i_1 -1} = \cdots = s_{p_1 -1} \cdots s_{p_r -1}$ and the left hand side equals $(f_q(\pi))^{-1}$, which completes the proof. $\square$

Proposition 8.5. *Recall from [12] and Section 3.2 the map $f: A_{n+1} \to S_n$. Then f is the restriction $f = f_2|_{A_{n+1}}$ of f_2 to A_{n+1}.*

Proof. Let $\pi \in A_{n+1}$, and let $\pi = a_{i_1}^{\epsilon_1} \cdots a_{i_r}^{\epsilon_r}$ be its A-canonical presentation, where all $\epsilon_j = \pm 1$. By definition, $f(\pi) = s_{i_1} \cdots s_{i_r}$. Replace each a_j in the above presentation by $a_j = s_1 s_{j+1}$ then, by commuting moves 'push' each s_1 as much as possible to the left. After some cancellations, an s_1 cannot move any more to the left if it is already the left-most factor, or if it is preceded by an s_2 on its left. It follows that

$$\pi = b s_{i_1+1} \cdots s_2 s_1 \cdots s_2 s_1 \cdots s_{i_r+1} \cdots$$

where $b \in \{1, s_1\}$, and this is an S-canonical presentation. Then $f_2(\pi) = s_{i_1} \cdots s_{i_r}$ and the proof follows. $\square$

Restricting the maps f_q to A_{n+q-1} we get more 'f-pairs' (see [12, Section 5]) with corresponding statistics, equi-distributions and generating-functions-identities for the alternating groups.

The main result here is

Proposition 8.6. *For every $\pi \in S_{n+q-1}$*

(1) $\mathrm{Del}_q(\pi) - q + 1 = \mathrm{Del}_1(f_q(\pi))$, *and in particular,* $\mathrm{del}_q(\pi) = \mathrm{del}_1(f_q(\pi))$.
(2) $\mathrm{Des}_q(\pi) - q + 1 = \mathrm{Des}_1(f_q(\pi))$ *and in particular,* $\mathrm{des}_q(\pi) = \mathrm{des}_1(f_q(\pi))$.
(3) $\mathrm{inv}_q(\pi) = \mathrm{inv}_1(f_q(\pi)) = \ell_q(\pi) = \ell_1(f_q(\pi))$.

Here $\mathrm{Del}_q(\pi) - r = \{i - r \mid i \in \mathrm{Del}_q(\pi)\}$ *and similarly for* $\mathrm{Des}_q(\pi) - r$.

The proof is given below.

Remark 8.7. Recall that $R_j = \{1, s_j, s_j s_{j-1}, \ldots, s_j s_{j-1} \cdots s_1\}$.

(1) Let $w = w_1 \cdots w_{n+q-2}$ where all $w_j \in R_j$ be the canonical presentation of $w \in S_{n+q-1}$. Then $f_q(w) = f_q(w_1) \cdots f_q(w_{n+q-2})$ is the canonical presentation of $f_q(w)$. Note that $f_q(w_1) = \cdots = f_q(w_{q-1}) = 1$.
(2) In addition, let also $w' = w'_1 \cdots w'_{n+q-2}$, where also $w'_j \in R_j$. It is obvious that $f_q(w) = f_q(w')$ if and only if $f_q(w_j) = f_q(w'_j)$ for all j.

(3) The definition of a^k.l.t.r.min in $\sigma = [b_1, \ldots, b_n]$—and therefore also the definition of the set $\mathrm{Del}_q(\sigma)$—applies whenever the integers $b_1, \ldots, b_n$ are distinct.
(4) Let $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ be two sets of distinct integers, let M be an integer satisfying $b_j, c_j < M$ for all j, let $1 \le k \le n$ and denote

$$\sigma = [b_1, \ldots, b_n], \qquad \sigma^* = [b_1, \ldots, b_{k-1}, M, b_k, \ldots, b_n]$$

and

$$\eta = [c_1, \ldots, c_n], \qquad \eta^* = [c_1, \ldots, c_{k-1}, M, c_k, \ldots, c_n].$$

Then it is rather easy to verify that $\mathrm{Del}_q(\sigma) = \mathrm{Del}_q(\eta)$ if and only if $\mathrm{Del}_q(\sigma^*) = \mathrm{Del}_q(\eta^*)$.

Lemma 8.8. *Let $w, w' \in S_{n+q-1}$ satisfy $f_q(w) = f_q(w')$, then*

1. $\mathrm{Del}_q(w) = \mathrm{Del}_q(w')$.
2. $\mathrm{Des}_q(w) = \mathrm{Des}_q(w')$.

Proof. Since $f_1(w) = w$, we assume that $q \ge 2$.

(1) By the definition of f_q and by Remark 8.7 it suffices to prove the following claim:

Let $w_j, w'_j \in R_j$ satisfy $f_q(w_j) = f_q(w'_j)$, $q \le j \le n+q-2$, and let $w = w_q \cdots w_{n+q-2}$ and $w' = w'_q \cdots w'_{n+q-2}$. Then $\mathrm{Del}_q(w) = \mathrm{Del}_q(w')$.

The proof is by induction on $n \ge 1$. If $n = 1$, $w = w' = 1$.

The induction step:

Denote $m = n + q - 1$, so $w = w_q \cdots w_{m-1}$ and $w' = w'_q \cdots w'_{m-1}$, then denote $\sigma = w_q \cdots w_{m-2}$ and $\sigma' = w'_q \cdots w'_{m-2}$. Since both permutations are in $S_{m-1} \subseteq S_m$, we have

$$\sigma = [b_1, \ldots, b_{m-1}, m] \qquad \text{and} \qquad \sigma' = [c_1, \ldots, c_{m-1}, m].$$

By induction, $\mathrm{Del}_q(\sigma) = \mathrm{Del}_q(\sigma')$. If $w_{m-1} = 1$ then also $w'_{m-1} = 1$ and we are done.

Thus, assume both are $\neq 1$. Recall that $f_q(w_{m-1}) = f_q(w'_{m-1})$ and let $w_{m-1} = s_{m-1} \cdots s_k$ and $w'_{m-1} = s_{m-1} \cdots s_{k'}$. If $k > q$, it follows that $w_{m-1} = w'_{m-1}$ and we are done. So let $k, k' \le q$. By comparing both cases with the case $k = q$ we may assume that $k = q$ and $k' \le q$, hence $w'_{m-1} = w_{m-1} s_{q-1} \cdots s_{k'}$.

