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The Umbral Calculus

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1. INTRODUCTION

There are three known ways of describing a sequence of numbers a_0, a_1, a_2, a_3,...:

(1) By recursion. Here, a specific rule f is given whereby $a_n = f(a_{n-1}, a_{n-2}, \ldots)$. This description is used whenever the sequence is to be explicitly computed.

(2) By generating functions. Here, the description of the sequence is thrown back on that of the function

$$f(x) = a_0 + a_1x + a_2x^2/2! + \cdots.$$

This description has proved effective when the asymptotic properties of the sequence are sought.

(3) By transform methods. Here, the sequence is represented as the result of performing a definite integral, for example as a moment sequence, say

$$a_n = \int_0^1 x^n f(x)\, dx,$$

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and the properties of a_n are thrown back on "corresponding" properties of the function $f(x)$. Stripped of irrelevancies, this method reduces to representing the sequence a_n as the result of applying a linear functional L to the sequence of polynomials x^n. Adopting the physicists' notation, we write this action as $\langle L \mid x^n \rangle = a_n$.

In the nineteenth century—and among combinatorialists well into the twentieth—the linear functional L would be called an umbra, a term coined by Sylvester, that great inventor of unsuccessful terminology. Before knowledge of linear algebra became widespread, the action of a linear functional L would be conceived of as raising the index n to a power, and then "treating" the sequence a_n as a sequence of powers a^n, while reserving the right to lower the index at the proper time. No precise rules for lowering of indices were stated, nor could they be, as long as the underlying conceptual framework was missing. A baffling difficulty in the calculus of umbrae was the important "rule"

$$(a + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k a^{n-k},$$

which seemed to imply $a + a \neq 2a$.

If mathematicians had held back their tendency to disregard techniques, even though useful, that do not conform to the standards of rigor of the day, they might have been led, by an analysis of umbrae, to the concept of Hopf algebra. Unfortunately, this was a missed opportunity, and the concept was to emerge much later from algebraic topology. Briefly, it was recognized that linear functionals on polynomials can not only be added, but also multiplied according to the rule

$$\langle L_1 L_2 \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_1 \mid x^k \rangle \langle L_2 \mid x^{n-k} \rangle.$$

The resulting pairing of two rings leads to a powerful formalism, which it is the purpose of this work to develop.

A vast variety of special polynomial sequences occurs in combinatorics and in analysis. It was recognized in a previous work that these sequences of polynomials $p_n(x)$, which we have called of *binomial type*, satisfy the identity

$$p_n(x + a) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)\, p_{n-k}(x).$$

We show that these sequences can be defined by a simple orthogonalization device. They are related to a linear functional L such that $\langle L \mid 1 \rangle = 0$ by the biorthogonality conditions

$$\langle L^k \mid p_n(x) \rangle = n!\, \delta_{n,k}.$$

We develop the theory of sequences of binomial type starting from this condition. From the point of view of computation, the two most important problems are, first, that of effectively calculating a sequence of binomial type once L is given, and second, that of computing the *connection constants* $c_{n,k}$ between two sequences of binomial type $p_n(x)$ and $q_n(x)$:

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} p_k(x).$$

Our solution consists in describing the polynomial sequence $r_n(x) = \sum_k c_{n,k} x^k$ as a sequence of binomial type whose functional L is computed in a very simple way in terms of those of $p_n(x)$ and $q_n(x)$.

Polynomial sequences of binomial type turn out in large variety in problems of enumeration. Roughly speaking, problems of enumerating objects that are pieced together out of smaller objects which are not allowed to overlap—for example, the enumeration of trees—fall within the scope of the present theory. A sprinkling of examples given at the end is meant to foreshadow a more substantial development in this direction which we were forced to postpone.

The text has been supplemented by several examples from analysis which have occurred in various circumstances, mostly in connection with expansion of functions into series of polynomials, such as Taylor's, Newton's or Euler–MacLaurin's. That such expansions, as well as sundry other properties of special polynomial sequences, turn out to be special cases of a few exceedingly simple facts, is not only a pleasing realization. It is hoped that it will encourage the use of the simple general techniques of the umbral calculus, and discourage the collector's mentality that considers each polynomial sequence as an inviolable manifestation of a unique phenomenon.

Among the by-products of the present theory is an effective formalism for computation involving composition with formal power series and Lagrange inversion. A great deal of combinatorics depends on these computations, and the classical notation of the calculus offers little relief. A linear functional L on polynomials such that $\langle L \mid x^n \rangle = a_n$ corresponds to the formal power series whose nth coefficient is a_n, and this algebraic isomorphism leads to a swift technique for functional composition and inversion, as can be gleaned from the examples in Section 11.

Paradoxically, this identification of linear functionals with formal power series is one reason why a development along the present lines was overlooked. But it would be just as arbitrary to identify linear functionals with distributions, or with some yet-to-be-conceived gadget. The simplifying power of the present notation occurs out of the ease of handling adjoints of linear operators in the vector space duality between polynomials and functionals, and would be lost, had functionals been identified from the start with formal power series.

Another by-product of the present work is the theory of factor sequences, which allows for "polynomials" of negative degree, and which can be considered as an extension of the theory of factorial series to arbitrary sequences of binomial type. Thus we can define Hermite, Bernoulli, Stirling polynomials, etc., of negative degree. Whereas the generating functions of sequences of polynomials of binomial type, as well as the closely related Sheffer sequences, are expressed by exponentials, their analogs for factor sequences lead us to define an "integral" analog of the notion of formal power series, which we propose to call the Cigler transform, as it partially answers a question posed by J. Cigler.

Throughout, some definitions and elementary results could have been presented as special cases of Hopf algebra notions, but we have avoided this line, partly because Hopf algebras are still little known, and partly because it is left as a challenge to Hopf algebraists to generalize some of our notions, for example, factor sequences, the adjointness between shifts and derivations, and umbral composition, to their rarefied atmosphere.

A great many of the results in this work are new. Others are taken from our previous work on this subject. In the choice of examples, we have preferred to rely on established polynomial sequences rather than describe new sequences which could not be properly motivated. Altogether, this work may be compared to the archeologist's putting together of a dinosaur out of a few charred bones in the desert.

2. Survey

The notion of polynomial sequence of binomial type goes back to E. T. Bell and probably earlier. Steffensen was the first to observe that sequences associated with delta operators in the same way as D is to x^n are of binomial type, but failed to notice the converse of this fact, which was first stated and proved by Mullin and Rota.

The idea of associated and conjugate polynomials is first developed here. The history of the subject has been sketched in "Finite Operator Calculus."

The isomorphism between the umbral algebra and the algebra of shift-invariant operators, first seen by the Hopf algebraists, has not yet made much headway elsewhere. Thus Feller in his treatise on probability dedicates two separate chapters to Fourier transforms and to convolution operators, and correspondingly gives two proofs of the Central Limit Theorem, little realizing that they are really one and the same proof. The use of linear functionals and of the augmentation—that is, evaluation at zero—in place of operators results in substantial simplifications; computations of composition and inversion of power series become transparent in terms of the duality between the algebra of polynomials and the umbral algebra of linear functionals (Sections 6 and 11); the Lagrange inversion formula, for example, boils down to the computation of the adjoint of an operator.

