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The Logarithmic Binomial Formula

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1. INTRODUCTION. The algebra $\mathscr{P}$ of polynomials in a single variable x provides a simple setting in which to do the "polynomial" calculus. One of the nicest features of $\mathscr{P}$ is that it is closed under both differentiation and antidifferentiation. Furthermore, within the algebra $\mathscr{P}$, we have the well-known binomial formula

$$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z}, \quad n \geq 0 \tag{1}$$

which may have been known as early as A.D. 1100 in the works of Omar Khayyam. (Euclid knew the formula for $n = 2$ around 300 B.C.). To be sure, the formula, as we know it today, was stated by Pascal in his *Traite du Triangle Arithmetic* in 1665.

Now suppose we wish to include the negative powers of x in our setting. One possibility is to combine the positive and negative powers of x, by working in the algebra $\mathscr{A}$ of Laurent series of the form

$$\sum_{k=-\infty}^{n} a_k x^k.$$

This algebra is certainly closed under differentiation, and there is even a binomial formula for *negative* integral powers

$$(x+a)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k x^{n-k}, \qquad n \in \mathbb{Z}, \quad n < 0. \tag{2}$$

due to Newton (1676), which converges for $|x| > |a|$.

Recall that the binomial coefficients are defined for integers satisfying $n \geq k \geq 0$, or $k \geq 0 > n$, by

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

where $k! = k(k-1)\cdots 1$.

The algebra $\mathscr{A}$ does suffer from one drawback, however. It is not closed under antidifferentiation, since there is no Laurent series $f(x)$ with the property that $Df(x) = x^{-1}$. To correct this problem, we must introduce the logarithm $\log x$. Doing so produces some rather interesting consequences, and it is the purpose of this paper to explore some of those consequences.

In particular, we will be led to some fascinating new functions, first studied by Loeb and Rota in 1989, who called them *harmonic logarithms*. We will also be led to a generalization of the binomial formulas (1) and (2), which holds for *all* integers n. This generalization is called the *logarithmic binomial formula*.

2. THE HARMONIC LOGARITHMS. Our setting will be the set L of all finite linear combinations, with real coefficients, of terms of the form $x^i(\log x)^j$, where i

is any integer, and j is any nonnegative integer. That is, L is the real vector space with basis $\{x^i(\log x)^j | i, j \in \mathbb{Z},\ j \geq 0\}$. Under ordinary multiplication, L becomes an algebra over the real numbers. Furthermore, the formula

$$Dx^i(\log x)^j = ix^{i-1}(\log x)^j + jx^{i-1}(\log x)^{j-1} \tag{3}$$

shows that L is closed under differentiation, and the formulas

$$D^{-1}x^i(\log x)^j = \frac{1}{i+1}x^{i+1}(\log x)^j - \frac{j}{i+1}D^{-1}x^i(\log x)^{j-1}, \qquad i \neq -1$$

$$D^{-1}x^{-1}(\log x)^j = \frac{1}{j+1}(\log x)^{j+1} \tag{4}$$

can be used to give an inductive proof showing that L is closed under antidifferentiation. In fact, we can characterize L as follows.

Proposition 2.1. *The algebra L is the smallest algebra that contains both x and x^{-1}, and is closed under differentiation and antidifferentiation.* ∎

Formulas (3) and (4) indicate that, while the basis $\{x^i(\log x)^j\}$ may be suitable for studying the algebraic properties of L, it is not ideal for studying the properties of L that are related to the operators D and D^{-1}. To search for a more suitable basis for L, let us take another look at how the derivative acts on powers of x. If we let

$$\lambda_n^{(0)}(x) = \begin{cases} x^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

then

$$D\lambda_n^{(0)}(x) = n\lambda_{n-1}^{(0)}(x)$$

for all integers n. Thinking of the functions $\lambda_n^{(0)}(x)$ as a doubly infinite sequence

$$\begin{array}{ccccccccc} \cdots & \lambda_{-3}^{(0)}(x) & \lambda_{-2}^{(0)}(x) & \lambda_{-1}^{(0)}(x) & \lambda_0^{(0)}(x) & \lambda_1^{(0)}(x) & \lambda_2^{(0)}(x) & \lambda_3^{(0)}(x) & \cdots \\ \cdots & 0 & 0 & 0 & 1 & x & x^2 & x^3 & \cdots \end{array}$$

we see that applying the derivative operator D has the effect of shifting one position to the left, and multiplying by a constant.