Compare first σw_{m-1} with $\sigma' w_{m-1}$:

$$\sigma w_{m-1} = [b_1, \ldots, b_{q-1}, m, b_q, \ldots, b_{m-1}],$$
$$\sigma' w_{m-1} = [c_1, \ldots, c_{q-1}, m, c_q, \ldots, c_{m-1}],$$

and by induction and Remark 8.7(4), $\mathrm{Del}_q(\sigma w_{m-1}) = \mathrm{Del}_q(\sigma' w_{m-1})$. Compare now $\sigma' w_{m-1}$ with $\sigma' w'_{m-1} = (\sigma' w_{m-1}) s_{q-1} \cdots s_{k'}$:

$$\sigma' w_{m-1} = [c_1, \ldots\ldots\ldots\ldots, c_{q-1}, m, c_q, \ldots, c_{m-1}] \qquad \text{and}$$
$$\sigma' w'_{m-1} = [c_1, \ldots, c_{k'-1}, m, c_{k'}, \ldots, c_{q-1}, \ldots\ldots, c_{m-1}].$$

A simple argument now shows that $q < i$ is a^{q-1}.l.t.r.min $\mathrm{Del}_q(\sigma' w_{m-1}) = \mathrm{Del}_q(\sigma' w'_{m-1})$ and the proof of part 1 is complete.

(2) The proof is similar to that of part 1. Denote $m = n + q - 1$, then write $w = w_1 \cdots w_{m-1} = \sigma w_{m-1}$ where $\sigma = w_1 \cdots w_{m-2}$, and similarly $w' = w'_1 \cdots w'_{m-1} = \sigma' w'_{n-1}$.

We assume that $f_q(w_j) = f_q(w'_j)$ for all j. Thus $f_q(\sigma) = f_q(\sigma')$ and by induction, $\mathrm{Des}_q(\sigma) = \mathrm{Des}_q(\sigma')$. By an argument similar to that in the proof of part 1, it follows that $\mathrm{Des}_q(\sigma w_{m-1}) = \mathrm{Des}_q(\sigma' w_{m-1})$ and it remains to show that $\mathrm{Des}_q(\sigma' w_{m-1}) = \mathrm{Des}_q(\sigma' w'_{m-1})$. Again as in the proof of part 1, we may assume that $w_{m-1} = s_{m-1} \cdots s_q$ and $w'_{m-1} = s_{m-1} \cdots s_t$ where $t < q$. We prove the case $t = q - 1$, the other cases being proved similarly.

Write $\sigma' = [a_1, \ldots, a_{m-1}, m]$. Now $\sigma' w'_{m-1} = \sigma' w_{m-1} s_{q-1}$, hence

$$\sigma' w_{m-1} = [a_1, \ldots, a_{q-2}, a_{q-1}, m, a_q, \ldots, a_{m-1}],$$
$$\sigma' w'_{m-1} = [a_1, \ldots, a_{q-2}, m, a_{q-1}, a_q, \ldots, a_{m-1}].$$

Clearly, $q \in \mathrm{Des}(\sigma' w_{m-1})$ (therefore $q \in \mathrm{Des}_q(\sigma' w_{m-1})$), but it is possible that $q \notin \mathrm{Des}(\sigma' w'_{m-1})$. However, at most all the $q - 1$ integers $a_1, \ldots, a_{q-1}$ are smaller than a_q (but $m > a_q$), hence $q + 1 \in \mathrm{Del}_q(\sigma' w'_{m-1})$, which implies that $q \in \mathrm{Des}_q(\sigma' w'_{m-1})$.

For all other indices $i \neq q$ it is easy to check that $i \in \mathrm{Des}_q(\sigma' w_{m-1})$ if and only if $i \in \mathrm{Des}_q(\sigma' w'_{m-1})$, and the proof is complete. □

The Proof of Proposition 8.6. (1) Let $\pi \in S_{n+q-1}$ and let π' denote the permutation obtained from π by erasing—in the canonical presentation of π—all the appearances of the Coxeter generators $s_1, \ldots, s_{q-1}$. Clearly, $f_q(\pi) = f_q(\pi')$, hence suffices to prove that

(a) $\mathrm{Del}_q(\pi) = \mathrm{Del}_q(\pi')$, and
(b) $\mathrm{Del}_q(\pi') - q + 1 = \mathrm{Del}(f_q(\pi'))$, i.e. $\mathrm{Del}_q(\pi') = \mathrm{Del}(f_q(\pi')) + q - 1$.

Let $\pi = w_1 \cdots w_{q-1} w_q \cdots w_{m-1} (m = n + q - 1)$ be the canonical presentation of $\pi: w_j \in R_j$. Denote $\tau = w_1 \cdots w_{q-1}$ and $\sigma = w_q \cdots w_{m-1}$, then both are given in their canonical presentations. Clearly, $f(\tau) = 1$ and $\pi' = \sigma' = w'_q \cdots w'_{m-1}$, where for each j w'_j is obtained from w_j by erasing all the appearances of $s_1, \ldots, s_{q-1}$, and therefore $f_q(w_j) = f_q(w'_j)$. By Lemma 8.8, $\mathrm{Del}_q(\sigma) = \mathrm{Del}_q(\sigma') = \mathrm{Del}_q(\pi')$. Since $\pi = \tau\sigma$ and $\tau \in S_q$, by Remark 5.7 $\mathrm{Del}_q(\pi) = \mathrm{Del}_q(\sigma)$—and (a) is proved.