It remains a mystery why so many polynomial sequences occurring in various mathematical circumstances turn out to be of binomial type. The explanation we give in terms of automorphisms of the umbral algebra can be recast in terms of the Weyl algebra in one pair of generators, that is, the associative algebra freely generated by two variables P and Q subject to the identity $PQ - QP = I$. Every sequence of binomial type determines a module over the Weyl algebra, and such modules are easily characterized.

The Weyl algebra approach is followed by J. Cigler in the study of factor sequences—the name is ours—but at considerable expense: in Cigler's approach the analog of Proposition 10.2 fails and as a consequence the computation of associated factor sequences becomes difficult and sometimes impossible to state.

The theory of factor sequences is barely scratched here, and it suggests the reopening of a number of questions in the calculus of finite differences which have lain dormant since Nörlund and Pincherle. The analogy between differential and difference equations, long considered a baffling coincidence, can now be seen as a special case of a theory of Q-difference equations, when Q is an arbitrary delta operator, each Q leading to its own theory of isolated singularities much as in the case of linear differential equations with rational coefficients. The purely algebraic connection between factor sequences and formal power series (Section 11) may be useful in developing a purely algebraic theory of singularities of Q-difference equations. For example, the analogy between $\log x$, "the" solution of $Dy = 1/x$, and $\psi(x)$, "the" solution of $\Delta y = 1/x$, leads more generally to the study of the Q-equations $Qy = 1/x$. Similarly, R. M. Cohn's difference algebra, conceived as a difference analog of Ritt's Galois theory for differential equations, is a good candidate for extension to delta operators.

Again, the combined use of polynomials and factor sequences does away with notions of convergence, or even of asymptotic approximation. It seems furthermore that the notion of "formal" definite integral, which we propose to call the Cigler transform, relates to those asymptotic expansions which arise from stationary phase.

We cannot pass under silence a conceptual problem arising from factor sequences. Every sequence of binomial type is the sequence of eigenfunctions of the operator $\theta_L \mu(L)$ in a suitable Hilbert space. What, then, is the spectral nature of those "eigenfunctions of negative order" that are the associated factor sequence? Does this phenomenon call for a retouching of the notion of eigenfunction expansion? Hilbert space considerations could also be called in to give, with the aid of the adjointness between umbral operators and automorphisms (Section 6), a simple solution of the problem of conjugacy of formal power series which has a good chance of extending to the multivariate case.

The sprinkling of examples is not meant to be exhaustive, and we were forced to defer some applications of umbral techniques, such as: a general understanding of Turán-type inequalities by sums of sequences (we give two examples), a goal toward which Al-Salam, Carlitz, Toscano, and others have contributed some dazzling spade work; a theory of "formal" partial fraction expansions; and a structural study of the Laguerre polynomials. These polynomials play a role in far too many questions, and their formal analogies with Hermite polynomials have not been satisfactorily explained. One can, for example, develop Feynman diagram representations of integrals of products of Laguerre polynomials, in analogy with Hermite. Does this mean that the Laguerre polynomials are associated with a yet-to-be-discovered stochastic process, as Hermite polynomials are to Brownian motion?

The combinatorial examples given in Section 14 are meant only as hints. A more systematic correspondence between operations on polynomial sequences of binomial type and set-theoretic operation on partitionals can and will be presented elsewhere. For example, umbral composition corresponds to a set-theoretic "composition" of two stores. Polynomial sequences with alternating, though still integer, coefficients can be interpreted by a sieve that expresses one store as resulting from the composition of two stores.

There is, however, a more promising set-theoretic interpretation of polynomial sequences of binomial type. Let $\mathbf{B}$ be a ring of subsets of a set S, that is, a family of subsets closed under unions, intersections, and relative complements. The *Poisson algebra* of $\mathbf{B}$ is the Boolean algebra $p(\mathbf{B})$ generated by elements denoted by (A, n), where $A \in \mathbf{B}$ and n is a nonnegative integer, subject to the identities $(A \cap B, n) = \bigcup_{i=0} ((A, i) \cap (B, n-i))$ for A and B disjoint, and $(A, n)^c = \bigcup_{i \neq n} (A, i)$. If μ is a measure on B, a signed measure π on $p(\mathbf{B})$ is said to be μ-invariant when $\pi((A, n)) = \pi((B, n))$ if $\mu(A) = \mu(B)$. It can be shown—subject to mild restrictions—that every μ-invariant measure on a Poisson algebra $p(\mathbf{B})$ is of the form $\pi((A, n)) = p_n(\mu(A)) \exp(\lambda\mu(A))$, when λ is a constant and $p_n(x)$ is a sequence of polynomials of binomial type. On the basis of this result, the umbral calculus can be systematically interpreted as a calculus of measures on Poisson algebras, generalizing compound Poisson processes. This interpretation in turn suggests a vast generalization of the umbral calculus, corresponding to measures on a Poisson algebra that are not assumed μ-invariant.

In addition to reiterating the acknowledgments given in "Finite Operator Calculus," we wish to express our indebtedness to the work of J. Cigler, A. Garsia, and especially J. Delsarte, whose pioneering contributions we have unpardonably failed to mention in previous works.

3. The Umbral Algebra

Let P denote the commutative algebra of all polynomials in a single variable x, with coefficients in a field K of characteristic zero, which we often assume to be either the real or the complex field. Let P^* be the vector space of all linear functionals on P. We denote the action of a linear functional L on a polynomial $p(x)$ by

$$\langle L \mid p(x) \rangle.$$

A *polynomial sequence* $p_n(x)$, $n = 0, 1, 2,\ldots$, is a sequence of polynomials where $p_n(x)$ is of degree n. It is clear that two linear functionals L and M are equal if and only if

$$\langle L \mid p_n(x) \rangle = \langle M \mid p_n(x) \rangle$$

for all $p_n(x)$ in a polynomial sequence. We will frequently use this argument, which we call *the spanning argument*. By the spanning argument, a linear functional L is defined once $\langle L \mid p_n(x) \rangle$ is given for all $p_n(x)$ in a polynomial sequence.

We make the vector space P^* into an algebra by defining the *product* of two linear functionals L and M by

$$\langle LM \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid x^k \rangle \langle M \mid x^{n-k} \rangle.$$

It is straightforward to verify

Proposition 3.1. *The product of linear functionals is commutative and associative.*

For a constant a, the linear functional ϵ_a, defined by

$$\langle \epsilon_a \mid p(x) \rangle = p(a),$$

is called *evaluation* at a. We write ϵ in place of ϵ_0, and call this linear functional the *augmentation*. It is easy to see that $\epsilon_a \epsilon_b = \epsilon_{a+b}$. Furthermore,

Proposition 3.2. *The augmentation is an identity for the product defined above.*

Thus the vector space of linear functionals P^*, with the above product and identity, is an algebra, which will be called the *umbral algebra*.