If we introduce the notation

$$\lfloor n \rceil = \begin{cases} n & \text{for } n \neq 0 \\ 1 & \text{for } n = 0 \end{cases}$$

then the functions $\lambda_n^{(0)}(x)$ are uniquely defined by the following properties.

1) $\lambda_0^{(0)}(x) = 1$
2) $\lambda_n^{(0)}(x)$ has no constant term for $n \neq 0$
3) $D\lambda_n^{(0)}(x) = \lfloor n \rceil \lambda_{n-1}^{(0)}(x)$

Notice that the antiderivative behaves nicely on the functions $\lambda_n^{(0)}(x)$, except when applied to $\lambda_{-1}^{(0)}(x)$. With the understanding that D^{-1} produces no arbitrary constant terms, we can write

$$D^{-1}\lambda_n^{(0)}(x) = \begin{cases} \dfrac{1}{n+1}\lambda_{n+1}^{(0)}(x) & \text{for } n \neq -1 \\ 0 & \text{for } n = -1. \end{cases}$$

At this point, we have only the nonnegative powers of x. However, we can obtain the negative powers of x by introducing a second row of functions $\lambda_n^{(1)}(x)$, starting with $\lambda_0^{(1)}(x) = \log x$, and using conditions similar to 1)–3). In particular, the conditions

4) $\lambda_0^{(1)}(x) = \log x$
5) $\lambda_n^{(1)}(x)$ has no constant term
6) $D\lambda_n^{(1)}(x) = \lfloor n \rceil \lambda_{n-1}^{(1)}(x)$

uniquely define a doubly infinite sequence of functions $\lambda_n^{(1)}(x)$

$\cdots$	$\lambda_{-3}^{(1)}(x)$	$\lambda_{-2}^{(1)}(x)$	$\lambda_{-1}^{(1)}(x)$	$\lambda_0^{(1)}(x)$	$\lambda_1^{(1)}(x)$	$\lambda_2^{(1)}(x)$	$\lambda_3^{(1)}(x)$	$\cdots$
$\cdots$	x^{-3}	x^{-2}	x^{-1}	$\log x$	$x(\log x - 1)$	$x^2(\log x - 1 - \frac{1}{2})$	$x^3(\log x - 1 - \frac{1}{2} - \frac{1}{3})$	$\cdots$

Observing the pattern in these functions, it is not hard to determine the general form of $\lambda_n^{(1)}(x)$.

Proposition 2.2. *The functions $\lambda_n^{(1)}(x)$, uniquely defined by conditions 4)–6) above, are given by*

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\log x - h_n) & \text{for } n \ge 0 \\ x^n & \text{for } n < 0 \end{cases}$$

where

$$h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for $n > 0$ and $h_0 = 0$. ∎

Notice that the behavior of D^{-1} on the functions $\lambda_n^{(1)}(x)$ is even nicer than it is on the functions $\lambda_n^{(0)}(x)$, for assuming no arbitrary constant, we have for *all n*,

$$D^{-1}\lambda_n^{(1)}(x) = \frac{1}{\lfloor n+1 \rceil}\lambda_{n+1}^{(1)}(x).$$

The vector space formed using the functions $\lambda_n^{(0)}(x)$ and $\lambda_n^{(1)}(x)$ as a basis contains both the positive and negative powers of x, and is closed under differentiation and antidifferentiation, but it is not an algebra. For instance, the functions $(\log x)^t$, for $t > 1$, are not in this vector space. This prompts us to enlarge our class of functions still further.

Definition. For all integers n and nonnegative integers t, we define the *harmonic logarithms* $\lambda_n^{(t)}(x)$ of *order* t and *degree* n as the unique functions satisfying the following properties.

1) $\lambda_0^{(t)}(x) = (\log x)^t$
2) $\lambda_n^{(t)}(x)$ has no constant term, except that $\lambda_0^{(0)}(x) = 1$
3) $D\lambda_n^{(t)}(x) = \lfloor n \rceil \lambda_{n-1}^{(t)}(x)$ ∎

This definition allows us (at least in theory) to construct the harmonic logarithms by starting each row (that is, the harmonic logarithms of a fixed order), at $\lambda_0^{(t)}(x) = (\log x)^t$. We then differentiate to get $\lambda_n^{(t)}(x)$ for $n < 0$, and antidifferentiate to get $\lambda_n^{(t)}(x)$ for $n > 0$.