Part (b) follows from the following fact:

Let $\pi' = s_{i_1} \cdots s_{i_r}$ be the canonical presentation of the above π' (therefore all $i_j \geq q$), then $f_q(\pi') = s_{i_1-q+1} \cdots s_{i_r-q+1}$. If $f_q(\pi') = [a_1, \ldots, a_n]$, it follows that $\pi' = [1, \ldots, q-1, a_1 + q - 1, \ldots, a_n + q - 1]$. If $2 \leq i$, it then follows that i is a l.t.r.min of $f_q(\pi')$ if and only if $i + q - 1$ is a^{q-1}.l.t.r.min of π', which proves (b). □

(2) Recall that

$$\mathrm{Des}_q(\pi) = (\mathrm{Des}(\pi) \cap \{q, q+1, \ldots, n\}) \cup (\mathrm{Del}_q(\pi) - 1).$$

Special Case: Assume π does not involve any of $s_1, \ldots, s_{q-1}$. As above, if $f_q(\pi) = [a_1, \ldots, a_n]$ then $\pi = [1, \ldots, q-1, a_1 + q - 1, \ldots, a_n + q - 1]$, hence

$$\mathrm{Des}(\pi) \cap \{q, q+1, \ldots, n+q-1\} = \mathrm{Des}(f_q(\pi)) + q - 1.$$

By part 1

$$\mathrm{Des}_q(\pi) = ([\mathrm{Des}(f_q(\pi))] \cup [\mathrm{Del}(f_q(\pi)) - 1]) + q - 1.$$

Since for any $\sigma \in S_n \mathrm{Des}(\sigma) \supseteq \mathrm{Del}(\sigma) - 1$, it follows that the right hand side equals $\mathrm{Des}(f_q(\pi)) + q - 1$, and this completes the proof of this case.

The general case. Let $\pi \in S_{n+q-1}$ be arbitrary. Let π' be the permutation obtained from π by deleting all the appearances of $s_1, \ldots, s_{q-1}$ from its canonical presentation. Then $f_q(\pi) = f_q(\pi')$ and the proof easily follows from the above special case and from Lemma 8.8(2).

(3) By Proposition 4.2, $\mathrm{inv}_q(\pi) = \ell_q(\pi)$. By the definitions of ℓ_q and f_q, $\ell_q(\pi) = \ell(f_q(\pi))$, and finally, $\ell(\sigma) = \mathrm{inv}(\sigma)$ for any permutation σ. □

Remark 8.9. Proposition 6.1 now follows from Proposition 8.6, combined with Propositions 3.7 and 8.5.

Lemma 8.10. *For every* $\pi \in S_n$

$$\# f_q^{-1}(\pi) = q! \cdot q^{\mathrm{del}_1(\pi)} = (q-1)! \cdot q^{\mathrm{del}_1(\pi)+1}.$$

Moreover, let $g_q \colon A_{n+q-1} \to S_n$ *be the restriction* $g_q = f_q \mid_{A_{n+q-1}}$ *of* f_q *to* A_{n+q-1}. *Then*

$$\# g_q^{-1}(\pi) = \tfrac{1}{2} \# f_q^{-1}(\pi).$$

Proof. Denote $m = n + q - 1$, so $f_q \colon S_m \to S_n$. Consider the canonical presentation of $\pi \in S_n$ and write it as $\pi = \pi^{(n-1)} \cdot v_{n-1}$, where $\pi^{(n-1)} \in S_{n-1}$ and $v_{n-1} \in R_{n-1} = \{1, s_{n-1}, s_{n-1}s_{n-2}, \ldots, s_{n-1}s_{n-2} \cdots s_1\}$. Thus

$$\# f_q^{-1}(\pi) = \# f_q^{-1}(\pi^{(n-1)}) \cdot \# f_q^{-1}(v_{n-1}) = q! \cdot q^{\mathrm{del}_1(\pi^{(n-1)})} \# f_q^{-1}(v_{n-1})$$

(by induction). If $\mathrm{del}_1(v_{n-1}) = 0$ then $\# f_q^{-1}(v_{n-1}) = 1$. If $\mathrm{del}_1(v_{n-1}) = 1$ then $\# f_q^{-1}(v_{n-1}) = q$, since in that case $v_{n-1} = s_{n-1} \cdots s_1$ and

$$f_q^{-1}(v_{n-1}) = \{w_{m-1}, w_{m-1}s_{q-1}, \ldots, w_{m-1}s_{q-1} \cdots s_1\},$$

where $w_{m-1} = s_{m-1}s_{m-2} \cdots s_q$. The proof now follows.

The argument for g_q is similar. The factor $1/2$ comes from the fact that $\# f_q^{-1}(1) = \# S_q$ while $\# g_q^{-1}(1) = \# A_q$. □

Following [12], we introduce

Definition 8.11. Let m_1 and m_q be two statistics on the symmetric groups. We say that (m_1, m_q) is an f_q*-pair* if for all n and $\pi \in S_{n+q-1}$, $m_q(\pi) = m_1(f_q(\pi))$.

As a corollary of Proposition 8.6 and Remark 11.1, we have

Corollary 8.12. *The following are* f_q*-pairs:*

$(\mathrm{inv}_1, \mathrm{inv}_q)$, (ℓ_1, ℓ_q), $(\mathrm{del}_1, \mathrm{del}_q)$, $(\mathrm{des}_1, \mathrm{des}_q)$, *and* $(\mathrm{rmaj}_{1,n}, \mathrm{rmaj}_{q,n+q-1})$.

The same argument as in the proof of Proposition 5.6 in [12], together with Lemma 8.10, now proves

Proposition 8.13. *Let (m_1, m_q) be an f_q-pair of statistics on the symmetric groups. Then*

$$\sum_{\pi \in S_{n+q-1}} t_1^{m_q(\pi)} t_2^{\mathrm{del}_q(\pi)} = q! \sum_{\sigma \in S_n} t_1^{m_1(\sigma)} t_2^{\mathrm{del}_1(\sigma)}.$$

Restricting f_q to A_{n+q-1} we obtain similarly, that

$$\sum_{\pi \in A_{n+q-1}} t_1^{m_q(\pi)} t_2^{\mathrm{del}_q(\pi)} = \tfrac{1}{2} q! \sum_{\sigma \in S_n} t_1^{m_1(\sigma)} t_2^{\mathrm{del}_1(\sigma)}.$$

Remark 8.14. As in [12], Proposition 8.13 allows us to lift equi-distribution theorems from S_n to S_{n+q-1}, as well as to A_{n+q-1}. This is demonstrated in Theorem 11.3. We leave the formulation and the proof of the corresponding A_{n+q-1} statement for the reader.

