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# The Theory of the Umbral Calculus III*

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*Contents.* 

## 1. INTRODUCTION

This is the third in a series of papers intended to develop the theory of the umbral calculus. The purpose of this paper is to generalize the theory presented in the first paper. Familiarity with the first two papers is not essential. However, some knowledge of Sections 1–5 of the first paper (hereinafter denoted by UCI) is recommended since, due to space considerations, we have been somewhat terse here whenever the line of reasoning is virtually identical with that in UCI. For instance, we have chosen to omit some of the simpler proofs of the peripheral results whenever these may be taken from UCI mutatis mutandis. Also, we have been somewhat shorter with the examples, discussing in some detail the cases of Hermite, Laguerre, Gegenbauer, and the $q$-case but only mentioning briefly the Chebyshev and Jacobi cases.

The underlying theme of the first paper is that the special polynomial sequences which we have termed Sheffer sequences possess a unified theory. From this theory many seemingly unrelated results in the classical literature emerge as special cases of a general result. Moreover, the umbral calculus provides a cohesive approach to the study of further properties of these sequences, as well as bringing to light some new sequences which are strongly related to the important classical ones.

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In this paper we replace the algebra $P$ of polynomials in $x$ by the field $\Lambda$ of formal Laurent series in $x$ of the form

$$p(x) = \sum_{j=-\infty}^{m} b_j x^j$$

for any integer $m$. The algebra $\mathscr{F}$ of formal power series in $t$, which gave us our representation of linear functionals and linear operators on $P$, is replaced by the field $\Gamma$ of formal Laurent series in $t$ of the form

$$f(t) = \sum_{k=n}^{\infty} a_k t^k,$$

for any integer $n$. Then $\Gamma$ is used to represent all the continuous linear functionals and some continuous linear operators on $\Lambda$ in much the same manner as in UCI. In fact, we still have

$$\langle t^k \mid x^n \rangle = c_n \delta_{n,k}$$

and

$$t^k x^n = \frac{c_n}{c_{n-k}} x^{n-k}$$

but now $n$ and $k$ range over *all* integers.

To make a long story short, the entire theory developed in UCI can be adapted to this new setting. Each Sheffer sequence of polynomials

$$s_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

gives rise to a sequence of Laurent series

$$\bar{s}_n(x) = \sum_{k=-\infty}^{n} \bar{a}_{n,k} x^k,$$

where $n$ ranges over all integers and for which $a_{n,k} = \bar{a}_{n,k}$ whenever $n, k \geqslant 0$.

In some special cases, and only for $n < 0$, the sequence $\bar{s}_n(x)$ has been studied under the name factor sequence. Several simple examples have appeared in the classical literature but it was not until the last few years that the factor sequence counterparts of, say, the Hermite and Laguerre polynomials, have emerged (see [2]). However, except for some brief work by this author the full sequence $\bar{s}_n(x)$ has not been studied.

## 2. Formal Laurent Series

In this section we shall set down a few basic definitions concerning formal Laurent series. Let $K$ be a field of characteristic zero. Let $\Gamma$ be the field of formal Laurent series over $K$ of the form

$$f(t) = \sum_{k=m}^{\infty} a_k t^k, \tag{2.1}$$

where $m$ is any integer. Addition and multiplication in $\Gamma$ are purely formal. The *degree* of $f(t)$ is the smallest integer $k$ for which $a_k \neq 0$. It is readily seen that $\deg f(t)\, g(t) = \deg f(t) + \deg g(t)$. We shall denote the multiplicative inverse of $f(t)$ in $\Gamma$ by $f^{-1}(t)$ or $1/f(t)$. It is clear that $\deg f^{-1}(t) = -\deg f(t)$. We shall write $f^{-1}(t)^k$ as $f(t)^{-k}$.

Let $g_k(t)$ be a sequence in $\Gamma$ for which $\lim_{k\to\infty} \deg g_k(t) = \infty$. Then for any sequence of constants $a_k$ the sum

$$\sum_{k=m}^{\infty} a_k\, g_k(t)$$

is a well-defined element of $\Gamma$. In case $\deg g_k(t) = k$ the sequence $g_k(t)$ forms a pseudobasis for $\Gamma$. In other words, for any $f(t)$ in $\Gamma$ there exists a unique sequence of constants $a_k$ and integer $m$ for which

$$f(t) = \sum_{k=m}^{\infty} a_k\, g_k(t).$$

If $\deg g(t) = 1$ and $f(t)$ has the form (2.1), then the composition

$$f(g(t)) = \sum_{k=m}^{\infty} a_k\, g(t)^k$$

is a well-defined element of $\Gamma$. A series $f(t)$ has a compositional inverse, denoted by $\bar{f}(t)$, and satisfying $f(\bar{f}(t)) = \bar{f}(f(t)) = t$, if and only if $\deg f(t) = 1$. We call any series $f(t)$ with $\deg f(t) = 1$ a *delta series*.

It will be convenient to define a notion of convergence in $\Gamma$. The sequence $f_n(t)$ *converges* to 0 if $\lim_{n\to\infty} \deg f_n(t) = \infty$. In case $f(t) = 0$ we set $\deg f(t) = \infty$. If $f_n(t)$ converges to 0 we write $f_n(t) \to 0$. The sequence $f_n(t)$ *converges* to $f(t)$ if $f_n(t) - f(t) \to 0$.

A linear operator $T$ on $\Gamma$ is *continuous* if and only if $Tf_n(t) \to 0$ whenever $f_n(t) \to 0$. In particular, a linear functional $L$ is continuous if and only if $Lf_n(t)$ is eventually equal to 0 whenever $f_n(t) \to 0$.

In case $\deg f_n(t) = n$, where $n$ ranges over all integers then a continuous linear operator $T$ on $\Gamma$ is uniquely defined by the values $Tf_n(t)$. Moreover, these values may be assigned arbitrarily provided $Tf_n(t) \to 0$ as $n \to \infty$.

Let $\Lambda$ be the field of formal Laurent series over $K$ of the form

$$p(x) = \sum_{j=-\infty}^{n} b_j x^j$$

for any integer $n$. Addition and multiplication are formal and the *degree* of $p(x)$ is the largest integer $j$ for which $b_j \neq 0$. If $p_j(x)$ is a sequence in $\Lambda$ for which $\lim_{j\to\infty} \deg p_j(x) = -\infty$ and if $b_j$ is a sequence of constants, then

$$\sum_{j=-\infty}^{n} b_j p_j(x)$$

is a well-defined element of $\Lambda$.

We shall say that the sequence $p_n(x)$ in $\Lambda$ *converges* to 0 if $\lim_{n\to\infty} \deg p_n(x) = -\infty$. Here if $p(x) = 0$ we set $\deg p(x) = -\infty$. In this case we write $p_n(x) \to 0$. The sequence $p_n(x)$ *converges* to $p(x)$ if $p_n(x) - p(x) \to 0$.

A linear operator $T$ on $\Lambda$ is *continuous* if $Tp_n(x) \to 0$ whenever $p_n(x) \to 0$. A linear functional $L$ is therefore continuous if and only if $Lp_n(x)$ is eventually 0 whenever $p_n(x) \to 0$.

If $\deg p_n(x) = n$ for all integers $n$, then $p_n(x)$ is a pseudobasis for $\Lambda$ and a continuous linear operator $T$ on $\Lambda$ is uniquely defined by the values $Tp_n(x)$. Moreover, these values may be assigned arbitrarily provided $Tp_n(x) \to 0$ as $n \to -\infty$. We shall use this fact repeatedly.

## 3. Linear Functionals

Let $\Lambda^*$ be the vector space of all continuous linear functionals on $\Lambda$. We use the notation $\langle L \mid p(x) \rangle$ for the action of $L$ in $\Lambda^*$ on $p(x)$ in $\Lambda$.

Let $c_n$ be a fixed sequence of nonzero constants. If $f(t)$ in $\Gamma$ has the form

$$f(t) = \sum_{k=m}^{\infty} a_k t^k,$$

then we define the continuous linear functional $f(t)$ on $\Lambda$ by

$$\langle f(t) \mid x^n \rangle = c_n a_n \tag{3.1}$$

for all integers $n$, where $a_n = 0$ if $n < m$. In view of the remarks at the end of Section 2, since $\deg x^n = n$ and $c_n a_n \to 0$ as $n \to -\infty$ we conclude that (3.1) defines a unique element of $\Lambda^*$.

Notice that (3.1) gives

$$\langle t^k \mid x^n \rangle = c_n \delta_{n,k}$$

for all integers $n$ and $k$.

We have used the same notation $f(t)$ for a formal Laurent series and a continuous linear functional on $\Lambda$. No confusion should arise since $f(t) = g(t)$ as formal Laurent series if and only if $f(t) = g(t)$ as continuous linear functionals.

Any continuous linear functional $L$ on $\Lambda$ has the property that $\langle L \mid x^n \rangle \to 0$ as $n \to -\infty$. In other words there exists an integer $n_0$, depending on $L$, for which $n < n_0$ implies $\langle L \mid x^n \rangle = 0$. Now consider the series

$$f_L(t) = \sum_{k=n_0}^{\infty} \frac{\langle L \mid x^k \rangle}{c_k} t^k.$$

Then

$$\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$$

for all integers $n$. Therefore, since both $L$ and $f_L(t)$ are continuous we conclude that $L = f_L(t)$ as linear functionals. The upshot is that the map $\psi: L \to f_L(t)$ is a vector space isomorphism from the vector space $\Lambda^*$ onto the field $\Gamma$. As is usual in the umbral calculus, we shall obscure this isomorphism and think of $\Lambda^*$ as being identical with $\Gamma$. Hence $\Lambda^*$ is a field. Let us give some basic facts.