The umbral algebra is related to the algebra of functions of a real variable under convolution. Let f and g be functions with the property that

$$\int_{-\infty}^{\infty} f(x)\, x^n\, dx \qquad \text{and} \qquad \int_{-\infty}^{\infty} g(x)\, x^n\, dx$$

are defined for all integers $n \geqslant 0$. Define linear functionals L_f and L_g by

$$\langle L_f \mid p(x)\rangle = \int_{-\infty}^{\infty} f(x)\, p(x)\, dx,$$

$$\langle L_g \mid p(x)\rangle = \int_{-\infty}^{\infty} g(x)\, p(x)\, dx;$$

then the product $L_f L_g$ is the linear functional

$$\langle L_f L_g \mid p(x)\rangle = \int_{-\infty}^{\infty} h(x)\, p(x)\, dx,$$

where the function $h(x)$ is the convolution of the functions $f(x)$ and $g(x)$:

$$h(x) = \int_{-\infty}^{\infty} f(t)\, g(x-t)\, dt.$$

A major portion of the sequel is concerned with the study of a special class of polynomial sequences. A polynomial sequence $p_n(x)$ is said to be of *binomial type* if $p_0(x) = 1$ and if it satisfies the *binomial identity*,

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\, p_{n-k}(y)$$

for all n, x, and y. For example, the sequence $p_n(x) = x^n$ is of binomial type.

The binomial identity yields, by iteration, the *multinomial identity*:

$$p_n(x_1 + x_2 + \cdots + x_k) = \sum \binom{n}{j_1, \ldots, j_k} p_{j_1}(x_1) \cdots p_{j_k}(x_k),$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, \ldots, j_k)$ for which $j_1 + \cdots + j_k = n$.

The product of two linear functionals can be computed by using any sequences of binomial type in place of the sequence x^n. In particular,

PROPOSITION 3.3. *If $p_n(x)$ is a sequence of binomial type and if L and M are linear functionals, then*

$$\langle LM \mid p_n(x)\rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle \langle M \mid p_{n-k}(x)\rangle.$$

Proof. Let $P[x, y]$ be the vector space of polynomials in the variables x and y. A linear functional L on P defines a linear operator L_x on $P[x, y]$ as follows. If $p(x, y) = \sum_{i,j} a_{i,j} x^i y^j$, then

$$L_x p(x, y) = \sum_{i,j} a_{i,j} \langle L \mid x^i\rangle\, y^j.$$

Similarly, the linear operator L_y is defined by

$$L_y p(x, y) = \sum_{i,j} a_{i,j} x^i \langle L \mid x^j \rangle.$$

In this notation the identity defining the product of two linear functionals L and M becomes

$$\langle LM \mid x^n \rangle = L_x M_y (x + y)^n.$$

By the spanning argument, the same identity holds for any polynomial $p(x)$:

$$\langle LM \mid p(x) \rangle = L_x M_y p(x + y).$$

The conclusion now follows by setting $p(x) = p_n(x)$ and expanding the right side by the binomial identity.

One proves similarly, using the multinomial identity,

PROPOSITION 3.4. *If $p_n(x)$ is a sequence of binomial type, and if $L_1, L_2, \ldots, L_k$ are linear functionals, then*

$$\langle L_1 L_2 \cdots L_k \mid p_n(x) \rangle = \sum \binom{n}{j_1, \ldots, j_k} \langle L_1 \mid p_{j_1}(x) \rangle \cdots \langle L_k \mid p_{j_k}(x) \rangle, \qquad (*)$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, \ldots, j_k)$ for which $j_1 + \cdots + j_k = n$.

One of the key properties of the product of linear functionals is

PROPOSITION 3.5. *Let L be a linear functional such that $\langle L \mid 1 \rangle = \langle L \mid x \rangle = \cdots = \langle L \mid x^{m-1} \rangle = 0$. Then*

$$\langle L^k \mid x^n \rangle = 0 \qquad \textit{for} \quad n < km.$$

Moreover,

$$\langle L^k \mid x^{km} \rangle = \frac{(km)!}{(m!)^k} \langle L \mid x^m \rangle^k.$$

Proof. Identity $(*)$, with $p_n(x) = x^n$ and $L_i = L$ for all $i = 1, 2, \ldots, k$ gives

$$\langle L^k \mid x^n \rangle = \sum \binom{n}{j_1, \ldots, j_k} \langle L \mid x^{j_1} \rangle \cdots \langle L \mid x^{j_k} \rangle,$$

where the sum ranges over all k-tuples of nonnegative integers $(j_1, \ldots, j_k)$ with $j_1 + \cdots + j_k = n$. If $n < km$, then $j_1 + \cdots + j_k < km$ and each term in the identity has a factor of the form $\langle L \mid x^{j_i} \rangle$ with $j_i < m$, and therefore equals zero. This establishes the first assertion.

When $n = km$, the only possible nonzero term in the identity comes when $j_i = m$ for all $i = 1, 2, \ldots, k$. The second assertion follows.

A frequently used special case of the preceding proposition is

COROLLARY 1. *If L is a linear functional such that $\langle L \mid 1 \rangle = 0$, then $\langle L^k \mid p(x) \rangle = 0$ for $k > \deg p(x)$.*

The umbral algebra P^* is a topological algebra under the topology defined as follows. A sequence L_n of linear functionals converges to a linear functional L whenever, given a polynomial $p(x)$, there exists an index n_0, depending on $p(x)$, such that for all $n > n_0$,

$$\langle L_n \mid p(x) \rangle = \langle L \mid p(x) \rangle.$$

Equivalently, an infinite series $\sum_{n \geqslant 0} L_n$ of linear functionals converges if and only if, given a polynomial $p(x)$, there is an index n_0 such that, for $n > n_0$,

$$\langle L_n \mid p(x) \rangle = 0.$$

In other words, the series $\sum_{n \geqslant 0} L_n$ converges if and only if the sequence L_n converges to zero. Under this topology, P^* becomes a complete topological algebra.

PROPOSITION 3.6. *For a linear functional L, and for a sequence of constants a_k, the following are equivalent*:

(i) $\langle L \mid 1 \rangle = 0$,

(ii) *the sequence L^k converges to zero,*

(iii) *the series $\sum_{k=0}^{\infty} a_k L^k$ converges.*

Proof. The equivalence of (ii) and (iii) follows from the definition of convergence, as remarked above. To see that (i) and (ii) are equivalent, notice that, if $\langle L \mid 1 \rangle = 0$, then by Corollary 1 to Proposition 3.5, $\langle L^k \mid p(x) \rangle = 0$ whenever $k > \deg p(x)$. Thus L^k converges to zero. Conversely, if $\langle L \mid 1 \rangle \neq 0$, then $\langle L^n \mid 1 \rangle = \langle L \mid 1 \rangle^n \neq 0$ for all $n \geqslant 0$, and so L^n cannot converge to zero.

In the sequel, the umbral algebra is always understood as a topological algebra.

As a final remark, if $p_n(x)$ is a polynomial sequence, then a sequence of linear functionals L_k such that

$$\langle L_k \mid p_n(x) \rangle = \delta_{k,n}$$

is not a basis for the vector space P^*, but only a pseudobasis. That is, every linear functional L can be uniquely expressed as a convergent series

$$L = \sum_{k=0}^{\infty} a_k L_k ,$$

where $a_k = \langle L \mid p_k(x)\rangle$. For the condition $\langle L_k \mid p_n(x)\rangle = \delta_{k,n}$ assures convergence of the series and its convergence to L follows by the spanning argument.

4. Delta Functionals

A *delta functional* is a linear functional L with the property that $\langle L \mid 1\rangle = 0$ and $\langle L \mid x\rangle \neq 0$.