In fact with the understanding that D^{-1} produces no arbitrary constants, we can write

$$\lambda_n^{(t)}(x) = a_{n,t}D^{-n}(\log x)^t$$

where the $a_{n,t}$ are constants. These constants can easily be determined using the definition of harmonic logarithm. It turns out that $a_{n,t}$ does not depend on t, and that $a_{n,t} = \lfloor n \rceil!$, where the latter are defined by

$$\lfloor n \rceil! = \begin{cases} n! & \text{for } n \geq 0 \\ \dfrac{(-1)^{-n-1}}{(-n-1)!} & \text{for } n < 0 \end{cases}$$

Loeb and Rota have called $\lfloor n \rceil!$ the *Roman factorial*. The notation $\lfloor n \rceil!$ was suggested by Donald Knuth. Thus, we have

Proposition 2.3. *The harmonic logarithms have the form*

$$\lambda_n^{(t)}(x) = \lfloor n \rceil! D^{-n} (\log x)^t. \qquad \blacksquare$$

Many of the well-known properties of the ordinary factorials carry over to the numbers $\lfloor n \rceil!$. Some of the more important of these properties are listed in Box 1.

Proposition 2.3 can be used to derive an explicit formula for the harmonic logarithms. However, since we do not need this formula yet, and since it is a bit involved, we prefer to postpone it until later. We should mention now, however, that the harmonic logarithms $\lambda_n^{(t)}(x)$ do form a basis for the algebra L.

Properties of the numbers $\lfloor n \rceil!$
1) $\lfloor n \rceil! = \lfloor n \rceil \lfloor n-1 \rceil!$
2) $\dfrac{\lfloor n \rceil!}{\lfloor n-k \rceil!} = \lfloor n \rceil \lfloor n-1 \rceil \cdots \lfloor n-k+1 \rceil$, for $k > 0$
3) $\lfloor n \rceil! \lfloor -n-1 \rceil! = (-1)^{n+(n<0)}$, where $(n<0)$ is 1 if $n < 0$ and 0 if $n \geq 0$.

Box 1

Using the definition of harmonic logarithm, along with Property 2 in Box 1, we get

$$D^k \lambda_n^{(t)}(x) = \frac{\lfloor n \rceil!}{\lfloor n-k \rceil!} \lambda_{n-k}^{(t)}(x)$$

which shows that the higher derivatives behave on all harmonic logarithms in the same way as they behave on the powers of x.

From the definition of $\lfloor n \rceil!$, it seems a natural step to generalize the binomial coefficients by setting

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{\lfloor n \rceil!}{\lfloor k \rceil! \lfloor n-k \rceil!}$$

for *all* integers n and k. Loeb and Rota have called the numbers $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rceil$ the *Roman coefficients*. The notation $\left\lfloor \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rceil$ was also suggested by Knuth, and is read "Roman n choose k."

The Roman coefficients agree with the ordinary binomial coefficients whenever the latter are defined. That is, whenever $n \geq k \geq 0$, or $k \geq 0 > n$, we have

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \binom{n}{k}.$$

On the other hand, we also have, for example

$$\left\lfloor \begin{matrix} n \\ -1 \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ n+1 \end{matrix} \right\rceil = \frac{1}{\lfloor n+1 \rceil} \quad \text{and} \quad \left\lfloor \begin{matrix} 0 \\ k \end{matrix} \right\rceil = \frac{(-1)^{k+(k>0)}}{\lfloor k \rceil}$$

showing that the Roman coefficients are not always integers, nor are they always nonnegative. Perhaps the most interesting question about these coefficients is "What, if anything, do they count, or measure?" The temptation to think that they do count, or measure, something is further enforced by their algebraic properties, which in many cases are direct generalizations of those of the ordinary binomial coefficients. Box 2 contains a small sampling.

Properties of the numbers $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil$
1) For all integers n, k and r, $$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ n-k \end{matrix} \right\rceil \quad \text{and} \quad \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ r \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ r \end{matrix} \right\rceil \left\lfloor \begin{matrix} n-r \\ k-r \end{matrix} \right\rceil.$$ 2) **(Pascal's formula)** For any two distinct, nonzero integers n and k, $$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n-1 \\ k \end{matrix} \right\rceil + \left\lfloor \begin{matrix} n-1 \\ k-1 \end{matrix} \right\rceil.$$ 3) **(Knuth's rotation/reflection law)** $$(-1)^{k+(k>0)} \left\lfloor \begin{matrix} -n \\ k-1 \end{matrix} \right\rceil = (-1)^{n+(n>0)} \left\lfloor \begin{matrix} -k \\ n-1 \end{matrix} \right\rceil.$$

Box 2

3. THE LOGARITHMIC BINOMIAL FORMULA. Now let us turn to the logarithmic binomial formula. For any positive real number a, we can expand the function $\lambda_n^{(t)}(x+a)$ in a Taylor series that is valid for $|x| < a$

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \frac{\left[D^k \lambda_n^{(t)}(x)\right]_{x=a}}{k!} x^k = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a) x^k.$$

Thus, we have the following logarithmic binomial theorem.