PROPOSITION 3.1. *If* $\deg f(x) > \deg p(x)$, *then*

$$\langle f(t) \mid p(x) \rangle = 0.$$

PROPOSITION 3.2. *Let* $\deg f_k(t) = k$ *and* $\deg p_j(x) = j$. *Then*

$$\begin{aligned}\left\langle \sum_{k=m}^{\infty} a_k f_k(t) \,\middle|\, \sum_{j=-\infty}^{n} b_j p_j(x) \right\rangle &= \sum_{k=m}^{\infty} \sum_{j=-\infty}^{n} a_k b_j \langle f_k(t) \mid p_j(x) \rangle \\ &= \sum_{k=m}^{n} \sum_{j=m}^{n} a_k b_j \langle f_k(t) \mid p_j(x) \rangle.\end{aligned}$$

PROPOSITION 3.3. *For* $f(t)$ *in* $\Gamma$

$$f(t) = \sum_{k=m}^{\infty} \frac{\langle f(t) \mid x^k \rangle}{c_k} t^k,$$

*where* $m = \deg f(t)$.

PROPOSITION 3.4. *For $f(t)$ and $g(t)$ in $\Gamma$*

$$\langle f(t)\, g(t) \mid x^n \rangle = \sum_{k=m}^{n-s} \frac{c_n}{c_k c_{n-k}} \langle f(t) \mid x^k \rangle \langle g(t) \mid x^{n-k} \rangle,$$

*where $m = \deg f(t)$ and $s = \deg g(t)$.*

PROPOSITION 3.5. *If $p(x)$ is in $\Lambda$, then*

$$p(x) = \sum_{j=-\infty}^{n} \frac{\langle t^j \mid p(x) \rangle}{c_j} x^j,$$

*where $n = \deg p(x)$.*

PROPOSITION 3.6. *If $\deg f_k(t) = k$ and $\langle f_k(t) \mid p(x) \rangle = 0$ for all integers $k$, then $p(x) = 0$.*

PROPOSITION 3.7. *If $\deg p_n(x) = n$ and $\langle f(t) \mid p_n(x) \rangle = 0$ for all integers $n$, then $f(t) = 0$.*

If $y$ is a constant and $m$ is an integer we define the *evaluation series* (or *evaluation functional*) of degree $m$ by

$$\varepsilon_{y,m}(t) = \sum_{k=m}^{\infty} \frac{y^k}{c_k} t^k.$$

Then

$$\begin{aligned} \langle \varepsilon_{y,m}(t) \mid x^n \rangle &= y^n \qquad \text{if} \quad n \geqslant m, \\ &= 0 \qquad \text{if} \quad n < m. \end{aligned}$$

PROPOSITION 3.8. *Let $p(x)$ and $q(x)$ be in $\Lambda$. Then $p(x) = q(x)$ if and only if $\varepsilon_{y,m}(t)\, p(x) = \varepsilon_{y,m}(t)\, q(x)$ for all $y$ in $K$ and all integers $m$.*

*Proof.* Let $p(x) = \sum_{j=-\infty}^{n} a_j x^j$. Then

$$\langle \varepsilon_{y,m}(t) \mid p(x) \rangle = \sum_{j=m}^{n} \frac{a_j}{c_j} y^j.$$

Thus if $\langle \varepsilon_{y,m}(t) \mid p(x) \rangle = 0$ for all $y$ in $K$ we deduce that $a_j = 0$ for $j = n, n-1, \ldots, m$. Since this holds for all integers $m$ we deduce that $p(x) = 0$. An appeal to linearity proves the result.

When we are considering a particular delta series in $\Gamma$ as a linear functional we shall refer to it as a *delta functional*.

## 4. Linear Operators

If $f(t)$ in $\Gamma$ has the form

$$f(t) = \sum_{k=m}^{\infty} a_k t^k$$

we define the continuous linear operator $f(t)$ on $\Gamma$ by

$$f(t)\, x^n = \sum_{k=m}^{\infty} \frac{c_n}{c_{n-k}} a_k x^{n-k} = \sum_{k=-\infty}^{n-m} \frac{c_n}{c_k} a_{n-k} x^k. \tag{4.1}$$

In particular,

$$t^k x^n = \frac{c_n}{c_{n-k}} x^{n-k}$$

for all integers $n$ and $k$. Notice that we use juxtaposition for the action of a linear operator.

We use the same notation $f(t)$ for a formal Laurent series, a linear functional and a linear operator. Again it is easy to see that $f(t) = g(t)$ as formal Laurent series if and only if $f(t) = g(t)$ as continuous linear operators.

It is straightforward to verify that

$$\begin{aligned} f(t)[g(t)\, p(x)] &= [f(t)\, g(t)]\, p(x) \\ &= [g(t) f(t)]\, p(x) \\ &= g(t)[f(t)\, p(x)] \end{aligned}$$

for all $f(t)$ and $g(t)$ in $\Gamma$ and $p(x)$ in $\Lambda$.

Notice that by Proposition 3.3 Eq. (4.1) becomes

$$f(t)\, x^n = \sum_{k=-\infty}^{n-m} \frac{c_n}{c_k c_{n-k}} \langle f(t) \mid x^{n-k} \rangle\, x^k. \tag{4.2}$$

THEOREM 4.1. *For any $f(t)$ and $g(t)$ in $\Gamma$ we have*

$$\langle f(t)\, g(t) \mid p(x) \rangle = \langle f(t) \mid g(t)\, p(x) \rangle$$

*for all $p(x)$ in $\Lambda$.*

*Proof.* The continuity of $f(t)$ and $g(t)$ as linear functionals and linear operators allows us to prove this by considering only the case $p(x) = x^n$. But then replacing $f(t)$ by $g(t)$ in (4.2) and applying the linear functional $f(t)$ gives

$$\langle f(t) \mid g(t)\, x^n \rangle = \sum_{k=-\infty}^{n-m} \frac{c_n}{c_k c_{n-k}} \langle g(t) \mid x^{n-k} \rangle \langle f(t) \mid x^k \rangle.$$

Finally, from Proposition 3.4 and degree considerations we obtain the result.

When we are considering a particular delta series as a linear operator we shall use the term *delta operator.*

It is not hard to see that not all continuous linear operators on $\Lambda$ take the form $f(t)$ in $\Gamma$. We shall postpone the characterization of all such operators until Theorem 5.5 when we can provide a more natural proof.

## 5. Sheffer Sequences

By a *sequence* $p_n(x)$ in $\Lambda$ we shall mean that $n$ ranges over *all* integers and $\deg p_n(x) = n$.

THEOREM 5.1. *Let $f(t)$ be a delta series and let* $\deg g(t) = 0$. *Then the identity*

$$\langle g(t) f(t)^k \mid s_n(x) \rangle = c_n \delta_{n,k}, \tag{5.1}$$

*valid for all integers $n$ and $k$, determines a unique sequence of Laurent series in $\Lambda$.*

*Proof.* The uniqueness follows from Proposition 3.6. For the existence suppose $s_n(x) = \sum_{j=-\infty}^{n} b_{n,j} x^j$ and $g(t) f(t)^k = \sum_{i=k}^{\infty} a_{k,i} t^i$, where $a_{k,k} \neq 0$. Then (5.1) becomes

$$\begin{aligned} c_n \delta_{n,k} &= \left\langle \sum_{i=k}^{\infty} a_{k,i} t^i \,\middle|\, \sum_{j=-\infty}^{n} b_{n,j} x^j \right\rangle \\ &= \sum_{i=k}^{n} a_{k,i} b_{n,i} c_i. \end{aligned}$$

One may readily solve this triangular system of equations [for $k = n$, $n-1, \ldots$] to obtain the coefficients $b_{n,j}$.

The sequence $s_n(x)$ is the *Sheffer sequence* for the pair $(g(t), f(t))$ and we say $s_n(x)$ is *Sheffer for* $(g(t), f(t))$. In case $g(t) = 1$ we call the Sheffer sequence for $(1, f(t))$ the *associated sequence* for $f(t)$.

THEOREM 5.2 (the Expansion Theorem). *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. Then for any $h(t)$ in $\Lambda$*

$$h(t) = \sum_{k=m}^{\infty} \frac{\langle h(t) \mid s_k(x)\rangle}{c_k} g(t) f(t)^k,$$

*where $m = \deg h(t)$.*

*Proof.* Simply apply the right side to $s_n(x)$ to obtain $h(t)\, s_n(x)$. The result follows by continuity.

COROLLARY 1. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. Then for any Laurent series $p(x)$ in $\Lambda$*

$$p(x) = \sum_{j=-\infty}^{n} \frac{\langle g(t) f(t)^j \mid p(x)\rangle}{c_j} s_j(x),$$

*where $n = \deg p(x)$.*

*Proof.* Setting $h(t) = \varepsilon_{y,m}(t)$ in the Expansion Theorem gives

$$\varepsilon_{y,m}(t) = \sum_{k=m}^{\infty} \frac{\langle \varepsilon_{y,m}(t) \mid s_k(x)\rangle}{c_k} g(t) f(t)^k.$$

Applying this to $p(x)$ gives

$$\langle \varepsilon_{y,m}(t) \mid p(x)\rangle = \sum_{k=m}^{n} \frac{\langle g(t) f(t)^k \mid p(x)\rangle}{c_k} \langle \varepsilon_{y,m}(t) \mid s_k(x)\rangle$$

$$= \left\langle \varepsilon_{y,m}(t) \;\middle|\; \sum_{k=-\infty}^{n} \frac{\langle g(t) f(t)^k \mid p(x)\rangle}{c_k} s_k(x) \right\rangle.$$

Since this holds for all $y$ in $K$ and all integers $m$ the result follows from Proposition 3.8.