In this section we establish four main results. We show that to every delta functional one can associate two sequences of polynomials of binomial type. Using one of the sequences, we generalize Taylor's formula. We also establish an isomorphism between the topological algebra of linear functionals and the algebra of formal power series.

We begin by examining a classical special case. Consider the delta functional A, called the *generator*, defined by $\langle A \mid x^n\rangle = \delta_{n,1}$. For the sequence $p_n(x) = x^n$, Proposition 3.4 gives $\langle A^k \mid x^n\rangle = n!\, \delta_{n,k}$. In other words, the sequence x^n and the powers of the linear functional A form a *biorthogonal set*. This idea of biorthogonality will now be generalized.

A polynomial sequence $p_n(x)$ is the *associated sequence* for a delta functional L when

$$\langle L^k \mid p_n(x)\rangle = n!\, \delta_{n,k} \qquad (*)$$

for all integers n, $k \geqslant 0$ (we set $L^0 = \epsilon$). Proposition 3.4 gives

Lemma 1. *If $p_n(x)$ is a sequence of binomial type and L is a delta functional then*

$$\langle L^n \mid p_n(x)\rangle = n!\langle L \mid p_1(x)\rangle^n \neq 0.$$

This allows us to prove

Proposition 4.1. *Every delta functional has a unique associated sequence.*

Proof. Let $p_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$ be a polynomial sequence. We show that $(*)$ uniquely defines the coefficients $a_{n,k}$. For $k = n$, $(*)$ gives

$$n! = a_{n,n}\langle L^n \mid x^n\rangle$$

and in view of the lemma, this uniquely defines $a_{n,n}$. We now proceed by induction. Assuming that $a_{n,n}$, $a_{n,n-1}, \ldots, a_{n,n-i}$ have been defined, we show the same is true for $a_{n,n-i-1}$. By Proposition 3.5,

$$\left\langle L^{n-i-1} \,\middle|\, \sum_{k=0}^{n} a_{n,k} x^k \right\rangle = \sum_{k=n-i-1}^{n} a_{n,k} \langle L^{n-i-1} \mid x^k \rangle,$$

and this, together with (*), yields

$$a_{n,n-i-1} \langle L^{n-i-1} \mid x^{n-i-1} \rangle = n!\, \delta_{n,n-i-1} - \sum_{k=n-i}^{n} a_{n,k} \langle L^{n-i-1} \mid x^k \rangle.$$

Since $\langle L^{n-i-1} \mid x^{n-i-1} \rangle \neq 0$, $a_{n,n-i-1}$ is uniquely defined. Q.E.D.

Since $L^0 = \epsilon$, (*) implies that $p_0(x) = 1$ and $p_n(0) = 0$ for $n > 0$.

We wish to show that the associated sequence for a delta functional is a sequence of binomial type, and conversely. To this end, we derive the following generalization of Taylor's expansion:

THEOREM 1 (Expansion Theorem). *Let M be a linear functional and let L be a delta functional with associated sequence $p_n(x)$. Then*

$$M = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!} L^k.$$

Proof. The result follows from the spanning argument, noting that

$$\sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!} \langle L^k \mid p_n(x) \rangle = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x) \rangle}{k!}\, n!\, \delta_{n,k} = \langle M \mid p_n(x) \rangle.$$

The following uniqueness assertion is implicit in the preceding proof.

COROLLARY 1. *Let M be a linear functional and let L be a delta functional. Suppose that*

$$M = \sum_{k=0}^{\infty} a_k L^k$$

for a_k in K. Then $a_k = \langle M \mid p_k(x) \rangle / k!$, where $p_k(x)$ is the associated sequence for L.

The Expansion Theorem says that every linear functional is in the closure of the linear span of the sequence of powers of a delta functional L. Thus, if $\langle L^k \mid p(x) \rangle = 0$ for all $k \geqslant 0$, we have $\langle M \mid p(x) \rangle = 0$ for all linear functionals M. This implies that $p(x) = 0$. We will use this argument many times in the sequel.

We come now to a main result:

THEOREM 2. (a) *Every associated sequence is a sequence of binomial type.*

(b) *Every sequence of binomial type is an associated sequence.*

Proof. (a) Let $p_n(x)$ be the associated sequence for the delta functional L. For nonnegative integers i and j, the definition of associated sequence gives

$$\langle L^i L^j \mid p_n(x)\rangle = \sum_{k=0}^{\infty} \binom{n}{k} \langle L^i \mid p_k(x)\rangle\langle L^j \mid p_{n-k}(x)\rangle. \tag{$**$}$$

Now if M and N are linear functionals with expansions

$$M = \sum_{k=0}^{\infty} a_k L^k$$

and

$$N = \sum_{k=0}^{\infty} b_k L^k,$$

a continuity argument together with $(**)$ implies

$$\begin{aligned}
\langle MN \mid p_n(x)\rangle &= \Big\langle \sum_i a_i L^i \sum_j b_j L^j \mid p_n(x)\Big\rangle \\
&= \sum_{i,j} a_i b_j \langle L^i L^j \mid p_n(x)\rangle \\
&= \sum_{i,j} a_i b_j \sum_{k=0}^{n} \binom{n}{k} \langle L^i \mid p_k(x)\rangle\langle L^j \mid p_{n-k}(x)\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} \Big\langle \sum_i a_i L^i \mid p_k(x)\Big\rangle\Big\langle \sum_j b_j L^j \mid p_{n-k}(x)\Big\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} \langle M \mid p_k(x)\rangle\langle N \mid p_{n-k}(x)\rangle.
\end{aligned}$$

Letting $M = \epsilon_a$, $N = \epsilon_b$ and recalling that $\epsilon_a \epsilon_b = \epsilon_{a+b}$, we conclude that

$$p_n(a+b) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)\, p_{n-k}(b)$$

for all a and b, as desired.

(b) Let $p_n(x)$ be a sequence of binomial type. Define a sequence of linear functionals L_k by the biorthogonality conditions

$$\langle L_k \mid p_n(x)\rangle = n!\, \delta_{n,k}.$$

In particular, $\langle L_1 \mid 1\rangle = 0$ and $\langle L_1 \mid p_1(x)\rangle \neq 0$. Thus L_1 is a delta functional. The proof will be complete if we show that $L_i = L_1{}^i$, for $i \geqslant 0$, or equivalently, that $L_i L_j = L_{i+j}$ for $i, j \geqslant 0$. But this follows from the spanning argument since

$$\langle L_i L_j \mid p_n(x) \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_i \mid p_k(x) \rangle \langle L_j \mid p_{n-k}(x) \rangle$$

$$= \sum_{k=0}^{n} \binom{n}{k} k!\, \delta_{i,k} (n-k)!\, \delta_{j,n-k} = n!\, \delta_{n,i+j} = \langle L_{i+j} \mid p_n(x) \rangle$$

for all $n \geqslant 0$. Thus part (b) is proved.

Our first goal has been achieved, and we turn to further corollaries of the expansion theorem.