9. Dashed patterns

Dashed patterns in permutations were introduced in [2]. For example, the permutation σ contains the pattern $(1-32)$ if $\sigma = [\ldots, a, \ldots, c, b, \ldots]$ for some $a < b < c$; if no such a, b, c exist then σ is said to avoid $(1-32)$. In [3] the author shows connections between the number of permutations avoiding $(1-32)$ and various combinatorial objects, like the Bell and the Stirling numbers, as well as the number of left-to-right-minima in permutations. In this and in the next sections we obtain the q-analogues for these connections and results.

In Section 5.2 it was observed that, always, $\mathrm{Del}_q(\pi) - 1 \subseteq \mathrm{Des}_q(\pi)$. It is proved in Proposition 9.3 that equality holds exactly for permutations avoiding a certain set of dashed-patterns.

Definition 9.1. 1. Given q, denote by

$$\mathrm{Pat}(q) = \{(\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q+2)(q+1)) \mid \sigma \in S_q\}$$

the set with these $q!$ dashed patterns.

For example, $\mathrm{Pat}(2) = \{(1-2-43), (2-1-43)\}$.

2. Denote by $\mathrm{Avoid}_q(m)$, $m = n + q - 1$, the set of permutations in S_m avoiding all the $q!$ patterns in $\mathrm{Pat}(q)$, and let $h_q(m)$ denote the number of the permutations in S_m avoiding $\mathrm{Pat}(q)$. Thus $h_q(m) = \#\mathrm{Avoid}_q(m)$ is the number of the permutations in S_{n+q-1} avoiding $\mathrm{Pat}(q)$. Note that $h_q(m) = n!$ if $m \le q + 1$. As usual, define $h_q(0) = 1$.

Connections between $h_q(n)$ and the q-Bell and q-Stirling numbers are given in Section 10.

Remark 9.2. A permutation $\pi \in S_{n+q-1}$ does satisfy one of the patterns in $\mathrm{Pat}(q)$ if and only if there exists a subsequence

$$1 \le i_1 < i_2 < \cdots < i_{q+1} < n + q - 1,$$

such that $\pi(i_{q+1}) > \pi(i_{q+1} + 1)$ and for every $1 \le j \le q$, $\pi(i_j) < \pi(i_{q+1} + 1)$. In such a case, $i_{q+1} + 1$ (namely, $\pi(i_{q+1} + 1)$) is not an a^{q-1}.l.t.r.min in π.

Proposition 9.3. *A permutation* $\pi \in S_{n+q-1}$ *avoids* $\mathrm{Pat}(q)$ *exactly when* $\mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)$*:*

$$\mathrm{Avoid}_q(n+q-1) = \{\pi \in S_{n+q-1} | \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}.$$

In particular,

$$h_q(n+q-1) = \#\{\pi \in S_{n+q-1} | \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}.$$

Proof. (1) Recall from Section 5.2 that, always, $\mathrm{Del}_q(\pi) - 1 \subseteq \mathrm{Des}_q(\pi)$. Let $\pi = [b_1, \ldots, b_{n+q-1}] \in S_{n+q-1}$ satisfy $\mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)$, which implies that $\mathrm{Des}(\pi) \cap \{q, \ldots, n+q-1\} \subseteq \mathrm{Del}_q(\pi) - 1$, and show that π avoids $\mathrm{Pat}(q)$. If not, by Remark 9.2 we obtain a descent in π at i_{q+1}, while $i_{q+1} + 1$ is not a^{q-1}.l.t.r.min in π; thus i_{q+1} is in $\mathrm{Des}(\pi) \cap \{q, \ldots, n+q-1\}$ but not in $\mathrm{Del}_q(\pi) - 1$, a contradiction.

(2) Denote $\pi = [b_1, \ldots, b_{n+q-1}]$. Assume now that $\pi \in \mathrm{Avoid}_q(n)$, let $k \in \mathrm{Des}(\pi) \cap \{q, \ldots, n+q-1\}$ (so $b_k > b_{k+1}$) and show that $k+1 \in \mathrm{Del}_q(\pi)$, that is, $k+1$ (namely b_{k+1}) is a^{q-1}.l.t.r.min in π. If not, there exist q (or more) b_j's in π, smaller than and left of b_{k+1}—hence also left of b_k. Together with $b_k > b_{k+1}$ this shows that $\pi \notin \mathrm{Avoid}_q(n+q-1)$, a contradiction. □

Corollary 9.4. *The covering map* f_q *maps* $\mathrm{Avoid}_q(S_{n+q-1})$ *to* $\mathrm{Avoid}_1(S_n)$*:*

$$f_q\colon \mathrm{Avoid}_q(S_{n+q-1}) \to \mathrm{Avoid}_1(S_n).$$

Similarly,

$$f_2 : \mathrm{Avoid}_q(S_{n+q-1}) \to \mathrm{Avoid}_{q-1}(S_{n+q-2}).$$

Proof. This follows straightforwardly from Propositions 8.6 and 9.3. □

10. *q*-Bell and *q*-Stirling numbers

10.1. *The q-Bell numbers*

Recall that $S(n,k)$ are the Stirling numbers of the second kind, i.e. the numbers of k-partitions of the set $[n] = \{1, \ldots, n\}$. Recall also that the Bell number $b(n)$ is the total number of the partitions of $[n]$: $b(n) = \sum_k S(n,k)$.

Definition 10.1. Define the *q-Bell numbers* $b_q(n)$ by

$$b_q(n) = \sum_k q^k S(n,k).$$

Remark 10.2. Let $q \geq 1$ be an integer and consider partitions of $[n]$ into k subsets, where each subset is coloured by one of q colours. The number of such q-coloured k-partitions is obviously $q^k S(n,k)$. It follows that the total number of such q-coloured partitions of $[n]$ is the nth q-Bell number $b_q(n)$.

Proposition 10.8 below shows that

$$\#\{\sigma \in S_{n+q-1} \mid \mathrm{Del}_q(\sigma) - 1 = \mathrm{Des}_q(\sigma) \text{ and } \mathrm{del}_q(\sigma) = k-1\}$$
$$= (q-1)!q^k S(n,k),$$

and therefore

$$(q-1)!b_q(n) = \#\{\pi \in S_{n+q-1} | \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}.$$

The q-Bell numbers are studied first.