The next result follows immediately from Theorems 4.1 and 5.1.

THEOREM 5.3. *The sequence $s_n(x)$ is Sheffer for $(g(t), g(t))$ if and only if the sequence $g(t)\, s_n(x)$ is the associated sequence for $f(t)$.*

Theorem 5.3 says that each associated sequence generates a class of Sheffer sequences, one for each $g(t)$ of degree 0.

We would like to characterize Sheffer sequences in terms of linear operators in $\Gamma$.

THEOREM 5.4. *A sequence $p_n(x)$ is the associated sequence for $f(t)$ if and only if*

(i) $\langle t^0 \mid p_n(x)\rangle = c_0 \delta_{n,0}$,

(ii) $f(t)\, p_n(x) = (c_n/c_{n-1})\, p_{n-1}(x)$.

*for all integers n.*

*Proof.* Suppose $\langle f(t)^k \mid p_n(x)\rangle = c_n \delta_{n,k}$. Then $k=0$ gives (i). For (ii) we have

$$\langle f(t)^k \mid f(t)\, p_n(x)\rangle = c_n \delta_{n,k+1} = \left\langle f(t)^k \;\middle|\; \frac{c_n}{c_{n-1}}\, p_{n-1}(x)\right\rangle$$

and so in view of Proposition 3.6 we deduce (ii). Conversely, if (i) and (ii) hold, then

$$\begin{aligned}\langle f(t)^k \mid p_n(x)\rangle &= \left\langle t^0 \;\middle|\; \frac{c_n}{c_{n-k}}\, p_{n-k}(x)\right\rangle \\ &= \frac{c_n}{c_{n-k}}\, c_0 \delta_{n-k,0} \\ &= c_n \delta_{n,k}\end{aligned}$$

and thus $p_n(x)$ is associated to $f(t)$.

Now we may easily derive a characterization of those continuous linear operators on $\Lambda$ of the form $g(t)$ in $\Gamma$.

THEOREM 5.5. *A continuous linear operator $U$ on $\Lambda$ is of the form $g(t)$ in $\Gamma$ if and only if there exists a delta operator $f(t)$ for which*

$$Uf(t)\, p(x) = f(t)\, Up(x)$$

*for all $p(x)$ in $\Lambda$.*

*Proof.* If $U$ has the form $g(t)$, then $U$ commutes with any delta operator. Conversely, let $p_n(x)$ be the associated sequence for $f(t)$. Then since $p_n(x) \to 0$ as $n \to -\infty$ and since $U$ is continuous we deduce the existence of an integer $m$ for which $n < m$ implies $\langle t^0 \mid Up_n(x)\rangle = 0$. Let $g(t)$ be defined by

$$g(t) = \sum_{k=m}^{\infty} \frac{\langle t^0 \mid Up_k(x)\rangle}{c_k}\, f(t)^k.$$

Then

$$\begin{aligned}
g(t)\, p_n(x) &= \sum_{k=m}^{\infty} \frac{\langle t^0 \mid Up_k(x)\rangle}{c_k} f(t)^k\, p_n(x) \\
&= \sum_{k=m}^{\infty} \frac{c_n}{c_k c_{n-k}} \langle t^0 \mid Up_k(x)\rangle\, p_{n-k}(x) \\
&= \sum_{k=m}^{\infty} \frac{\langle t^0 \mid Uf(t)^{n-k}\, p_n(x)\rangle}{c_{n-k}}\, p_{n-k}(x) \\
&= \sum_{k=m}^{\infty} \frac{\langle f(t)^{n-k} \mid Up_n(x)\rangle}{c_{n-k}}\, p_{n-k}(x) \\
&= \sum_{j=-\infty}^{n-m} \frac{\langle f(t)^j \mid Up_n(x)\rangle}{c_j}\, p_j(x) \\
&= Up_n(x).
\end{aligned}$$

The continuity of $U$ and $g(t)$ complete the proof.

THEOREM 5.6. *A sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ for some $g(t)$ of degree 0 if and only if*

$$f(t)\, s_n(x) = \frac{c_n}{c_{n-1}}\, s_{n-1}(x) \tag{5.2}$$

*for all integers $n$.*

*Proof.* Suppose $s_n(x)$ is Sheffer for $(g(t), f(t))$. Then $p_n(x) = g(t)\, s_n(x)$ is associated to $f(t)$ and so

$$\begin{aligned}
f(t)\, s_n(x) &= f(t)\, g^{-1}(t)\, p_n(x) \\
&= g^{-1}(t)\, f(t)\, p_n(x) \\
&= g^{-1}(t) \frac{c_n}{c_{n-1}}\, p_{n-1}(x) \\
&= \frac{c_n}{c_{n-1}}\, s_{n-1}(x).
\end{aligned}$$

For the converse, let $p_n(x)$ be the associated sequence for $f(t)$ and let $U$ be the continuous linear operator on $A$ defined by

$$Us_n(x) = p_n(x).$$

Then

$$\begin{aligned} Uf(t)\, s_n(x) &= \frac{c_n}{c_{n-1}}\, Us_{n-1}(x) \\ &= \frac{c_n}{c_{n-1}}\, p_{n-1}(x) \\ &= f(t)\, p_n(x) \\ &= f(t)\, Us_n(x) \end{aligned}$$

and by Theorem 5.5 there exists $g(t)$ in $\Gamma$ for which $g(t)\, s_n(x) = p_n(x)$. Thus $s_n(x)$ is Sheffer for $(g(t), f(t))$.

We are in a somewhat more enviable position here than we were in UCI since $\Gamma$ is a field. For example, we have

THEOREM 5.7. *The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ for some $g(t)$ of degree* 0 *if and only if*

$$f(t)^k\, s_n(x) = \frac{c_n}{c_{n-k}}\, s_{n-k}(x)$$

*for* all *integers n and k.*

THEOREM 5.8. *Let $s(x)$ be a series in $\Lambda$ with* $\deg s(x) = m$. *Let $f(t)$ be a delta series. Then there is a unique Sheffer sequence $s_n(x)$ with delta series $f(t)$ satisfying $s_m(x) = s(x)$. In fact we have*

$$s_n(x) = \frac{c_n}{c_m}\, f(t)^{m-n}\, s(x).$$

We can characterize Sheffer sequences by an identity which generalizes the binomial identity.

THEOREM 5.9 (the Sheffer Identity). *A sequence $s_n(x)$ is Sheffer for the pair $(g(t), f(t))$ for some $g(t)$ if and only if*

$$\varepsilon_{y,0}(t)\, s_n(x) = \sum_{k=-\infty}^{n} \frac{c_n}{c_k c_{n-k}} \langle \varepsilon_{y,0}(t) \mid p_{n-k}(x)\rangle\, s_k(x),$$

*where $p_n(x)$ is the associated sequence for $f(t)$.*

*Proof.* The Expansion Theorem for $\varepsilon_{y,0}(t)$ reads

$$\varepsilon_{y,0}(t) = \sum_{k=0}^{\infty} \frac{\langle \varepsilon_{y,0}(t) \mid p_k(x)\rangle}{c_k}\, f(t)^k.$$

Applying this to $s_n(x)$ and using Theorem 5.6 gives the Sheffer identity. Conversely, let $U$ be the continuous linear operator on $\Lambda$ defined by $Us_n(x) = p_n(x)$. Then it is sufficient to show that $U = h(t)$ for some $h(t)$ in $\Gamma$. Now

$$\begin{aligned}
\varepsilon_{y,0}(t)\, Us_n(x) &= \varepsilon_{y,0}(t)\, p_n(x) \\
&= \sum_{k=-\infty}^{n} \frac{c_n}{c_k c_{n-k}} \langle \varepsilon_{y,0}(t) \mid p_{n-k}(x) \rangle\, p_k(x) \\
&= U \sum_{k=-\infty}^{n} \frac{c_n}{c_k c_{n-k}} \langle \varepsilon_{y,0}(t) \mid p_{n-k}(x) \rangle\, s_k(x) \\
&= U\varepsilon_{y,0}(t)\, s_n(x).
\end{aligned}$$

Thus $U$ commutes with the delta operator $\varepsilon_{y,0}(t) - c_0^{-1}t^0$ and Theorem 5.5 concludes the proof.

Notice that if

$$p(x) = \sum_{j=-\infty}^{n} a_j x^j,$$

then

$$\langle \varepsilon_{y,0}(t) \mid p(x) \rangle = \sum_{j=0}^{n} a_j y^j.$$

This leads us to define the *polynomial part of* $p(x)$ as

$$\tilde{p}(x) = \sum_{j=0}^{n} a_j x^j,$$

where of course if $n < 0$, then $\tilde{p}(x) = 0$. The Sheffer identity may now be written

$$\varepsilon_{y,0}(t)\, s_n(x) = \sum_{k=-\infty}^{n} \frac{c_n}{c_k c_{n-k}} \tilde{p}_{n-k}(y)\, s_k(x).$$

## 6. Generating Functions

In order to derive the generating function for a Sheffer sequence we consider the vector space of all formal Laurent series over $K$ in the two variables $y$ and $t$ of the form

$$\sigma(y, t) = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{k,j} y^k t^j.$$

Recall that two formal Laurent series $\sigma(y, t)$ and $\rho(y, t)$ are equal if and only if the corresponding coefficients of $y^k t^j$ are equal for all integers $k$ and $j$.