COROLLARY 2. *Let M and N be linear functionals, and let L be a delta functional. Suppose*

$$M = \sum_{k=0}^{\infty} a_k L^k, \qquad a_k \in K,$$

and

$$N = \sum_{k=0}^{\infty} b_k L^k, \qquad b_k \in K.$$

Then if

$$MN = \sum_{k=0}^{\infty} c_k L^k, \qquad c_k \in K,$$

we have

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}\,.$$

The preceding corollary leads to a simple criterion for invertibility of a linear functional:

COROLLARY 3. *A linear functional M is invertible in the umbral algebra if and only if $\langle M \mid 1 \rangle \neq 0$.*

Proof. In the notation of the preceding corollary, if $a_0 = \langle M \mid 1 \rangle \neq 0$, then setting $c_0 = 1$ and $c_k = 0$ for $k \geqslant 1$ we may solve successively for the coefficients b_k, and thereby determine the series expansion for a linear functional N, which is inverse to M. Conversely, if $\langle M \mid 1 \rangle = 0$, then M is not invertible since it has a nontrivial null space.

Setting $M = \epsilon_y$ in the Expansion Theorem, we find

COROLLARY 4. *If L is a delta functional with associated sequence $p_n(x)$, then*

$$\epsilon_y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} L^k.$$

Any polynomial is a linear combination of a finite number of $p_n(x)$. The coefficients of such a linear combination are given by

COROLLARY 5. *If $p_n(x)$ is the associated sequence for the delta functional L, and if $p(x)$ is a polynomial, then*

$$p(x) = \sum_{k\geqslant 0} \frac{\langle L^k \mid p(x)\rangle}{k!} p_k(x).$$

By Corollary 1 to Proposition 3.5, all but a finite number of terms in the above sum are zero.

We proceed now to the next main result. By virtue of the Expansion Theorem, given a delta functional L we may associate to every linear functional M a formal power series in a single variable. In fact, if

$$M = \sum_{k=0}^{\infty} a_k L^k$$

we associate to M the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k.$$

We call $f(t)$ the *L-indicator* of the linear functional M. When L is the generator A, we call $f(t)$ simply the *indicator* of M.

Recall that the algebra $\mathbf{F}$ of formal power series can be made into a topological algebra by stipulating that a sequence $f_n(t)$ converges whenever the sequence of coefficients of each power of t converges in the discrete topology of the field K; that is, whenever the sequence of coefficients is eventually constant. In this topology we can show

THEOREM 3. *Let L be a delta functional. Then the mapping ϕ which associates to every linear functional*

$$M = \sum_{k=0}^{\infty} a_k L^k, \qquad a_k \in K,$$

the formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k t^k$$

is a continuous isomorphism of the umbral algebra onto the algebra of formal power series.

Proof. The Expansion Theorem, together with Corollary 1, shows that ϕ is linear, one-to-one and onto. Corollary 2 shows that ϕ is an algebra homomorphism.

To prove that ϕ is continuous, suppose L has associated sequence $p_n(x)$, and suppose M_n is a sequence of linear functionals converging to the linear functional M. If

$$M_n = \sum_{k=0}^{\infty} \alpha_k^{(n)} L^k$$

and

$$M = \sum_{k=0}^{\infty} \alpha_k L^k$$

we must show that

$$\phi(M_n) = \sum_{k=0}^{\infty} \alpha_k^{(n)} t^k$$

converges to

$$\phi(M) = \sum_{k=0}^{\infty} \alpha_k t^k.$$

By definition of convergence in P^*, for any fixed $j \geqslant 0$, there is an n_0 such that $n > n_0$ implies $\langle M_n \mid p_j(x)\rangle = \langle M \mid p_j(x)\rangle$. In other words, $n > n_0$ implies $\alpha_j^{(n)} = \alpha_j$. But this is the definition of convergence in $\mathbf{F}$, and thus $\phi(M_n)$ converges to $\phi(M)$.

COROLLARY 1. *A linear functional M is a delta functional if and only if the L-indicator of M has zero constant term and nonzero linear term.*

COROLLARY 2. *A linear functional M is a delta functional if and only if, for every delta functional L, there exists an invertible functional N such that $M = LN$.*

The following property of delta functionals will be repeatedly used:

PROPOSITION 4.2. *Let L be a linear functional with $\langle L \mid 1\rangle = 0$. Then the powers of L, including $L^0 = \epsilon$, span the space P^* if and only if L is a delta functional.*

Proof. If L is a delta functional, the Expansion Theorem shows that the powers of L span P^*. Conversely, suppose the powers of L span P^*. If $\langle L \mid x\rangle = 0$, then $\langle L^k \mid x\rangle = 0$ for all $k \geqslant 0$. But since $\langle A \mid x\rangle \neq 0$, the generator A cannot lie in the span of L^k. Thus $\langle L \mid x\rangle \neq 0$.

We now turn to the final main result of this section, which is another one-to-one correspondence between delta functionals and sequences of binomial type. We begin with a characterization of the coefficients of sequences of binomial type. Its verification is straightforward.

PROPOSITION 4.3. *A polynomial sequence*

$$q_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

is of binomial type if and only if

$$\binom{i+j}{i} a_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} a_{k,i} a_{n-k,j} \qquad (***)$$

for all $n \geqslant 0$, *and for all* $i, j \geqslant 0$.

We define the *conjugate sequence* of a delta functional L as the polynomial sequence

$$q_n(x) = \sum_{k \geqslant 0} \frac{\langle L^k \mid x^n \rangle}{k!} x^k.$$

By Proposition 3.5, each $q_n(x)$ is a polynomial of degree n.

THEOREM 4. (a) *Every conjugate sequence is a sequence of binomial type.*

(b) *Every sequence of binomial type is a conjugate sequence.*

Proof. (a) It follows directly from the definition of product of linear functionals that the coefficients of the conjugate sequence satisfy (***), thus proving part (a).

(b) Given a sequence $q_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$ of binomial type, we define a sequence of linear functionals L_k by

$$\langle L_k \mid x^n \rangle = k!\, a_{n,k} .$$

Then $\langle L_1 \mid 1 \rangle = a_{1,0} = 0$ and $\langle L_1 \mid x \rangle = a_{1,1} \neq 0$ so L_1 is a delta functional. Moreover, since the $a_{n,k}$ satisfy (***), we infer that

$$\langle L_{i+j} \mid x^n \rangle = \sum_{k=0}^{n} \binom{n}{k} \langle L_i \mid x^k \rangle \langle L_j \mid x^{n-k} \rangle.$$

Therefore $L_{i+j} = L_i L_j$. This implies that $L_k = L_1{}^k$ and $q_n(x)$ is the conjugate sequence for L_1 , proving part (b).

Thus we see that a delta functional L is associated with two sequences of binomial type, its associated sequence $p_n(x)$ and its conjugate sequence $q_n(x)$. We will say that $q_n(x)$ is *reciprocal* to $p_n(x)$. Should $p_n(x) = q_n(x)$, as in the case $L = A$, the sequence $p_n(x)$ is called *self-reciprocal.*

Similarly, a sequence $p_n(x)$ of binomial type is associated with two delta functionals, namely, the functional L, for which $p_n(x)$ is the associated sequence and the functional $\tilde{L}$, for which $p_n(x)$ is the conjugate sequence. We will say that $\tilde{L}$ is *reciprocal* to L. Should $L = \tilde{L}$, the linear functional L is called *self-reciprocal.*

If $p_n(x)$ is a sequence of binomial type, and if L is the linear functional satisfying

$$\langle L \mid p_n(x)\rangle = \delta_{n,1}$$

for $n \geqslant 0$, then by the spanning argument, L is the delta functional whose associated sequence is $p_n(x)$. Thus

$$\langle L^k \mid p_n(x)\rangle = n!\, \delta_{n,k}\,.$$

We generalize this with:

PROPOSITION 4.4. *Let $p_n(x)$ be a sequence of binomial type. Let L be a delta functional and let M be an invertible linear functional. Then $p_n(x)$ is the associated sequence for LM^{-1} if and only if*

$$\langle L \mid p_n(x)\rangle = n\langle M \mid p_{n-1}(x)\rangle$$

for $n \geqslant 1$.