When $q = 1$, by considering the subset in a k-partition of $[n]$ which contains n, one easily deduces the well-known recurrence relation

$$b_1(n) = \sum_k \binom{n-1}{k} b_1(n-k-1).$$

In the general q colours case, apply the same argument, now taking into account that each subset—and in particular the one containing n—can be coloured by q colours. This proves:

Lemma 10.3. *For each integer $q \geq 1$ we have the following recurrence relation*

$$b_q(n) = q \sum_k \binom{n-1}{k} b_q(n-k-1).$$

Remark 10.4. 1. Let $B_q(x) = \sum_{n=0}^{\infty} b_q(n) \frac{x^n}{n!}$ denote the exponential generating function of $\{b_q(n)\}$. As in page 42 in [15], Lemma 10.3 implies that $B'(x) = q e^x B_q(x)$. Together with $B(0) = 1$ (since, by definition, $b_q(0) = 1$), this implies that

$$B_q(x) = \exp(q e^x - q).$$

2. The classical formula

$$b_1(n) = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^n}{r!}$$

generalizes as follows:

$$b_q(n) = \frac{1}{e^q} \sum_{r=0}^{\infty} \frac{q^r r^n}{r!}.$$

The proof follows, essentially unchanged, the argument on page 21 in [15].

10.2. Connections with pattern-avoiding permutations

Recall that $\mathrm{Pat}(q) = \{(\sigma_1 - \sigma_2 - \cdots - \sigma_q - (q+2)(q+1)) \mid \sigma \in S_q\}$ and that $h_q(n)$ denotes the number of the permutations in S_n avoiding all these $q!$ patterns in $\mathrm{Pat}(q)$.

Proposition 10.5. *The q-Bell numbers $b_q(n)$ and the numbers $h_q(n+q-1)$ of permutations in S_{n+q-1} that avoid* $\mathrm{Pat}(q)$, *satisfy*

$$h_q(n+q-1) = (q-1)! \cdot b_q(n).$$

By Proposition 9.3 this implies that

$$(q-1)!b_q(n) = \#\{\pi \in S_{n+q-1} \mid \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}.$$

The proof requires the following recurrence.

Lemma 10.6. *If* $n \geq q$ *then*

$$h_q(n) = q \sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1).$$

Proof. The proof is by a rather standard argument.

Let $K \subseteq \{q+1, q+2, \ldots, n\}$ be a subset, with $|K| = k$, hence $0 \leq k \leq n-q$. Let κ be the word obtained by writing the numbers of K in an increasing order. Note that there are $\binom{n-q}{k}$ such K's—hence $\binom{n-q}{k}$ such κ's. Let $1 \leq i \leq q$ and let $\sigma^{(i)}$ be a permutation of the set $\{1, \ldots, i-1, i+1, \ldots, n\} \backslash K$, which avoids Pat$(q)$. By definition, since there are $n-1-k$ elements in that set, there are $h_q(n-k-1)$ such $\sigma^{(i)}$'s. Now construct (the word) $\eta^{(i)} = \sigma^{(i)} i \kappa$, then $\eta^{(i)} \in S_n$ and it avoids Pat(q) since there is no descent in the part $i\kappa$ of $\eta^{(i)}$ (see Remark 9.2). For each $1 \leq i \leq q$, the number of $\eta^{(i)}$'s thus constructed is $\sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1)$, hence

$$h_q(n) \leq q \sum_{k=0}^{n-q} \binom{n-q}{k} h_q(n-k-1).$$

Conversely, assume $\eta \in S_n$ avoids Pat(q). Among $1, \ldots, q$, let i appear the rightmost in η and write the word η as $\eta = \sigma i \kappa$, then none of $1, \ldots, q$ appears in κ. The numbers in κ are increasing since otherwise, if there is a descent in κ, Remark 9.2 would imply that η does satisfy one of the dashed patterns in Pat(q), a contradiction. Since η avoids Pat(q), obviously the part σ of η also avoids Pat(q). It follows that η is the above permutation $\eta = \eta^{(i)}$. This proves the reverse inequality and completes the proof. □

The proof of Proposition 10.5 now follows by induction on $n \geq 0$. The case $n = 0$ is clear. Assume $n \geq 1$, then by Lemma 10.6

$$h_q(n+q-1) = q \sum_{k=0}^{n-1} \binom{n-1}{k} h_q(n-1-k+q-1)$$

(by induction)

$$= q \sum_{k=0}^{n-1} \binom{n-1}{k} \cdot (q-1)! \cdot b_q(n-k-1)$$

$$= (q-1)! \cdot \left[q \sum_{k=0}^{n-1} \binom{n-1}{k} b_q(n-k-1) \right]$$

(by Lemma 10.3)

$$= (q-1)! \cdot b_q(n).$$

This proves the first equation of the proposition. Together with Definition 9.1 and Proposition 9.3, this implies that $h_q(n+q-1) = \#\{\pi \in S_{n+q-1} \mid \mathrm{Del}_q(\pi)-1 = \mathrm{Des}_q(\pi)\}$, hence

$$(q-1)! b_q(n) = \#\{\pi \in S_{n+q-1} \mid \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_q(\pi)\}. \quad \square$$

In the case $q = 1$,

$$b_1(n) = b(n) = \#\mathrm{Avoid}_1(n) = \#\{\sigma \in S_n \mid \mathrm{Del}_1(\sigma) - 1 = \mathrm{Des}_1(\sigma)\},$$

which appears in [3].

Let

$$H_q(x) = \sum_{n=0}^{\infty} h_q(n + q - 1)\frac{x^n}{n!}$$

be the exponential generating function of the $h_q(n + q - 1)$'s. By Remark 10.4(1) and Proposition 10.5 we have

Corollary 10.7.

$$H_q(x) = (q - 1)! \cdot \exp(q\mathrm{e}^x - q).$$

10.3. Stirling numbers of the second kind

The following refinement of the second equation of Proposition 10.5 is proved in this subsection.