If $h(t)$ is a delta series in $\Gamma$ we set

$$E^{yh(t)} = \sum_{k=-\infty}^{\infty} \frac{y^k}{c_k} h(t)^k$$

which is a well-defined formal Laurent series in the variables $y$ and $t$. Also, if $l(t)$ is a series in $\Gamma$, then the formal product

$$l(t)\, E^{yh(t)}$$

is a well-defined formal Laurent series in $y$ and $t$, the coefficient of $y^k t^j$ being the coefficient of $t^j$ in $(1/c_k)\, l(t)\, h(t)^k$.

We are now in a position to determine a generating function for Sheffer sequences.

THEOREM 6.1. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. Then*

$$\frac{1}{g(\bar{f}(t))} E^{y\bar{f}(t)} = \sum_{k=-\infty}^{\infty} \frac{s_k(y)}{c_k} t^k \tag{6.1}$$

*as formal Laurent series in $y$ and $t$. (Recall that $\bar{f}(t)$ is the compositional inverse of $f(t)$.)*

*Proof.* The Expansion Theorem for $\varepsilon_{y,m}(t)$ reads

$$\varepsilon_{y,m}(t) = \sum_{k=m}^{\infty} \frac{\langle \varepsilon_{y,m}(t) \mid s_k(x) \rangle}{c_k} g(t)\, f(t)^k$$

and so

$$g(\bar{f}(t))^{-1}\, \varepsilon_{y,m}(\bar{f}(t)) = \sum_{k=m}^{\infty} \frac{\langle \varepsilon_{y,m}(t) \mid s_k(x) \rangle}{c_k} t^k \tag{6.2}$$

for all $y$ in $K$ and all integers $m$. Now from degree considerations it follows that the coefficient of $t^k$ on both sides of (6.2) is a polynomial in $y$. Since Eq. (6.2) holds for all $y$ in $K$ we may consider (6.2) as an equality between formal Laurent series in the two variables $y$ and $t$. Finally, by comparing coefficients of $y^k$ in (6.2) one sees that (6.1) must hold.

The following characterization of Sheffer sequences is a consequence of the generating function.

THEOREM 6.2. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. Then*

$$s_n(x) = \frac{\langle g(\bar{f}(t))^{-1} f(t)^k \mid x^n \rangle}{c_k} x^k. \tag{6.3}$$

*Proof.* Applying (6.2) to $x^n$ gives

$$\begin{aligned}
\langle \varepsilon_{y,m}(t) \mid s_n(x) \rangle &= \sum_{k=m}^{\infty} \frac{\langle \varepsilon_{y,m}(t) \mid s_k(x) \rangle}{c_k} \langle t^k \mid x^n \rangle \\
&= \langle g(\bar{f}(t))^{-1} \varepsilon_{y,m}(\bar{f}(t)) \mid x^n \rangle \\
&= \sum_{k=m}^{n} \frac{y^k}{c_k} \langle g(\bar{f}(t))^{-1} f(t)^k \mid x^n \rangle \\
&= \left\langle \varepsilon_{y,m}(t) \;\middle|\; \sum_{k=-\infty}^{n} \frac{\langle g(\bar{f}(t))^{-1} f(t)^k \mid x^n \rangle}{c_k} x^k \right\rangle.
\end{aligned}$$

Since this holds for all $y$ in $K$ and all integers $m$ the result follows.

Equation (6.3) is the *conjugate representation* for $s_n(x)$. It is usually the most convenient method for computing Sheffer sequences provided an explicit form can be found for $\bar{f}(t)$.

## 7. RECURRENCE FORMULAS

Let $\mu$ be a continuous linear operator on $\Lambda$. The *adjoint* $\mu^*$ of $\mu$ is the linear operator on $\Gamma$ defined by

$$\mu^* f(t) = \sum_{k=m}^{\infty} \frac{\langle f(t) \mid \mu x^k \rangle}{c_k} t^k, \tag{7.1}$$

where $m$ is chosen so that $\deg \mu x^k < \deg f(t)$ for all $k < m$. Recall that such an integer $m$ must exist since $\mu$ is continuous.

PROPOSITION 7.1. *If $\mu$ is a continuous linear operator on $\Lambda$, then*

$$\langle \mu^* f(t) \mid p(x) \rangle = \langle f(t) \mid \mu p(x) \rangle$$

*for all $f(t)$ in $\Gamma$ and $p(x)$ in $\Lambda$.*

*Proof.* Since both $f(t)$ and $\mu^* f(t)$ are continuous linear functionals we need only check this for $p(x) = x^n$ for all $n$. But then from (7.1) applied to $x^n$ we obtain the desired result.

PROPOSITION 7.2. *The adjoint of a continuous linear operator on $\Lambda$ is continuous.*

*Proof.* Let $\mu$ be a continuous linear operator on $\Lambda$, and let $f_n(t) \to 0$ in $\Gamma$. We want to show that $\mu^* f_n(t) \to 0$. Now since $\mu$ is continuous there exists an integer $k_0$ such that $k < k_0$ implies $\deg \mu x^k < 0$. Also, there exists an integer $n_0$ such that $n > n_0$ implies $\deg f_n(t) > 0$. Now let $m$ be given. Then there exists an integer $n_{m,k_0}$ such that $n > n_{m,k_0}$ implies $\deg f_n(t) > \max_{k_0 \leqslant i \leqslant m} \{\deg \mu x^i\}$. Therefore, if we set $n_m = \max\{n_0, n_{m,k_0}\}$, then $n > n_m$ implies $\deg f_n(t) > \deg \mu x^k$ for all $k < m$. Thus $\langle \mu^* f_n(t) \mid x^k \rangle = \langle f_n(t) \mid \mu x^k \rangle = 0$ for all $n > n_m$ and for all $k < m$. In other words, for any integer $m$ there exists an integer $n_m$ such that $n > n_m$ implies $\deg \mu^* f_n(t) \geqslant m$. That is, $\mu^* f_n(t) \to 0$ as $n \to \infty$ and the proposition is proved.

If $p_n(x)$ is the associated sequence for $f(t)$, then the *umbral shift* $\theta_f$ associated to $f(t)$ [or $p_n(x)$] is the continuous linear operator on $\Lambda$ defined by

$$\theta_f p_n(x) = \frac{(n+1)\, c_n}{c_{n+1}}\, p_{n+1}(x)$$

for all integers $n$. Umbral shifts may be characterized by their adjoints.

THEOREM 7.3. *A continuous linear operator $\theta$ on $\Lambda$ is the umbral shift for $f(t)$ if and only if its adjoint $\theta^*$ is a continuous derivation on $\Gamma$ for which $\theta^* f(t) = t^0$.*

*Proof.* Suppose $\theta_f$ is the umbral shift for $f(t)$. Then Proposition 7.2 implies that $\theta_f^*$ is continuous. Moreover, if $p_n(x)$ is the associated sequence for $f(t)$ we have

$$\begin{aligned}\langle \theta_f^* f(t)^k \mid p_n(x) \rangle &= \frac{(n+1)\, c_n}{c_{n+1}} \langle f(t)^k \mid p_{n+1}(x) \rangle \\ &= k c_n \delta_{n,k-1} \\ &= \langle k f(t)^{k-1} \mid p_n(x) \rangle.\end{aligned}$$

Therefore $\theta_f^* f(t)^k = k f(t)^{k-1}$. The continuity of $\theta_f^*$ and the Expansion Theorem allow us to conclude that $\theta_f^*$ is a derivation. For the converse, suppose $\omega$ is a continuous derivation on $\Lambda$ for which $\omega f(t) = t^0$. Then $\omega f(t)^k = k f(t)^{k-1}$ and if $p_n(x)$ is associated to $f(t)$ we have

$$\begin{aligned}\langle \omega f(t)^k \mid p_n(x) \rangle &= \langle k f(t)^{k-1} \mid p_n(x) \rangle \\ &= k c_n \delta_{n,k-1} \\ &= \langle f(t)^k \mid \theta_f\, p_n(x) \rangle \\ &= \langle \theta_f^* f(t)^k \mid p_n(x) \rangle.\end{aligned}$$

Thus $\omega = \theta_f^*$. This completes the proof.

We remark that

$$\theta_t^* g(t) = g'(t)$$

and

$$\theta_t t = xD,$$

where $g'(t)$ is the formal derivative of $g(t)$ with respect to $t$ and where $D$ is the formal derivative with respect to $x$.

Next we have the chain rule.

THEOREM 7.4. *If $\theta_f$ and $\theta_g$ are umbral shifts, then*

$$\theta_f^* = (\theta_f^* g(t))\, \theta_g^*.$$

*Proof.* Since $\theta_f^*$ is a derivation we have

$$\begin{aligned} \theta_f^* g(t)^k &= k g(t)^{k-1}\, \theta_f^* g(t) \\ &= (\theta_f^* g(t))\, \theta_g^* g(t)^k \end{aligned}$$

and an appeal to continuity completes the proof.

We can now relate two umbral shifts.

THEOREM 7.5. *If $\theta_f$ and $\theta_g$ are umbral shifts, then*

$$\theta_f = \theta_g \circ (\theta_g^* f(t))^{-1}.$$

*Proof.* For any $p(x)$ in $A$

$$\begin{aligned} \langle t^k \mid \theta_f\, p(x)\rangle &= \langle \theta_f^* t^k \mid p(x)\rangle \\ &= \langle (\theta_g^* f(t))^{-1} (\theta_f^* t^k) \mid p(x)\rangle \\ &= \langle t^k \mid \theta_g \circ (\theta_g^* f(t))^{-1}\, p(x)\rangle \end{aligned}$$

from which the result follows.

From this we obtain our first recurrence formula.