Proof. If $p_n(x)$ is the associated sequence for LM^{-1}, we have

$$\begin{aligned}
\langle L \mid p_n(x)\rangle &= \langle (LM^{-1})\, M \mid p_n(x)\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} \langle LM^{-1} \mid p_k(x)\rangle\langle M \mid p_{n-k}(x)\rangle \\
&= n\langle M \mid p_{n-1}(x)\rangle,
\end{aligned}$$

the last equality since $\langle LM^{-1} \mid p_k(x)\rangle = \delta_{k,1}$. Conversely, if $\langle L \mid p_n(x)\rangle = n\langle M \mid p_{n-1}(x)\rangle$, for $n \geqslant 1$, we have

$$\begin{aligned}
\langle LM^{-1} \mid p_n(x)\rangle &= \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle\langle M^{-1} \mid p_{n-k}(x)\rangle \\
&= \sum_{k=1}^{n} \binom{n}{k} k\langle M \mid p_{k-1}(x)\rangle\langle M^{-1} \mid p_{n-k}(x)\rangle \\
&= n\langle MM^{-1} \mid p_{n-1}(x)\rangle = n\langle \epsilon \mid p_{n-1}(x)\rangle \\
&= n\delta_{n,1} = \delta_{n,1}\,.
\end{aligned}$$

By the remark preceding the proposition, $p_n(x)$ is the associated sequence for LM^{-1}.

5. Examples

We begin a continuing discussion of some notable examples. We label each installment by the symbol $a.b$, where a is the example number and b is the installment number.

First we give examples of delta functionals, along with their associated sequences, indicators, and some applications of the Expansion Theorem. Derivation of the associated sequences is deferred to Section 8, and computation of the indicators, being straightforward, is omitted.

1.1. The sequence x^n is the associated sequence for the generator A, whose indicator is the formal power series t. The binomial identity is the binomial formula, and expansion of the evaluation ϵ_y in powers of the generator is Taylor's formula

$$\epsilon_y = \sum_{k\geqslant 0} \frac{y^k}{k!} A^k$$

since $\langle A^k \mid p(x)\rangle = p^{(k)}(0)$.

2.1. The *falling factorial sequence* $(x/a)_n$, where $(y)_n = y(y-1)\cdots(y-n+1)$ is the falling factorial, is the associated sequence for the *forward difference functional* $\epsilon_a - \epsilon$, whose indicator is the formal power series $e^{at} - 1$.

For $a = 1$, the binomial identity becomes

$$(x+y)_n = \sum_{k=0}^{n} \binom{n}{k} (x)_k (y)_{n-k} \,.$$

Expansion of the evaluation ϵ_y in terms of the forward difference functional gives Newton's expansion

$$\epsilon_y = \sum_{k\geqslant 0} \frac{(y/a)_k}{k!} (\epsilon_a - \epsilon)^k. \tag{$*$}$$

Using the expansion

$$(\epsilon_a - \epsilon)^k = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \epsilon_a{}^j$$

and the fact that $\epsilon_a{}^j = \epsilon_{aj}$, and applying the result to a polynomial $p(x)$ gives

$$p(y) = \sum_{k\geqslant 0} \binom{y/a}{k} \sum_{j=0}^{n} \binom{k}{j} (-1)^{k-j} p(aj).$$

By way of orientation, we derive one of the classical formulas for numerical differentiation. This results from the expansion of A in powers of $\epsilon_a - \epsilon$:

$$p'(0) = \langle A \mid p(x)\rangle = \sum_{k\geqslant 0} \frac{\langle A \mid (x/a)_k\rangle}{k!} \langle(\epsilon_a - \epsilon)^k \mid p(x)\rangle$$

$$= \sum_{k\geqslant 0} \frac{(-1)^{k+1}}{ak} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} p(aj).$$

3.1. The *rising factorial sequence* $\langle x/a\rangle_n$, where $\langle y\rangle_n = y(y+1)\cdots(y+n-1)$, is the associated sequence for the *backward difference functional* $\epsilon - \epsilon_{-a}$, whose indicator is the formal power series $1 - e^{-at}$. The identities are similar to those of the forward difference functional, and we mention only

$$\langle x + y\rangle_n = \sum_{k=0}^{n} \binom{n}{k} \langle x\rangle_k \langle y\rangle_{n-k}\,.$$

If L is any delta functional, the *Abelization* of L is the delta functional $\epsilon_a L$. The associated polynomials for $\epsilon_a L$ can be explicitly computed (Section 7) in terms of the associated polynomials for L. We give two examples.

4.1. The Abel polynomials $A_n(x, a) = x(x - an)^{n-1}$ are easily verified to be the associated polynomials for the *Abel functional* $\epsilon_a A$, where

$$\langle \epsilon_a A \mid p(x)\rangle = p'(a).$$

The indicator of the Abel functional is the series te^{at}.

Theorem 2 gives a proof of Abel's identity:

$$(x + y)(x + y - an)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} xy(x - ak)^{k-1} (y - a(n-k))^{n-k-1}.$$

Expansion of the evaluation ϵ_y in powers of $\epsilon_a A$ gives

$$\epsilon_y = \sum_{k\geqslant 0} \frac{y(y - ak)^{k-1}}{k!} A^k \epsilon_{ka}\,,$$

or

$$p(y) = \sum_{k\geqslant 0} \frac{y(y - ak)^{k-1}}{k!} p^{(k)}(ka),$$

and, when $p(y) = e^y$, we obtain the beautiful

$$e^y = \sum_{k\geqslant 0} \frac{y(y - ak)^{k-1}}{k!} e^{ka},$$

which is easily justified by a limiting process.

5.1. The *Gould polynomials*

$$G_n(x, a, b) = \frac{x}{x - an} \left(\frac{x - an}{b}\right)_n$$

are the associated polynomials for the delta functional $\epsilon_a(\epsilon_b - \epsilon)$, the *difference-Abel functional*, whose indicator is $e^{at}(e^{bt} - 1)$.

The binomial identity, resulting from Theorem 2, is Vandermonde convolution

$$\frac{x + y}{x + y - an} \binom{(x + y - an)/b}{n}$$
$$= \sum_{k=0}^{n} \frac{x}{x - ak} \frac{y}{y - a(n - k)} \binom{(x - ak)/b}{k} \binom{(y - a(n - k))/b}{n - k}.$$

Corollary 5 to Theorem 1 gives the interesting expansion

$$p(y) = \sum_{k\geqslant 0} \frac{y}{y - ak} \binom{(y - ak)/b}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} p(ak + bj).$$

6.1. The *central difference functional* $\delta_a = \epsilon_{a/2} - \epsilon_{-a/2}$, whose indicator is the series $e^{at/2} - e^{-at/2} = 2 \sinh at/2$, is a special case of the preceding example. For $a = 1$, the associated sequence

$$x^{[n]} = x(x + n/2 - 1)_{n-1}$$

gives the *Steffensen polynomials*.