Proposition 10.8.

$$\#\{\sigma \in S_{n+q-1} \mid \mathrm{Del}_q(\sigma) - 1 = \mathrm{Des}_q(\sigma) \textit{ and } \mathrm{del}_q(\sigma) = k - 1\}$$
$$= (q - 1)!q^k S(n, k).$$

Deduce that

$$\sum_{\{\pi \in S_n \mid \mathrm{Del}_1(\pi)-1=\mathrm{Des}_1(\pi)\}} q^{\mathrm{del}_1(\pi)} = \frac{1}{q} \cdot b_q(n),$$

and more generally,

$$\sum_{\{\sigma \in S_{n+q-1} \mid \mathrm{Del}_q(\sigma)-1=\mathrm{Des}_q(\sigma)\}} q^{\mathrm{del}_q(\sigma)} = \frac{(q-1)!}{q} \cdot \sum_k q^{2k} S(n, k)$$
$$= \frac{(q-1)!}{q} \cdot b_{q^2}(n).$$

Proof. We first prove the case $q = 1$ namely, that

$$\#\{\sigma \in S_n \mid \mathrm{Del}_1(\sigma) - 1 = \mathrm{Des}_1(\sigma) \quad \text{and} \quad \mathrm{del}_1(\sigma) = k - 1\} = S(n, k).$$

Recall that $S(n, k)$ is the number of partitions of $[n]$ into k non-empty subsets. Given such a partition $D_1 \cup \cdots \cup D_k = [n]$, assume w.l.o.g. that the numbers in each D_i are increasing: D_i is $\{d_{i,1} < d_{i,2} < \cdots\}$, and also, the minimal elements $d_{1,1}, d_{2,1}, \ldots$ are decreasing: $d_{1,1} > d_{2,1} > \cdots > d_{k,1}$. Corresponding to that partition we construct the permutation $\sigma = [D_1, D_2, \ldots]$, namely $\sigma = [d_{1,1}, d_{1,2}, \ldots, d_{2,1}, d_{2,2}, \ldots, d_{k,1}, d_{k,2} \ldots]$.

Now $\mathrm{Del}_1(\sigma)$, the l.t.r.min of σ, are exactly at the $(k-1)$ positions of $d_{2,1}, d_{3,1}, \ldots, d_{k,1}$, and obviously the descents occur at $\mathrm{Del}_1(\sigma) - 1$. This establishes an injection of the set of the k partitions of $[n]$ into the above set, which implies that

$$\mathrm{card}\{\sigma \in S_n \mid \mathrm{Del}_1(\sigma) - 1 = \mathrm{Des}_1(\sigma) \quad \text{and} \quad \mathrm{del}_1(\sigma) = k-1\} \geq S(n,k).$$

Since the sum on all k's of both sides equals $b(n)$, this implies the case $q = 1$.

The general q case follows from Proposition 8.6, and from Lemma 8.10:

Let $\pi \in S_n$. By Proposition 8.6,

$$\mathrm{Del}_1(\pi) - 1 = \mathrm{Des}_1(\pi) \qquad \text{if and only if } \mathrm{Del}_q(f_q^{-1}(\pi)) - 1 = \mathrm{Des}_q(f_q^{-1}(\pi)),$$

and also, $\mathrm{del}_1(\pi) = k-1$ if and only if $\mathrm{del}_q(f_q^{-1}(\pi)) = k-1$. Denote $D_q(n,k) = \{\sigma \in S_{n+q-1} \mid \mathrm{Del}_q(\sigma) - 1 = \mathrm{Des}_q(\sigma)$ and $\mathrm{del}_q(\sigma) = k-1\}$, so that $D_1(n,k) = \{\pi \in S_n \mid \mathrm{Del}_q(\pi) - 1 = \mathrm{Des}_1(\pi)$ and $\mathrm{del}_1(\pi) = k-1\}$. It follows that

$$D_q(n,k) = \bigcup_{\pi \in D_1(n,k)} f_q^{-1}(\pi),$$

a disjoint union. By Lemma 8.10, $\# f_q^{-1}(\pi) = (q-1)! \cdot q^k$ for all $\pi \in D_1(n,k)$, and the proof now follows easily from the case $q = 1$. □

10.4. *Stirling numbers of the first kind*

Let $c(n,k)$ be the signless Stirling numbers of the first kind.

Proposition 10.9. $c(n,k) = \#\{\pi \in S_n \mid \mathrm{del}_S(\pi) = \mathrm{del}_1(\pi) = k-1\}$, *namely,* $c(n,k)$ *equals the number of permutations in* S_n *with* $k-1$ *l.t.r.min.*

For the proof, see Proposition 5.8 in [12].

The following is a q-analogue of Proposition 10.9.

Proposition 10.10.

$$\#\{\pi \in S_{n+q-1} \mid \mathrm{del}_q(\pi) = k-1\} = c_q(n,k),$$

where $c_q(n,k) = q^k(q-1)!c(n,k)$.

Proof. The proof is essentially identical to the proof of Proposition 10.8, with the set $D_q(n,k)$ being replaced here by the set $H_q(n,k) = \{\pi \in S_{n+q-1} \mid \mathrm{del}_q(\pi) = k-1\}$. Then $H_1(n,k) = \{\pi \in S_n \mid \mathrm{del}_1(\pi) = k-1\}$, and by Proposition 5.8 in [12], $\# H_1(n,k) = c(n,k)$, the *signless Stirling number of the first kind*. The proof now follows. □

11. Equi-distribution

11.1. *MacMahon type theorems for* q*-statistics*

Recall the definition of $\mathrm{rmaj}_{q,n+q-1}$ from Definition 5.9.

Remark 11.1. Note that for $\pi \in S_{n+q-1}$,

$$\text{rmaj}_{q,n+q-1}(\pi) = \text{rmaj}_{1,n}(f_q(\pi)) = \text{rmaj}_{S_n}(f_q(\pi)).$$

This follows since by Proposition 8.6(2), $i \in \text{Des}_q(\pi)$ if and only if $i - q + 1 \in \text{Des}_1(f_q(\pi))$.

The following is a q-analogue of MacMahon's equi-distribution theorem.

Theorem 11.2. *For every positive integer n and q*

$$\sum_{\pi \in S_{n+q-1}} t^{\text{rmaj}_{q,n+q-1}(\pi)} = \sum_{\pi \in S_{n+q-1}} t^{\text{inv}_q(\pi)} = q!(1+tq)(1+t+t^2q)\cdots(1+t+\cdots+t^{n-2}+t^{n-1}q).$$

This theorem is obtained from the next one by substituting $t_2 = 1$.