THEOREM 7.6. *If $p_n(x)$ is associated to $f(t)$, then*

$$\frac{(n+1)\, c_n}{c_{n+1}}\, p_{n+1}(x) = \theta_t (f'(t))^{-1}\, p_n(x),$$

*where $f'(t)$ is the formal derivative of $f(t)$ with respect to $t$.*

*Proof.* This follows from Theorem 7.5 by taking $g(t) = t$ and applying to $p_n(x)$.

One may deduce from Theorem 7.6 a recurrence formula for the polynomial part $\tilde{p}_n(x)$ of an associated sequence. Since

$$\theta_t x^n = \frac{(n+1)\, c_n}{c_{n+1}}\, x^{n+1}$$

we see that if $n < 0$, then the polynomial part of $\theta_t x^n$ is 0. Hence if

$$p(x) = \sum_{j=-\infty}^{n} a_j x^j,$$

then

$$\begin{aligned}\widetilde{\theta_t p(x)} &= \sum_{j=-\infty}^{n} a_j \widetilde{\theta_t x^j} \\ &= \sum_{j=0}^{n} a_j \widetilde{\theta_t x^j} \\ &= \sum_{j=0}^{n} a_j \theta_t x^j \\ &= \theta_t \tilde{p}(x).\end{aligned}$$

Thus we have for $n > 0$

$$\tilde{p}_{n+1}(x) = \frac{c_{n+1}}{(n+1)\, c_n}\, \theta_t \widetilde{(f'(t))^{-1}\, p_n(x)}. \tag{7.2}$$

But the polynomial part of $(f'(t))^{-1} p_n(x)$ is easily computed by modifying the action of the linear operator $t$ on $x^n$ so that $t^k x^n = 0$ if $k > n$. Put another way, if $g(t) = \sum_{k=0}^{\infty} b_k t^k$, then $\widetilde{g(t)\, x^n} = (\sum_{k=0}^{n} b_k t^k)\, x^n$. Thus (7.2) is the same recurrence formula for associated sequences which appears in UCI.

We would like to derive the analog of Theorem 7.6 for Sheffer sequences. To this end we have the following formula for the adjoint of an umbral shift.

THEOREM 7.7. *Let $\theta_f$ be an umbral shift. Then*

$$\theta_f^* h(t) = h(t)\, \theta_f - \theta_f h(t)$$

*for all $h(t)$ in $\Gamma$.*

*Proof.* If $g(t)$ is in $\Gamma$ and $p(x)$ is in $\Lambda$, then

$$\begin{aligned}\langle g(t) \mid (\theta_f^* h(t))\, p(x)\rangle &= \langle (\theta_f^* h(t))\, g(t) \mid p(x)\rangle \\ &= \langle \theta_f^*(h(t)\, g(t)) - h(t)\, \theta_f^* g(t) \mid p(x)\rangle \\ &= \langle g(t) \mid (h(t)\, \theta_f - \theta_f h(t))\, p(x)\rangle\end{aligned}$$

from which the conclusion follows.

THEOREM 7.8. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. Then*

$$\frac{(n+1)\, c_n}{c_{n+1}}\, s_{n+1}(x) = \left(\theta_t - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)}\, s_n(x).$$

*Proof.* Let $p_n(x)$ be the associated sequence for $f(t)$. Then

$$\begin{aligned}\frac{(n+1)\, c_n}{c_{n+1}}\, s_{n+1}(x) &= \frac{(n+1)\, c_n}{c_{n+1}}\, g^{-1}(t)\, p_{n+1}(x) \\ &= g^{-1}(t)\, \theta_t (f'(t))^{-1}\, g(t)\, s_n(x). \qquad (7.3)\end{aligned}$$

But

$$\begin{aligned}g^{-1}(t)\, \theta_t\, g(t) &= [g^{-1}(t)\, \theta_t - \theta_t\, g^{-1}(t)]\, g(t) + \theta_t \\ &= [\theta_t^* g^{-1}(t)]\, g(t) + \theta_t \\ &= -\frac{g'(t)}{g(t)} + \theta_t\end{aligned}$$

and inserting this in (7.3) gives the result.

THEOREM 7.9. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$. If*

$$T = \left(\theta_t - \frac{g'(t)}{g(t)}\right) \frac{f(t)}{tf'(t)} = \left(xD - \frac{tg'(t)}{g(t)}\right) \frac{f(t)}{tf'(t)},$$

*then*

$$Ts_n(x) = ns_n(x).$$

*In other words $s_n(x)$ is a formal eigenfunction for $T$ with eigenvalue $n$. Conversely, if $p(x)$ in $A$ is a solution to*

$$Tp(x) = np(x)$$

*for some $n$, then $p(x) = cs_n(x)$ for some constant $c$.*

*Proof.* The two forms for $T$ are equivalent since $\theta_t t = xD$. The fact that $Ts_n(x) = ns_n(x)$ follows from Theorem 7.8 since $s_n(x) =$

$(c_n/c_{n+1}) f(t) s_{n+1}(x)$. For the converse, let $Tp(x) = np(x)$. Then there exists constants $a_j$ such that

$$p(x) = \sum_{j=-\infty}^{m} a_j s_j(x).$$

If $n \neq 0$ applying the continuous linear operator $T$ to both sides and dividing by $n$ yields

$$p(x) = \sum_{j=-\infty}^{m} a_j \frac{j}{n} s_j(x)$$

from which we deduce that $a_j = a_j(j/n)$ for all $j$ and so $a_j = a_n \delta_{n,j}$. In case $n = 0$ applying $T$ gives

$$0 = \sum_{j=-\infty}^{m} a_j j s_j(x)$$

from which we deduce that $a_j = a_0 \delta_{0,j}$. In either case, $p(x) = a_n s_n(x)$ which completes the proof.

The following result is easily proved.

THEOREM 7.10. *Let $T$ be a linear operator on $\Lambda$ of the form*

$$T = (\theta_t - h(t))\, l(t),$$

*where $l(t)$ is a delta operator with leading coefficient equal to* 1 *and* $\deg h(t) = 0$. *Then a solution to*

$$Ts_n(x) = ns_n(x)$$

*is given by the nth series in the Sheffer sequence for the pair*

$$\left(\exp \int h(t)\, dt, t \exp \int (l^{-1}(t) - t^{-1})\, dt\right).$$

## 8. Transfer Formulas

The conjugate representation for a Sheffer sequence is most useful when the delta series $\bar{f}(t)$ can be explicitly computed. In this section we give a formula for associated sequences which does not involve $\bar{f}(t)$. This formula, in connection with Theorem 5.3, gives us a powerful formula for computing Sheffer sequences.

THEOREM 8.1 (the Transfer Formula). *Let $p_n(x)$ be the associated sequence for $f(t)$. Then*

$$p_n(x) = \frac{c_n}{c_{-1}} f'(t) f(t)^{-n-1} x^{-1}$$
$$= f'(t) \left(\frac{f(t)}{t}\right)^{-n-1} x^n.$$

*Proof.* The two formulas on the right are easily seen to be equivalent. Now suppose $q_n(x) = (c_n/c_{-1}) f'(t) f(t)^{-n-1} x^{-1}$. We shall check the conditions of Theorem 5.4. If $n \neq 0$, then

$$\langle t^0 \mid q_n(x)\rangle = \left\langle t^0 \,\middle|\, \frac{c_n}{c_{-1}} f'(t) f(t)^{-n-1} x^{-1}\right\rangle$$
$$= \frac{c_n}{c_{-1}} \langle f'(t) f(t)^{-n-1} \mid x^{-1}\rangle$$
$$= \frac{c_n}{c_{-1}} \frac{1}{-n} \langle (f(t)^{-n})' \mid x^{-1}\rangle$$
$$= \frac{c_n}{-nc_{-1}} \langle \theta_t^* f(t)^{-n} \mid x^{-1}\rangle$$
$$= \frac{c_n}{-nc_{-1}} \langle f(t)^{-n} \mid \theta_t x^{-1}\rangle$$
$$= 0.$$

In case $n = 0$, we have

$$\langle t^0 \mid q_0(x)\rangle = \frac{c_0}{c_{-1}} \langle f'(t) f(t)^{-1} \mid x^{-1}\rangle$$

and it is easy to see that the coefficient of $t^{-1}$ in $f'(t) f(t)^{-1}$ is equal to 1. Thus

$$\langle t^0 \mid q_0(x)\rangle = c_0.$$

and part (i) of Theorem 5.4 is established. For part (ii) we have

$$f(t)\, q_n(x) = \frac{c_n}{c_{-1}} f'(t) f(t)^{-n} x^{-1}$$
$$= \frac{c_n}{c_{n-1}} q_{n-1}(x).$$

This completes the proof.

An alternate form of the Transfer Formula may be derived.

THEOREM 8.2 (the Transfer Formula). *Let $p_n(x)$ be the associated sequence for $f(t)$. Then*

$$p_n(x) = \frac{c_n}{nc_{-1}} \theta_t f(t)^{-n} x^{-1}$$

*for all $n \neq 0$.*

*Proof.* From Theorems 8.1 and 7.7 we have for $n \neq 0$,

$$\begin{aligned} p_n(x) &= \frac{c_n}{c_{-1}} f'(t) f(t)^{-n-1} x^{-1} \\ &= \frac{c_n}{-nc_{-1}} [f(t)^{-n}]' x^{-1} \\ &= \frac{c_n}{-nc_{-1}} [f(t)^{-n} \theta_t - \theta_t f(t)^{-n}] x^{-1} \\ &= \frac{c_n}{nc_{-1}} \theta_t f(t)^{-n} x^{-1}. \end{aligned}$$

## 9. UMBRAL COMPOSITION AND TRANSFER OPERATORS

Let $p_n(x)$ be associated to $f(t)$. The *transfer operator* for $p_n(x)$ or $f(t)$ is the continuous linear operator $\lambda_f$ on $\Lambda$ defined by

$$\lambda_f x^n = p_n(x).$$

From Eq. (7.1) we obtain

$$\lambda_f^* g(t) = \sum_{k=m}^{\infty} \frac{\langle g(t) \mid p_k(x) \rangle}{c_k} t^k$$

and so

$$\lambda_f^* f(t)^n = t^n \tag{9.1}$$

for all integers $n$.