By expanding a polynomial $p(x)$ in terms of the Steffensen polynomials (Corollary 5 to Theorem 1), we obtain the interpolation formula

$$p(y) = \sum_{k\geqslant 0} \frac{y}{y + k/2} \binom{y + k/2}{k} \langle \delta^k \mid p(x) \rangle.$$

7.1. The (basic) Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n - 1}{k - 1} (-x)^k$$

are the associated polynomials for the *Laguerre functional*

$$\langle l \mid p(x) \rangle = \int_{-\infty}^{0} e^t p'(t)\, dt.$$

From $\langle l \mid x^n \rangle = -n!$, we infer that the indicator of l is the formal power series $t/(t - 1)$.

Expanding the polynomial $p_n(x) = x^n$ in terms of the Laguerre polynomials gives the remarkable

$$y^n = \sum_{k\geqslant 0} (-1)^k \frac{n!}{k!} \binom{n-1}{k-1} L_k(y).$$

We now use Proposition 3.4 to derive some identities. Taking, for example, all $L_i = \epsilon_a - \epsilon$, the forward difference functional, and $p_n(x)$ any sequence of binomial type, we find

$$\langle(\epsilon_a - \epsilon)^k \mid p_n(x)\rangle = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} p_{i_1}(a) \cdots p_{i_k}(a).$$

Expanding $(\epsilon_a - \epsilon)^k$ by the binomial theorem

$$(\epsilon_a - \epsilon)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \epsilon_{ia}$$

gives the identity

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} p_n(ia) = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} p_{i_1}(a) \cdots p_{i_k}(a).$$

For $p_n(x) = x^n$, this specializes to

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^n = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k}.$$

The right side counts the number of ways of placing n balls into k boxes, with no box empty. It thus equals $k!\, S(n, k)$, where $S(n, k)$ are the Stirling numbers of the second kind.

Setting $p_n(x) = (x/b)_n$, the falling factorial sequence, and then replacing a/b by r, we obtain the binomial identity:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \binom{ir}{n} = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{r}{i_1} \cdots \binom{r}{i_k}$$

valid for all $r > 0$. With $p_n(x) = \langle x/b\rangle_n$, the rising factorial sequence, a similar identity is obtained where the multiset coefficients replace the binomial coefficients.

The difference–Abel functional $\epsilon_a(\epsilon_b - \epsilon)$ gives other remarkable identities by the same use of Proposition 3.4. For example,

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} p_n(ak + bi)$$

$$= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1, \ldots, i_k} [p_{i_1}(a+b) - p_{i_1}(a)] \cdots [p_{i_k}(a+b) - p_{i_k}(a)]$$

for any sequence $p_n(x)$ of binomial type. In particular, for $p_n(x) = x^n$ with $a + b = -1$ and $a = 1$:

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (k - 2i)^n$$

$$= \sum_{i_1+\cdots+i_k=n} \binom{n}{i_1, \ldots, i_k} [(-1)^{i_1} - 1] \cdots [(-1)^{i_k} - 1]$$

$$= (-2)^k \sum_{\substack{i_1+\cdots+i_k=n \\ i_j \text{odd}}} \binom{n}{i_1, \ldots, i_k}.$$

Except for the factor $(-2)^k$, the right side counts the number of ways of placing n balls into k boxes, subject to the condition that each box contain an odd number of balls.

We next consider some examples of conjugate sequences.

1.2. The conjugate sequence for the generator A is clearly the sequence x^n.

2.2. The conjugate sequence for the forward difference functional $\epsilon_a - \epsilon$ can be obtained from $\langle (\epsilon_a - \epsilon)^k \mid x^n \rangle = a^k k!\, S(n, k)$, where $S(n, k)$ are the Stirling numbers of the second kind. In fact, for $a = 1$:

$$\phi_n(x) = \sum_{k=0}^{n} S(n, k)\, x^k.$$

These are the exponential polynomials.

3.2. The backward difference functional gives a variant of the exponential polynomials, namely, for $a = 1$:

$$q_n(x) = \sum_{k=0}^{n} (-1)^{n-k} S(n, k)\, x^k.$$

4.2. The conjugate sequence for the Abel functional $\epsilon_a A$ is easily computed to be

$$\mu_n(x, a) = \sum_{k=0}^{n} \binom{n}{k} (ak)^{n-k} x^k.$$

5.2. The conjugate sequence for the difference–Abel functional $\epsilon_a(\epsilon_b - \epsilon)$ has not occurred in the literature. It is:

$$g_n(x, a, b) = \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} (ak)^i\, b^{n-i} S(n-i, k)\, x^k.$$

We call these the *conjugate Gould polynomials.*

6.2. The conjugate polynomials for the central difference functional δ_a are found by the same methods to be the *Carlitz–Riordan polynomials*

$$K_n(x) = a^n \sum_{k=0}^{n} \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-k-i}\, k^i 2^{n-i} S(n-i, k)\, x^k.$$

7.2. For the Laguerre functional l, we find

$$\begin{aligned}\langle l^k \mid x^n \rangle &= \sum_{\substack{i_1+\cdots+i_k=n \\ i_j>0}} \binom{n}{i_1, \ldots, i_k} (-1)^k\, i_1! \cdots i_k! \\ &= (-1)^k\, n! \binom{n-1}{k-1}.\end{aligned}$$

Thus the conjugate sequence of the Laguerre functional is the same as the associated sequence, namely, the basic Laguerre polynomials. The explanation of this remarkable fact is given in the sequel.

8.2. *The Bell polynomials.* For the first time we require a field other than the real or complex field. Let k be a field of characteristic zero, to which a sequence of independent transcendentals $x_1, x_2, \ldots$ has been adjoined. Over this field, define the *generic delta functional* L by

$$\begin{aligned}\langle L \mid x^n \rangle &= x_n \qquad \text{for} \quad n \geqslant 1, \\ \langle L \mid 1 \rangle &= 0.\end{aligned}$$

The conjugate polynomials for the generic delta functional are the Bell polynomials. An explicit formula for the coefficients is obtained from Proposition 3.4:

$$B_{n,k} = B_{n,k}(x_1, x_2, \ldots) = \frac{\langle L^k \mid x^n \rangle}{k!} = \sum \frac{n!}{c_1!\, c_2! \cdots} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots,$$

where the sum ranges over all nonnegative integers $c_1, c_2, \ldots$ satisfying $c_1 + 2c_2 + \cdots = n$ and $c_1 + c_2 + \cdots = k$.