Proposition 3.1. (*Logarithmic binomial theorem*) *For all integers n,*

$$\lambda_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \lambda_{n-k}^{(t)}(a) x^k$$

valid for $|x| < a$. ∎

Boxes 3–5 describe the logarithmic binomial formula of orders one and two.

The First Order Logarithmic Binomial Formula
Let $t = 0$. We have $\lambda_{n-k}^{(0)}(a) = a^{n-k}$ for $n \geq k$, and $\lambda_{n-k}^{(0)}(a) = 0$ for $n < k$. Furthermore, since $\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \binom{n}{k}$ when $n \geq k \geq 0$, the logarithmic binomial formula is $$(x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$ which is equivalent to the classical binomial formula (1).

Box 3

The Second Order Logarithmic Binomial Formula of Negative Degree

Let $t = 1$ and $n < 0$. Since

$$\lambda_n^{(1)}(x) = \begin{cases} x^n(\log x - h_n) & \text{for } n \geq 0 \\ x^n & \text{for } n < 0 \end{cases}$$

the logarithmic binomial formula is

$$(x + a)^n = \sum_{k=0}^{\infty} \left\lfloor {n \atop k} \right\rfloor a^{n-k} x^k.$$

Interchanging the roles of x and a, and noting that $\left\lfloor {n \atop k} \right\rfloor = \binom{n}{k}$ when $k \geq 0 > n$, we get the classical binomial formula (2).

Box 4

The Second Order Logarithmic Binomial Formula of Nonnegative Degree

When $t = 1$ and $n \geq 0$, the logarithmic binomial formula gives some interesting new results. Extending the definition of the harmonic logarithms of order 1, when $n > 0$, by taking

$$\lambda_n^{(1)}(0) = \lim_{x \to 0+} \lambda_n^{(1)}(x) = 0$$

we have

$$\lambda_n^{(1)}(x + a) = \sum_{k=0}^{\infty} \left\lfloor {n \atop k} \right\rfloor \lambda_{n-k}^{(1)}(a) x^k$$

which is valid

1) For $|x| < a$, when $n < 0$,
2) For $|x| \leq a$, $x \neq -a$, when $n = 0$,
3) For $|x| \leq a$, when $n > 0$, where $\lambda_n^{(1)}(0) = 0$.

Taking $a = 1$ leads to a nice expansion of the function $(x + 1)^n \log(x + 1)$ when $n > 0$

$$(x + 1)^n \log(x + 1) = \sum_{k=0}^{n} \binom{n}{k} (h_n - h_{n-k}) x^k + \sum_{k=n+1}^{\infty} \left\lfloor {n \atop k} \right\rfloor x^k$$

valid for $|x| \leq 1$, where the left side is equal to 0 for $x = -1$.

Taking $x = -1$ in this expression, we get the following beautiful summation (for $n > 0$)

$$\sum_{k=0}^{\infty} (-1)^k \left\lfloor {n \atop k} \right\rfloor = \frac{(-1)^{n+1}}{n}.$$

Box 5

4. AN EXPLICIT FORMULA FOR THE HARMONIC LOGARITHMS. Although the harmonic logarithms are ideally suited to differentiation and antidifferentiation, their expression in terms of powers of x and $\log x$ is not so simple.

Proposition 4.1. *The harmonic logarithms $\lambda_n^{(t)}(x)$ are given by the formula*

$$\lambda_n^{(t)}(x) = x^n \sum_{j=0}^{t} (-1)^j (t)_j c_n^{(j)} (\log x)^{t-j}$$

where $(t)_j = t(t-1)\cdots(t-j+1)$, $(t)_0 = 1$ and where the constants $c_n^{(j)}$ are uniquely determined by the initial conditions

$$c_n^{(0)} = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \quad \text{and} \quad c_0^{(j)} = \begin{cases} 1 & \text{for } j = 0 \\ 0 & \text{for } j \neq 0 \end{cases}$$

and the recurrence relation (for $j > 0$)

$$nc_n^{(j)} = c_n^{(j-1)} + [n]c_{n-1}^{(j)}. \qquad \blacksquare$$

The numbers $c_n^{(j)}$ are known as the *harmonic numbers*, and have some rather fascinating properties as shown, for example, in Boxes 6–8. Notice the intriguing pattern in the first few harmonic numbers of positive degree n (in Box 7). It is also interesting to contrast the asymptotic behavior of the harmonic logarithms of positive and negative orders (in Boxes 7 and 8).