We can characterize transfer operators by their adjoints.

THEOREM 9.1. *The continuous linear operator $\lambda$ on $\Lambda$ is the transfer operator for $f(t)$ if and only if its adjoint $\lambda^*$ is a continuous automorphism of $\Gamma$ for which $\lambda^* f(t) = t$.*

*Proof.* Suppose $\lambda$ is the transfer operator for $f(t)$. Then $\lambda^*$ is continuous and Eq. (9.1) together with the Expansion Theorem implies that $\lambda^*$ is an automorphism of $\Gamma$. For the converse, suppose $\omega$ is a continuous automorphism of $\Gamma$ for which $\omega f(t) = t$. Then if $\lambda_f$ is the transfer operator for $p_n(x)$, we have $\lambda_f^* f(t)^n = t^n = \omega f(t)^n$ and an appeal to continuity proves that $\omega = \lambda_f^*$. This completes the proof.

Some properties of transfer operators are contained in the next result.

THEOREM 9.2. (a) *Let $\lambda_f x^n \to p_n(x)$ be a transfer operator and let $q_n(x)$ be associated to $g(t)$. Then $\lambda q_n(x)$ is associated to $\lambda^{*-1} g(t)$.*

(b) *Let $p_n(x)$ be associated to $f(t)$ and $q_n(x)$ be associated to $g(t)$. Then the continuous linear operator $\lambda p_n(x) \to q_n(x)$ is a transfer operator and $\lambda^* g(t) = f(t)$.*

Suppose $p_n(x)$ and $q_n(x)$ are sequences in $\Lambda$ and $q_n(x) = \sum_{j=-\infty}^{n} q_{n,j} x^j$. Then the *umbral composition* of $q_n(x)$ with $p_n(x)$ is the sequence

$$q_n(\underline{p}(x)) = \sum_{j=-\infty}^{n} q_{n,j} p_j(x).$$

Notice that if $\lambda x^n \to p_n(x)$, then

$$q_n(\underline{p}(x)) = \lambda q_n(x).$$

THEOREM 9.3. *Let $p_n(x)$ be associated to $f(t)$ and $q_n(x)$ be associated to $g(t)$. Then $q_n(\underline{p}(x))$ is associated to $g(f(t))$.*

*Proof.* Let $\lambda x^n \to p_n(x)$ be the transfer operator for $f(t)$. Then by Theorem 9.2(a) we see that $q_n(\underline{p}(x)) = \lambda q_n(x)$ is associated to $\lambda^{*-1} g(t) = g(f(t))$. This concludes the proof.

We would like to extend this to Sheffer sequences.

THEOREM 9.4. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$ and let $r_n(x)$ be Sheffer for $(h(t), l(t))$. Then $r_n(\underline{s}(x))$ is Sheffer for the pair*

$$(g(t)\, h(f(t)), l(f(t))).$$

*Proof.* Let $\lambda_f x^n \to p_n(x)$ be the transfer operator for $f(t)$ and let $\mu x^n \to s_n(x)$. Then $\mu = g^{-1}(t)\, \lambda_f$ and

$$\begin{aligned}\langle g(t)\, h(f(t))\, l(f(t))^k \mid r_n(\underline{s}(x))\rangle &= \langle g(t)\, h(f(t))\, l(f(t))^k \mid \mu r_n(x)\rangle \\ &= \langle h(f(t))\, l(f(t))^k \mid \lambda_f r_n(x)\rangle \\ &= \langle h(t)\, l(t)^k \mid r_n(x)\rangle \\ &= c_n \delta_{n,k}.\end{aligned}$$

This completes the proof.

Suppose $r_n(x)$ and $s_n(x)$ are two sequences in $\Lambda$ related by

$$r_n(x) = \sum_{k=-\infty}^{n} a_{n,k} s_k(x).$$

The *connection-constants problem* is to determine the constants $a_{n,k}$. In case $s_n(x)$ are Sheffer sequences we can give a solution to this problem.

THEOREM 9.5. *Let $s_n(x)$ be Sheffer for $(g(t), f(t))$ and let $r_n(x)$ be Sheffer for $(h(t), l(t))$. Suppose*

$$r_n(x) = \sum_{k=-\infty}^{n} a_{n,k} s_k(x). \tag{9.2}$$

*Then the sequence*

$$t_n(x) = \sum_{k=-\infty}^{n} a_{n,k} s_k(x).$$

*is the Sheffer sequence for the pair*

$$\left(\frac{h(\bar{f}(t))}{g(\bar{f}(t))}, l(\bar{f}(t))\right).$$

*Proof.* Equation (9.2) can be written as $r_n(x) = t_n(\underline{s}(x))$. Now if $t_n(x)$ is Sheffer for $(X(t), Y(t))$, then by Theorem 9.4 we have

$$h(t) = g(t)\, X(f(t)), \qquad l(t) = Y(f(t)).$$

The result follows by solving these equations for $X(t)$ and $Y(t)$.

## 10. EXAMPLES: HERMITE AND LAGUERRE

Let

$$\begin{aligned} c_n &= n!\,, & n &\geqslant 0, \\ &= \frac{(-1)^{n-1}}{(-n-1)!}, & n &< 0. \end{aligned}$$

The constants $c_n/c_{n-k}$ are extensions of the lower factorials $(n)_k$. In particular,

$$\begin{aligned} \frac{c_n}{c_{n-1}} &= n, & n &\neq 0, \\ &= 1, & n &= 0. \end{aligned}$$

We shall have occasion to use the following list:

| $n, k$ | $\dfrac{c_n}{c_{n-k}}$ |
|---|---|
| $0 \leqslant k \leqslant n$ | $(n)_k$ |
| $n < 0 \leqslant k$ | $(n)_k$ |
| $0 \leqslant n < k$ | $(-1)^{n-k+1}\, n!(k-n-1)!\,.$ |

Since

$$\begin{aligned} tx^n &= nx^{n-1}, \qquad & n \neq 0, \\ &= x^{-1}, \qquad & n = 0, \end{aligned}$$

we may connect the operator $t$ with the derivative

$$tp(x) = Dp(x) + \langle t \mid p(x) \rangle\, x^{-1}. \tag{10.1}$$

Also,

$$\begin{aligned} \theta_t x^n &= x^{n+1}, \qquad & n \neq -1, \\ &= 0, \qquad & n = -1, \end{aligned}$$

and so

$$\theta_t\, p(x) = xp(x) - \langle t^{-1} \mid p(x) \rangle. \tag{10.2}$$

Since

$$\varepsilon_{y,0}(t) = e^{yt},$$

then if $n < 0$ we obtain

$$\varepsilon_{y,0}(t)\, x^n = (x+y)^n \tag{10.3}$$

and if $n \geqslant 0$ an easy calculation gives

$$\varepsilon_{y,0}(t)\, x^n = (x+y)^n + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{n!(k-1)!}{(n+k)!}\, y^{n+k} x^{-k}. \tag{10.4}$$

In view of (10.3) the Sheffer identity is, for $n < 0$,

$$s_n(x+y) = \sum_{k=-\infty}^{n} \binom{n}{n-k} \tilde{p}_{n-k}(y)\, s_k(x).$$

For $n \geqslant 0$, this identity does not take such a simple form. However, by taking polynomial parts in (10.4) we obtain

$$\tilde{s}_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \tilde{p}_{n-k}(y)\, \tilde{s}_k(x),$$

which is the Sheffer identity for polynomials appearing in [5].

(1) The Sheffer sequence for the pair $(e^{-vt^2/2}, t)$ is the *Hermite sequence of variance v*

$$H_n^{(v)}(x) = e^{-vt^2/2} x^n. \tag{10.5}$$

Each term in the sequence is called an *Hermite series*. By expanding $e^{-vt^2/2}$ we see that

$$H_n^{(v)}(x) = \sum_{k=0}^{\infty} \left(\frac{-v}{2}\right)^k \frac{c_n}{k!\, c_{n-2k}} x^{n-2k}.$$

For $n < 0$ we obtain

$$H_n^{(v)}(x) = \sum_{k=0}^{\infty} \left(\frac{-v}{2}\right)^k \frac{(n)_{2k}}{k!} x^{n-2k}$$

and for $n \geqslant 0$

$$H_n^{(v)}(x) = \sum_{k=0}^{[n/2]} \left(\frac{-v}{2}\right)^k \frac{(n)_{2k}}{k!} x^{n-2k} + \sum_{k=[n/2]+1}^{\infty} \left(\frac{-v}{2}\right)^k (-1)^{n+1} \frac{n!(2k-n-1)!}{k!} x^{n-2k}.$$

It follows from (10.5) that for $n \geqslant 0$ the polynomials $\tilde{H}_n^{(v)}(x)$ are the Hermite polynomials.