Theorem 11.3. *For every positive integer n and q*

$$\sum_{\pi \in S_{n+q-1}} t_1^{\text{rmaj}_{q,n+q-1}(\pi)} t_2^{\text{del}_q(\pi)} = \sum_{\pi \in S_{n+q-1}} t_1^{\text{inv}_q(\pi)} t_2^{\text{del}_q(\pi)} = q!(1+t_1t_2q)(1+t_1+t_1^2t_2q)\cdots \times (1+t_1+\cdots+t_1^{n-2}+t_1^{n-1}t_2q).$$

Proof. By Proposition 8.6 and Remark 11.1, $(\text{rmaj}_{S_n}, \text{rmaj}_{q,n+q-1})$ and $(\text{inv}, \text{inv}_q)$ are f_q-pairs. The proof now follows from Proposition 8.13 and Theorem 3.3. □

The following is a q-analogue of Foata–Schützenberger's equi-distribution theorem [7, Theorem 1].

Theorem 11.4. *For every positive integer n and q and every subset* $B \subseteq [q, n+q-1]$

$$\sum_{\{\pi \in S_{n+q-1} \mid \text{Des}_q(\pi^{-1})=B\}} t^{\text{inv}_q(\pi)} = \sum_{\{\pi \in S_{n+q-1} \mid \text{Des}_q(\pi^{-1})=B\}} t^{\text{rmaj}_{q,n+q-1}(\pi)}.$$

This theorem is obtained from the next one by substituting $B_2 = [q, n+q-1]$.

Theorem 11.5. *For every positive integer n and q and every subsets* $B_1 \subseteq [q, n+q-1]$ *and* $B_2 \subseteq [q, n+q-1]$

$$\sum_{\{\pi \in S_{n+q-1} \mid \text{Des}_q(\pi^{-1})=B_1, \text{Del}_q(\pi^{-1})=B_2\}} t^{\text{inv}_q(\pi)} = \sum_{\{\pi \in S_{n+q-1} \mid \text{Des}_q(\pi^{-1})=B_1, \text{Del}_q(\pi^{-1})=B_2\}} t^{\text{rmaj}_{q,n+q-1}(\pi)}.$$

Proof. By Proposition 8.6 and Remark 11.1, it suffices to prove that for every subset $B_1 \subseteq [n-1]$ and $B_2 \subseteq [n-1]$

$$\sum_{\{\pi \in S_{n+q-1} | \mathrm{Des}_1(f_q(\pi^{-1}))=B_1, \mathrm{Del}_1(f_q(\pi^{-1}))=B_2\}} t^{\mathrm{inv}_1(f_q(\pi))} = \sum_{\{\pi \in S_{n+q-1} | \mathrm{Des}_1(f_q(\pi^{-1}))=B_1, \mathrm{Del}_1(f_q(\pi^{-1}))=B_2\}} t^{\mathrm{rmaj}_{1,n}(f_q(\pi))}.$$

By Proposition 8.4 $f_q(\pi^{-1}) = f_q(\pi)^{-1}$. Thus, denoting $\sigma = f_q(\pi)$, it suffices to prove that

$$\sum_{\{\sigma \in S_n | \mathrm{Des}_1(\sigma^{-1})=B_1, \mathrm{Del}_1(\sigma^{-1})=B_2\}} \# f_q^{-1}(\sigma) \cdot t^{\mathrm{inv}_1(\sigma)} = \sum_{\{\sigma \in S_n | \mathrm{Des}_1(\sigma^{-1})=B_1, \mathrm{Del}_1(\sigma^{-1})=B_2\}} \# f_q^{-1}(\sigma) \cdot t^{\mathrm{rmaj}_{1,n}(\sigma)}.$$

By Propositions 5.2 and 5.5, for every $\sigma \in S_n$ with $\mathrm{Del}_1(\sigma^{-1}) = B_2$, $\mathrm{del}_1(\sigma) = \#B_2$. Thus, by Lemma 8.10, $\# f_q^{-1}(\sigma) = (q-1)! \cdot q^{\#B_2+1}$ for all permutations in the sums. Hence, the theorem is reduced to

$$(q-1)! \cdot q^{\#B_2+1} \cdot \sum_{\{\sigma \in S_n | \mathrm{Des}_1(\sigma^{-1})=B_1, \mathrm{Del}_1(\sigma^{-1})=B_2\}} t^{\mathrm{inv}_1(\sigma)} = (q-1)! \cdot q^{\#B_2+1} \cdot \sum_{\{\sigma \in S_n | \mathrm{Des}_1(\sigma^{-1})=B_1, \mathrm{Del}_1(\sigma^{-1})=B_2\}} t^{\mathrm{rmaj}_{1,n}(\sigma)}.$$

Theorem 3.2 completes the proof. □

Theorem 11.4 implies q-analogues of two classical identities, due to [7, 14].

Corollary 11.6. *For every positive integer n and q*

(1) $\sum_{\pi \in S_{n+q-1}} t_1^{\mathrm{inv}_q(\pi)} t_2^{\mathrm{des}_q(\pi^{-1})} = \sum_{\pi \in S_{n+q-1}} t_1^{\mathrm{rmaj}_{q,n+q-1}(\pi)} t_2^{\mathrm{des}_q(\pi^{-1})}$, *and*

(2) $\sum_{\pi \in S_{n+q-1}} t_1^{\mathrm{inv}_q(\pi)} t_2^{\mathrm{rmaj}_{q,n+q-1}(\pi^{-1})} = \sum_{\pi \in S_{n+q-1}} t_1^{\mathrm{rmaj}_{q,n+q-1}(\pi)} t_2^{\mathrm{rmaj}_{q,n+q-1}(\pi^{-1})}$.

11.2. Equi-distribution on $\mathrm{Avoid}_q(n)$

The main theorem on equi-distribution on permutations avoiding patterns is the following.