Some values of the harmonic numbers $C_n^{(j)}$

$$\begin{array}{cc}
 & n = 0 \\
 & \downarrow \\
j = 0 \rightarrow & \left[\begin{matrix}
\cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\
\cdots & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & \frac{3}{2} & \frac{11}{6} & \frac{25}{12} & \frac{137}{60} & \cdots \\
\cdots & -\frac{137}{60} & -\frac{25}{12} & -\frac{11}{6} & -\frac{3}{2} & -1 & 0 & 0 & 1 & \frac{7}{4} & * & * & * & \cdots \\
\cdots & -\frac{15}{8} & -\frac{35}{24} & -1 & -\frac{1}{2} & 0 & 0 & 0 & 1 & \frac{15}{8} & * & * & * & \cdots \\
\cdots & -\frac{17}{24} & -\frac{5}{12} & -\frac{1}{6} & 0 & 0 & 0 & 0 & 1 & \frac{31}{16} & * & * & * & \cdots \\
\cdots & -\frac{3}{24} & -\frac{1}{24} & 0 & 0 & 0 & 0 & 0 & 1 & \frac{63}{32} & * & * & * & \cdots \\
\cdots & -\frac{1}{120} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{127}{64} & * & * & * & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix}\right]
\end{array}$$

Columns sum to n Columns approach n

Box 6

The harmonic numbers of positive degree $n > 0$

1) $c_n^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

$$c_n^{(2)} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)$$

$$c_n^{(3)} = 1 + \frac{1}{2}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right)\right] + \frac{1}{3}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right)\right] + \cdots$$

$$\cdots + \frac{1}{n}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2}\right) + \frac{1}{3}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \cdots + \frac{1}{n}\left(1 + \frac{1}{2} + \cdots \frac{1}{n}\right)\right].$$

2) In general, for $n > 0$ and $j > 0$, we have

$$c_n^{(j)} = \sum_{i=1}^{n} \frac{1}{i} c_i^{(j-1)}.$$

3) For $n > 0$,

$$c_n^{(j)} = \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} i^{-j}.$$

4) (Asymptotic behavior) For each $n > 0$, the sequence $c_n^{(j)}$ forms a nondecreasing sequence in j which is strictly increasing for $n > 1$. Furthermore, we have for each $n \geq 0$,

$$\lim_{j \to \infty} c_n^{(j)} = n.$$

Box 7

The harmonic numbers of negative degree $n < 0$
1) For $n < 0$, $$c_n^{(j)} = (-1)^j \lfloor n \rceil ! s(-n, j).$$ Where the numbers $s(n, j)$ are the famous Stirling numbers of the first kind, defined for all nonnegative integers n and j, by the condition $$x(x-1)\cdots(x-n+1) = \sum_{j=0}^{n} s(n, j) x^j.$$
2) (Asymptotic behavior) For each $n < 0$, we have $c_n^{(j)} = 0$ for $j > -n$, and so only a finite number of the $c_n^{(j)}$ are nonzero. Furthermore, their *sum* (not limit) is $$\sum_{j=0}^{\infty} c_n^{(j)} = \sum_{j=0}^{-n} c_n^{(j)} = n.$$

Box 8

5. CONCLUDING REMARKS. We have merely scratched the surface in the study of the algebra L and its differential operators. For example, the harmonic logarithms $\lambda_n^{(t)}(x)$ have a very special relationship with the derivative operator, spelled out in the definition of these functions. Loeb and Rota show that there are other, at least formal, functions that bear an analogous relationship to other operators, such as the forward difference operator Δ defined by $\Delta p(x) = p(x+1) - p(x)$. The functions associated with the operator Δ are denoted by $(x)_n^{(t)}$ and called the *logarithmic lower factorial functions*. In general, the sequences $p_n^{(t)}(x)$ associated with various operators can be characterized in several ways, for example as sequences of *logarithmic binomial type*, satisfying the identity

$$p_n^{(t)}(x+a) = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} p_k^{(0)}(a) p_{n-k}^{(t)}(x).$$

The properties of the Roman coefficients seem to indicate that they are a worthy generalization of the binomial coefficients. (This is not to suggest that there may not be other worthy generalizations.) As mentioned earlier, it would be a further confirmation of this fact to discover a nice combinatorial, or probabilistic, interpretation of these coefficients.

For further details on the matters discussed in this paper, with complete proofs, we refer the interested reader to reference 5.

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