The generating function for $H_n^{(v)}(x)$ is

$$e^{vt^2/2} E^{xt} = \sum_{k=-\infty}^{\infty} \frac{H_k^{(v)}(x)}{c_k} t^k.$$

From Theorem 5.6 we obtain

$$\begin{aligned} tH_n^{(v)}(x) &= \frac{c_n}{c_{n-1}} H_{n-1}^{(v)}(x) \\ &= nH_{n-1}^{(v)}(x), \qquad n \neq 0, \\ &= H_{-1}^{(v)}(x), \qquad n = 0. \end{aligned} \tag{10.6}$$

Next we consider the recurrence formula. From Theorem 7.8 we obtain

$$(1-\delta_{n,-1})\, H_{n+1}^{(\nu)}(x) = (\theta_t + \nu t)\, H_n^{(\nu)}(x).$$

In view of (10.2),

$$(1-\delta_{n,-1})\, H_{n+1}^{(\nu)}(x) = xH_n^{(\nu)}(x) + \nu t H_n^{(\nu)}(x) - \langle t^{-1} \mid H_n^{(\nu)}(x)\rangle.$$

From (10.6) we get

$$xH_n^{(\nu)}(x) - (1-\delta_{n,-1})\, H_{n+1}^{(\nu)}(x) + \frac{c_n}{c_{n-1}}\, \nu H_{n-1}^{(\nu)}(x) = \langle t^{-1} \mid H_n^{(\nu)}(x)\rangle,$$

where

$$\begin{aligned} \langle t^{-1} \mid H_n^{(\nu)}(x)\rangle &= \left(\frac{-\nu}{2}\right)^{(n+1)/2} (-1)^{n+1} \frac{n!}{((n+1)/2)!} && \text{for } n \geqslant 1,\ n \text{ odd}, \\ &= 1 && \text{for } n = -1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

By taking polynomial parts of the above recurrence, and noticing that $\widetilde{xH_n^{(\nu)}}(x) = x\tilde{H}_n^{(\nu)}(x) + \langle t^{-1} \mid H_n^{(\nu)}(x)\rangle$ we obtain

$$x\tilde{H}_n^{(\nu)}(x) - \tilde{H}_{n+1}^{(\nu)}(x) + n\nu \tilde{H}_{n-1}^{(\nu)}(x) = 0$$

which is the classical three term recurrence for Hermite polynomials [3, p. 179].

From Theorem 7.9 we have

$$nH_n^{(\nu)}(x) = (xD + \nu t^2)\, H_n^{(\nu)}(x).$$

Combining this with (10.1) gives the second order $t$-operator equation

$$\nu t^2 H_n^{(\nu)}(x) + xtH_n^{(\nu)}(x) - nH_n^{(\nu)}(x) = \langle t \mid H_n^{(\nu)}(x)\rangle,$$

where

$$\begin{aligned} \langle t \mid H_n^{(\nu)}(x)\rangle &= \left(\frac{-\nu}{2}\right)^{n/2} \frac{n!}{(n/2)!} && \text{for } n \geqslant 0,\ n \text{ even}, \\ &= 0 && \text{otherwise.} \end{aligned}$$

We may use (10.1) to convert these $t$-operator equations into differential equations. As an example, if $n < 0$, then $tH_n^{(\nu)}(x) = DH_n^{(\nu)}(x)$ and so

$$\nu D^2 H_n^{(\nu)}(x) + xDH_n^{(\nu)}(x) - nH_n^{(\nu)}(x) = 0.$$

(2) The Sheffer sequence for the pair $((1-t)^{-\alpha-1}, t/(t-1))$ is the *Laguerre sequence* $L_n^{(\alpha)}(x)$ *of order* $\alpha$. Each term $L_n^{(\alpha)}(x)$ is called a *Laguerre series*. In case $\alpha = 1$, the sequence $L_n^{(-1)}(x) = L_n(x)$ is the associated sequence for $t/(t-1)$. By the Transfer Formula

$$\begin{aligned} L_n(x) &= -(t-1)^{n-1} x^n \\ &= \sum_{k=0}^{\infty} \binom{n-1}{k} \frac{c_n}{c_{n-k}} (-x)^{n-k} \end{aligned}$$

and so

$$\begin{aligned} L_n^{(\alpha)}(x) &= (1-t)^{\alpha+1} L_n(x) \\ &= (-1)^n (1-t)^{\alpha+n} x^n \\ &= \sum_{k=0}^{\infty} \binom{n+\alpha}{k} \frac{c_n}{c_{n-k}} (-x)^{n-k}. \end{aligned}$$

Thus for $n < 0$,

$$\begin{aligned} L_n^{(\alpha)}(x) = &\sum_{k=0}^{n} \binom{n+\alpha}{k} (n)_k (-x)^{n-k} \\ &- \sum_{k=n+1}^{\infty} \binom{n+\alpha}{k} n!(k-n-1)!\, x^{n-k}. \end{aligned}$$

The generating function for $L_n^{(\alpha)}(x)$ is

$$(1-t)^{-\alpha-1} E^{xt/(t-1)} = \sum_{k=-\infty}^{\infty} \frac{L_k^{(\alpha)}(x)}{c_k} t^k.$$

Theorem 5.6 implies that

$$\frac{t}{t-1} L_n^{(\alpha)}(x) = \frac{c_n}{c_{n-1}} L_{n-1}^{(\alpha)}(x) \tag{10.7}$$

or

$$tL_n^{(\alpha)}(x) - \frac{c_n}{c_{n-1}} tL_{n-1}^{(\alpha)}(x) + \frac{c_n}{c_{n-1}} L_{n-1}^{(\alpha)}(x) = 0.$$

Using (10.1) we obtain

$$\begin{aligned} &DL_n^{(\alpha)}(x) - \frac{c_n}{c_{n-1}} DL_{n-1}^{(\alpha)}(x) + \frac{c_n}{c_{n-1}} L_{n-1}^{(\alpha)}(x) \\ &= \left[ \binom{n+\alpha}{n} c_n + \binom{n+\alpha-1}{n-1} c_{n-1} \right] x^{-1}, \end{aligned}$$

where of course $\binom{i}{j}=0$ if $j<0$. We remark that the equation for Laguerre polynomials obtained by taking polynomial parts of the above appears in [4, Eq. (2), p. 202] (with a slight modification).

Theorem 7.9 gives

$$nL_n^{(\alpha)}(x)=[\theta_t(1-t)-(\alpha+1)]\,tL_n^{(\alpha)}(x).$$

Employing (10.2) we obtain, after some rearrangement, the second order $t$-operator equation

$$(x(1-t)t-(\alpha+1)t-n)\,L_n^{(\alpha)}(x)=(n+\alpha+1)_n,$$

where $(n+\alpha+1)_n=0$ if $n<0$.

As in the Hermite case we may use (10.1) to obtain second order differential equations.

As an example of umbral techniques Theorem 9.4 tells us that the umbral composition $L_n^{(\beta)}(\underline{L}^{(\alpha)}(x))$ is Sheffer for $((1-t)^{\beta-\alpha}, t)$. Hence

$$\begin{aligned} L_n^{(\beta)}(\underline{L}^{(\alpha)}(x)) &= (1-t)^{\alpha-\beta}\,x^n \\ &= \sum_{k=0}^{\infty}\binom{\alpha-\beta}{k}(-1)^k\frac{c_n}{c_{n-k}}\,x^{n-k}. \end{aligned}$$

For $n<0$ we obtain

$$L_n^{(\beta)}(\underline{L}^{(\alpha)}(x))=\sum_{k=0}^{\infty}\binom{\alpha-\beta}{k}(n)_k(-1)^k\,x^{n-k}$$

and for $n\geqslant 0$

$$\begin{aligned} L_n^{(\beta)}(\underline{L}^{(\alpha)}(x)) &= \sum_{k=0}^{n}\binom{\alpha-\beta}{k}(n)_k(-1)^k\,x^{n-k} \\ &\quad+\sum_{k=n+1}^{\infty}\binom{\alpha-\beta}{k}(-1)^{n+1}. \end{aligned}$$

In case $\alpha=\beta$ we have

$$L_n^{(\alpha)}(\underline{L}^{(\alpha)}(x))=x^n$$

for all integers $n$. Thus the Laguerre sequence is self-inverse under umbral composition.

## 11. Examples: Gegenbauer

One may recall from UCI that the Gegenbauer polynomials arose from the case

$$c_n = \frac{1}{\binom{-\lambda}{n}}$$

for $n \geqslant 0$, where $-\lambda$ is not a nonnegative integer. Thus

$$\frac{c_n}{c_{n-1}} = \frac{-n}{\lambda + n - 1}.$$

We now require that $\lambda$ not be an integer. Then we set

$$\begin{aligned} c_n &= \frac{1}{\binom{-\lambda}{n}}, && n \geqslant 0, \\ &= -n\binom{\lambda - 1}{-n}, && n < 0. \end{aligned}$$

In this case

$$\begin{aligned} \frac{c_n}{c_{n-1}} &= \frac{-n}{\lambda + n - 1}, && n \neq 0, \\ &= \frac{1}{\lambda - 1}, && n = 0. \end{aligned}$$

Therefore

$$tp(x) = -(\lambda + xD)^{-1} Dp(x) + \frac{1}{\lambda - 1} \langle t^0 \mid p(x) \rangle x^{-1} \tag{11.1}$$

for all $p(x)$ in $\nu$, which is readily verified by taking $p(x) = x^n$.