Theorem 11.7. *For every positive integer n and q and every subset $B \subseteq [q, \ldots, n+q-2]$*

$$\sum_{\{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1) | \mathrm{Des}_q(\pi^{-1})=B\}} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)} = \sum_{\{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1) | \mathrm{Des}_q(\pi^{-1})=B\}} t^{\mathrm{inv}_q(\pi)}$$

Proof. Substituting $B_1 = B_2 - 1 = B$ in Theorem 11.5 we obtain, for every subset $B \subseteq [q, n+q-1]$

$$\sum_{\{\pi \in S_{n+q-1} | \mathrm{Des}_q(\pi^{-1}) = \mathrm{Del}_q(\pi^{-1}) - 1 = B\}} t^{\mathrm{inv}_q(\pi)} = \sum_{\{\pi \in S_{n+q-1} | \mathrm{Des}_q(\pi^{-1}) = \mathrm{Del}_q(\pi^{-1}) - 1 = B\}} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)}.$$

By Proposition 9.3

$$\{\pi \in S_{n+q-1} | \mathrm{Des}_q(\pi^{-1}) = \mathrm{Del}_q(\pi^{-1}) - 1 = B\} = \{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1) | \mathrm{Des}_q(\pi^{-1}) = B\}.$$

This completes the proof. □

Theorem 11.7 implies

Corollary 11.8. *For every positive integer n and q*

$$\sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t_1^{\mathrm{rmaj}_{q,n+q-1}(\pi)} t_2^{\mathrm{des}_q(\pi)} = \sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t_1^{\mathrm{inv}_q(\pi)} t_2^{\mathrm{des}_q(\pi)}$$

The following is an extension of MacMahon's theorem to permutations avoiding patterns.

Theorem 11.9. *For every positive integer n and q*

$$\sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t^{\mathrm{rmaj}_{q,n+q-1}(\pi)} = \sum_{\pi^{-1} \in \mathrm{Avoid}_q(n+q-1)} t^{\mathrm{inv}_q(\pi)}$$

Proof. Substitute $t_2 = 1$ in Corollary 11.8. □

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Appendix. $\mathrm{Des}_2 = \mathrm{Des}_A$: the proof

Lemma A.1. *Let* $w = [b_1, \ldots, b_{n+1}] \in A_{n+1}$. *Let* $1 \le i \le n-1$, *then* $i \in \mathrm{Des}_A(w)$ *if and only if one of the following two conditions hold.*

1. $b_{i+1} > b_{i+2}$, *or*
2. $b_{i+1} < b_{i+2}$ *and* $b_1, b_2, \ldots, b_i > b_{i+2}$.

In particular, $1 \in \mathrm{Des}_A(w)$ *if and only if* $b_1 > b_3$ *(and/) or* $b_2 > b_3$.

Proof. The basic tool is the formula

$$\ell_A(w) = \ell_S(w) - \mathrm{del}_S(w).$$

Assume first that $2 \le i \le n-1$, then $v = wa_i = [b_2, b_1, \ldots, b_{i+2}, b_{i+1}, \ldots]$. Now compare $\ell_S(w)$ with $\ell_S(v)$, and $\mathrm{del}_S(w)$ with $\mathrm{del}_S(v)$, then apply the above formula, and the proof follows. Here are the details.

The case $2 \le i \le n-1$ and $b_{i+1} > b_{i+2}$.

If $b_1 < b_2$ then $\ell_S(w) = \ell_S(v)$. Now, $\mathrm{del}(\sigma)$ is the number of l.t.r.min in σ. Interchanging $b_1 < b_2$ in w adds one such l.t.r.min, while interchanging $b_{i+1} > b_{i+2}$ reduces that (del_S) number by one, or leaves it unchanged. In particular, $\mathrm{del}_S(w) \le \mathrm{del}_S(v)$. It follows that $\ell_A(w) = \ell_S(w) - \mathrm{del}_S(w) \ge \ell_S(v) - \mathrm{del}_S(v) = \ell_A(v)$, i.e. $\ell_A(wa_i) \le \ell_A(w)$, hence $i \in \mathrm{Des}_A(w)$.

Similarly for the other cases. If $b_1 > b_2$ (and $b_{i+1} > b_{i+2}$), verify that $\ell_S(w) = \ell_S(v) + 2$, while $\mathrm{del}_S(w) \le \mathrm{del}_S(v) + 2$, and again this implies that $i \in \mathrm{Des}_A(w)$. This completes the proof of 2.a.

The case $2 \le i \le n-1$ and $b_{i+1} < b_{i+2}$.

First, assume $b_1 < b_2$, then $\ell_S(v) = \ell_S(w) + 2$. If $b_1, b_2, \ldots, b_i > b_{i+2}$ then also $\mathrm{del}_S(v) = \mathrm{del}_S(w) + 2$, hence $\ell_A(wa_i) = \ell_A(v) = \ell_A(w)$, and $i \in \mathrm{Des}_A(w)$. If $b_j < b_{i+2}$ for some $1 \le j \le i$ then $\mathrm{del}_S(v) = \mathrm{del}_S(w) + 1$ and it follows that $i \notin \mathrm{Des}_A(w)$.

If $b_1 > b_2$ then $\ell_S(v) = \ell_S(w)$. Assuming that $b_1, b_2, \ldots, b_i > b_{i+2}$, deduce that also $\mathrm{del}_S(v) = \mathrm{del}_S(w)$, hence $i \in \mathrm{Des}_A(w)$. If $b_j < b_{i+2}$ for some $1 \le j \le i$ then $\mathrm{del}_S(v) = \mathrm{del}_S(w) - 1$, so $\ell_A(wa_i) = \ell_A(v) = \ell_A(w) - 1$ and $i \notin \mathrm{Des}_A(w)$.

Finally assume that $i = 1$, then $v = wa_1 = ws_1s_2 = [b_2, b_3, b_1, b_4, b_5, \ldots]$. Obviously, $\ell_S(w) - \ell_S(v)$ depends only on the order relations among b_1, b_2, b_3, and similarly for $\mathrm{del}_S(w) - \mathrm{del}_S(v)$. We can therefore assume that $\{b_1, b_2, b_3\} = \{1, 2, 3\}$, then check the $3! = 6$ possible cases of $w = [b_1, b_2, b_3, \ldots]$. For example, assume $w = [1, 3, 2, \ldots]$, then $wa_1 = [3, 2, 1, \ldots] = v$, so $\ell_S(v) = \ell_S(w) + 2$ while $\mathrm{del}_S(v) = \mathrm{del}_S(w) + 2$, hence $1 \in \mathrm{Des}_A(w)$.

Similarly for the remaining five cases. □

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