The *Gegenbauer sequence* $G_n(x)$ is the Sheffer sequence for

$$g(t) = \left( \frac{2}{1 + \sqrt{1 - t^2}} \right)^{\lambda}, \qquad f(t) = \frac{\sqrt{1 - t^2} - 1}{t}.$$

Each term is called a *Gegenbauer series.* A simple computation yields

$$\bar{f}(t) = \frac{-2t}{1 + t^2}$$

and

$$g(\bar{f}(t)) = (1+t^2)^{\lambda}.$$

Thus the conjugate representation (Theorem 6.2) gives

$$\begin{aligned} G_n(x) &= \sum_{k=-\infty}^{n} \frac{(-2)^k}{c_k} \langle t^k (1+t^2)^{-\lambda-k} \mid x^n \rangle x^k \\ &= \sum_{j=0}^{\infty} \binom{-\lambda+2j-n}{j} \frac{c_n}{c_{n-2j}} (-2x)^{n-2j} \end{aligned}$$

Evaluating $c_n/c_{n-2j}$ gives for $n<0$,

$$G_n(x) = \sum_{j=0}^{\infty} \frac{(-1)^j(-n+2j-1)_{2j}}{j!(\lambda+n-1)j} (-2x)^{n-2j}$$

and for $n \geqslant 0$,

$$G_n(x) = \sum_{j=0}^{[n/2]} \frac{\binom{-\lambda}{n-2j}\binom{-\lambda+2j-n}{j}}{\binom{-\lambda}{n}} (-2x)^{n-2j}$$

$$+ \sum_{j=[n/2]+1}^{\infty} \frac{\binom{-\lambda+2j-n}{j}}{(-n+2j)\binom{-\lambda}{n}\binom{\lambda-1}{-n+2j}} (-2x)^{n-2j}.$$

The generating function for $G_n(x)$ is

$$(1+t^2)^{-\lambda} E^{x(-2t/(1+t^2))} = \sum_{k=-\infty}^{\infty} \frac{G_k(x)}{c_k} t^k.$$

Our first recurrence comes from Theorem 5.6. Since $f(t)\, G_n(x) = (c_n/c_{n-1})\, G_{n-1}(x)$ we deduce that

$$\sqrt{1-t^2}\, G_n(x) = \frac{c_n}{c_{n-1}} t G_{n-1}(x) + G_n(x). \tag{11.2}$$

On the other hand

$$\begin{aligned}\sqrt{1-t^2}\, G_n(x) &= (1+\sqrt{1-t^2})\, G_n(x) - G_n(x) \\ &= \frac{c_n}{c_{n+1}} (1+\sqrt{1-t^2}) f(t)\, G_{n+1}(x) - G_n(x) \\ &= -\frac{c_n}{c_{n+1}} tG_{n+1}(x) - G_n(x).\end{aligned}$$

Equating the two expressions for $\sqrt{1-t^2}\, G_n(x)$ gives the recurrence

$$\frac{c_n}{c_{n+1}}\, tG_{n+1}(x) + \frac{c_n}{c_{n-1}} tG_{n-1}(x) + 2G_n(x) = 0.$$

Since

$$\frac{g'(t)}{g(t)} = -\lambda_0 tf'(t)$$

and

$$\frac{f(t)}{f'(t)} = t\sqrt{1-t^2}.$$

Theorem 9.7 gives

$$nG_n(x) = \theta_t t\sqrt{1-t^2}\, G_n(x) + \frac{c_n}{c_{n-1}} \lambda tG_{n-1}(x).$$

Using (11.2) we obtain

$$\frac{c_n}{c_{n-1}} \theta_t t^2 G_{n-1}(x) + \theta_t tG_n(x) + \frac{c_n}{c_{n-1}} \lambda tG_{n-1}(x) - nG_n(x) = 0.$$

Employing (11.1) and the fact that $\theta_t t = xD$ we obtain

$$\begin{aligned}&xDG_n(x) - \frac{c_n}{c_{n-1}} DG_{n-1}(x) - nG_n(x) \\ &\quad = \frac{n(\lambda-1)}{\lambda+n-1} \left[ \binom{-\lambda}{(n-1)/2} \Big/ \binom{-\lambda}{n-1} \right] x^{-1}\end{aligned}$$

with the usual proviso that $\binom{i}{j} = 0$ if $j < 0$ or $j$ is not an integer. Again taking polynomial parts given a familiar recurrence for the Gegenbauer polynomials ([4, (9), p. 279] with some modification).

## 12. EXAMPLES: CHEBYSHEV

We mention only briefly the case

$$c_n = (-1)^n$$

for all integers $n$. One may recall from UCI that the Chebyshev polynomials were obtained from this choice of $c_n$ for $n \geqslant 0$. Then

$$\frac{c_n}{c_{n-1}} = -1.$$

and so

$$t = -x^{-1}$$

and

$$\theta_t = -xDx.$$

The *Chebyshev sequence of the first kind* $T_n(x)$ is the Sheffer sequence for

$$g(t) = \frac{1}{\sqrt{1-t^2}}, \qquad f(t) = \frac{\sqrt{1-t^2}-1}{t}.$$

One easily obtains

$$\bar{f}(t) = \frac{-2t}{1+t^2}$$

and

$$g(\bar{f}(t)) = \frac{1+t^2}{1-t^2}.$$

The generating function for $T_n(x)$ is

$$\frac{1-t^2}{1+t^2} E^{x(-2t/(1+t^2))} = \sum_{k=-\infty}^{\infty} (-1)^k T_k(x) t^k.$$

The *Chebyshev sequence of the second kind* $U_n(x)$ is Sheffer for

$$g(t) = \frac{2-2\sqrt{1-t^2}}{t^2}, \qquad f(t) = \frac{\sqrt{1-t^2}-1}{t}.$$

Since $g(\bar{f}(t)) = 1 + t^2$ the generating function is

$$\frac{1}{1+t^2} E^{x(-2t/(1+t^2))} = \sum_{k=-\infty}^{\infty} (-1)^k U_k(x)\, t^k.$$

## 13. Examples: Jacobi

We shall not pursue a study of the Jacobi series at this time. Suffice it to say that by setting

$$\begin{aligned} c_n &= \frac{4^n(\alpha+n)_n}{(\alpha+\beta+2n)_{2n}}, && n \geqslant 0, \\ &= \frac{4^n(\alpha+\beta)_{-2n}}{(\alpha)_{-n}}, && n < 0, \end{aligned}$$

the Sheffer sequence for

$$g(t) = \left(\frac{2}{1+\sqrt{1+2t}}\right)^{1+\alpha+\beta}, \qquad f(t) = \frac{1+t-\sqrt{1+2t}}{t}$$

behaves in a Jacobi-like manner.

## 14. Examples: The $q$-Case

Let us take

$$\begin{aligned} c_n &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)^n}, && n \geqslant 0, \\ &= \frac{(-1)^{n-1} q^{\binom{-n}{2}}(1-q)^{-n-1}}{(1-q)(1-q^2)\cdots(1-q^{-n-1})}, && n < 0. \end{aligned}$$

Then one readily verifies that

$$\begin{aligned} \frac{c_n}{c_{n-1}} &= \frac{1-q^n}{1-q}, && n \neq 0, \\ &= 1, && n = 0. \end{aligned}$$

Therefore

$$\begin{aligned} tx^n &= \frac{1-q^n}{1-q} x^{n-1}, && n \neq 0, \\ &= x^{-1}, && n = 0. \end{aligned} \tag{14.1}$$

The *q-derivative* $D_q$ is the continuous linear operator on satisfying

$$D_q x^n = \frac{1-q^n}{1-q} x^{n-1}$$

for all integers $n$. Thus

$$D_q x^n = \frac{x^n - (qx)^n}{x - qx}$$

and so

$$D_q p(x) = \frac{p(x) - p(qx)}{x - qx}.$$

In view of (14.1) we have

$$tp(x) = D_q p(x) + \langle t^0 \mid p(x) \rangle x^{-1}.$$

Also,

$$\theta_t x^n = \frac{(n+1)(1-q)}{(1-q^{n+1})} x^{n+1}.$$

The $q$-binomial coefficient $\binom{n}{k}_q$ is defined in the literature for $n, k \geqslant 0$ as

$$\binom{n}{k}_q = \frac{c_n}{c_k c_{n-k}}.$$

Since the right side is defined for all integers $n$ and $k$ the $q$-binomial coefficient is automatically extended. The evaluation series is

$$\varepsilon_{y,0}(t) = \sum_{k=0}^{\infty} \binom{n}{k}_q y^k t^k.$$

The Sheffer sequence for $(\varepsilon_{y,0}(t), t)$ will be denoted by $s_n(x) = [x]_{y,n}$. Thus

$$[x]_{y,n} = \varepsilon_{y,0}(t)^{-1} x^n.$$

In UCI we found that

$$\varepsilon_{y,0}(t)^{-1} = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(1-q) \cdots (1-q^k)} q^{\binom{k}{2}} (-y)^k \frac{c_n}{c_{n-k}} x^{n-k}.$$

and so we obtain

$$[x]_{y,n} = \sum_{k \geqslant 0}^{\infty} \frac{(1-q)^k}{(1-q) \cdots (1-q^k)} q^{\binom{k}{2}} (-y)^k \frac{c_n}{c_{n-k}} x^{n-k}.$$

For $n < 0$, we get

$$[x]_{y,n} = \sum_{k=0}^{\infty} \frac{(1-q^{-n}) \cdots (1-q^{-n+k})}{(1-q) \cdots (1-q^k)} q^{kn} y^k x^{n-k}$$

and for $n \geqslant 0$,

$$\begin{aligned} x_{y,n} &= (x-y)(x-qy) \cdots (x-q^{n-1}y) \\ &\quad + \sum_{k=n+1}^{\infty} (1-q) \prod_{j=k+1}^{n} (1-q^j) \\ &\quad \times \prod_{i=1}^{k-n-1} (1-q^i)\, q^{n(2k-n-1)/2} y^k x^{n-k}. \end{aligned}$$

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