For the Bell polynomials, we use the notation

$$b_n(x; x_1, x_2, \ldots) = \sum_{k=0}^{n} B_{n,k} x^k.$$

All known identities for the Bell coefficients $B_{n,k}$ follow from the multiplication rules for delta functionals. We give a sampling:

$$\text{(a)}\quad kB_{n,k} = \sum_{j=k-1}^{n-1} \binom{n}{j} x_{n-j} B_{j,k-1},$$

rewritten in the present notation, becomes the trivial

$$\frac{\langle L^k \mid x^n \rangle}{(k-1)!} = \sum_{j=k-1}^{n-1} \binom{n}{j} \langle L \mid x^{n-j} \rangle \frac{\langle L^{k-1} \mid x^j \rangle}{(k-1)!}.$$

(b) Let the delta functional L_1 be defined by $\langle L_1 \mid x^n \rangle = x_{n+1}/(n+1)$, for $n \geqslant 1$ and $\langle L_1 \mid 1 \rangle = 0$. Then $L = L_1 A + x_1 A$. The conjugate sequence for L_1 is $b_n(x; x_2/2, x_3/3, \ldots)$. An identity relating this polynomial sequence to the Bell polynomials is derived as follows. We apply

$$L^k = \sum_{j=0}^{k} \binom{k}{j} x_1^{k-j} L_1{}^j A^k$$

to the polynomial x^n and simplify:

$$\langle L^k \mid x^n \rangle = \sum_{j=0}^{k} \binom{k}{j} x_1^{k-j} (n)_k \langle L_1{}^j \mid x^{n-k} \rangle.$$

Hence

$$B_{n,k}(x_1, x_2, \ldots) = \sum_{j=0}^{k} \frac{n!}{(n-k)!\,(k-j)!} x_1^{k-j} B_{n-k,j}(x_2/2, x_3/3, \ldots).$$

Similar identities can be obtained with the unique delta functional L_i such that $L = x_1 A + x_2 A^2/2! + \cdots + x_{i-1} A^{i-1}/(i-1)! + A^i L$.

(c) Consider now the field k with additional independent transcendentals $y_1, y_2, \ldots$ adjointed. The conjugate sequence of the delta functional L' given by $\langle L' \mid x^n \rangle = x_n + y_n$ is $b_n(x; x_1 + y_1, x_2 + y_2, \ldots)$. Setting $\langle L'' \mid x^n \rangle = y_n$, so that $L' = L + L''$, one obtains

$$\langle (L')^k \mid x^n \rangle = \sum_{j=0}^{k} \binom{k}{j} \langle L^j \mid x^n \rangle \langle (L'')^{k-j} \mid x^n \rangle,$$

whence we obtain

$$B_{n,k}(x_1 + y_1, x_2 + y_2, \ldots) = \sum_{j=0}^{k} B_{n,j}(x_1, x_2, \ldots)\, B_{n,k-j}(y_1, y_2, \ldots).$$

(d) From Proposition 3.4, one easily obtains

$$B_{n,k}(0, 0, \ldots, x_j, 0, \ldots) = 0,$$

unless $n = jk$, and

$$B_{jk,k} = \frac{(jk)!}{k!\,(j!)^k}\, x_j\,.$$

(e) Every delta functional can be obtained from the Bell generic delta functional by specializing the values of the x_i. Thus every formula for the Bell polynomials gives a formula for all conjugate sequence. For example, from (b) one obtains

$$\langle L^k \mid x^n\rangle = \sum_{j=0}^{k} \binom{k}{j} (n)_k \langle L \mid x\rangle^{k-j} \langle L_1{}^j \mid x^{n-k}\rangle,$$

where L is any delta functional and where $L = L_1 A + x_1 A$. Similarly, (c) gives the conjugate polynomials of the sum of two (or more) delta functionals in terms of the conjugate sequences of the summand.

6. Automorphisms and Derivations

Given two polynomial sequences $p_n(x)$ and $q_n(x)$, a frequently encountered problem is that of determining a matrix of constants $c_{n,k}$, which we call the *connection constants* of $p_n(x)$ with $q_n(x)$, such that

$$q_n(x) = \sum_{k=0}^{n} c_{n,k} p_k(x). \tag{$*$}$$

In this section, we give a solution to this problem when the polynomial sequences are of binomial type. The solution we propose takes a particularly simple form in the *umbral notation* we now introduce. If $r(x) = \sum_{k=0}^{n} c_k x^k$ is a polynomial, and $p_n(x)$ is a polynomial sequence, the *umbral composition* of $r(x)$ with $p_n(x)$ is the polynomial, written $r(\mathbf{p}(x))$, and defined by

$$r(\mathbf{p}(x)) = \sum_{k=0}^{n} c_k p_k(x).$$

If $r_n(x)$ and $p_n(x)$ are two polynomial sequences, the umbral composition of $r_n(x)$ with $p_n(x)$ is the polynomial sequence $r_n(\mathbf{p}(x))$. In this notation, $(*)$ becomes

$$q_n(x) = r_n(\mathbf{p}(x)),$$

where $r_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$.

Umbral composition is simply the result of applying a suitable linear operator to a polynomial sequence. In particular, if α is the linear operator on P defined by $\alpha x^n = p_n(x)$ for $n = 0, 1, 2, \ldots$, then $\alpha r_n(x) = r_n(\mathbf{p}(x))$, and $(*)$ becomes

$$q_n(x) = \alpha r_n(x).$$

Thus the constants $c_{n,k}$ are determined once the polynomials $r_n(x) = \alpha^{-1}q_n(x)$ are known.

We are therefore led to define an ***umbral operator*** as a linear operator α on P, given by $\alpha x^n = p_n(x)$, where $p_n(x)$ is a sequence of binomial type. When we wish to emphasize the delta functional L for which $p_n(x)$ is the associated sequence, we write α_L for α.

Before proceeding further, we recall some basic facts about adjoints of linear operators. Let T be a linear operator mapping P into itself. The adjoint T^* of T is the operator mapping P^* into itself uniquely defined by

$$\langle T^*(L) \mid p(x)\rangle = \langle L \mid Tp(x)\rangle$$

for all $L \in P^*$ and all $p(x) \in P$. The adjoint T^* of a linear operator T on P exists and is continuous. To see the latter, suppose L_n is a sequence of linear functionals converging to L. For any polynomial $p(x)$, we have

$$\langle T^*(L_n) \mid p(x)\rangle = \langle L_n \mid Tp(x)\rangle,$$

and by the definition of convergence in P^*, if n is large, this equals

$$\langle L \mid Tp(x)\rangle = \langle T^*(L) \mid p(x)\rangle.$$

Thus $T^*(L_n)$ converges to $T^*(L)$, and T^* is continuous.

On the other hand, suppose U is a linear operator mapping P^* into itself. Then the adjoint U^* maps P^{**} into itself. Thinking of P as a subspace of P^{**}, in general U^* will not map P into itself. The sufficient condition to ensure that U^* maps polynomials to polynomials is the continuity of U. We have

PROPOSITION 6.1. *A linear operator mapping P^* into itself is the adjoint of a linear operator mapping P into itself if and only if it is continuous.*

Proof. We have already seen that the adjoint of an operator mapping P into itself is continuous. For the converse, suppose U is a continuous operator mapping P^* into itself. Since the sequence of powers A^k converges to zero, so does the sequence $U(A^k)$. Thus the function

$$p_n(x) = \sum_{k=0}^{\infty} \frac{\langle U(A^k) \mid x^n\rangle}{k!} x^k$$

is a polynomial, and

$$\langle A^k \mid p_n(x)\rangle = \langle U(A^k) \mid x^n\rangle$$

for all $k \geqslant 0$.

If we define the operator V mapping P into itself by $Vx^n = p_n(x)$, then

$$\begin{aligned}\langle V^*(L) \mid x^n \rangle &= \langle L \mid p_n(x) \rangle \\ &= \langle U(L) \mid x^n \rangle, \qquad \text{for all } L \in P^*,\end{aligned}$$

the last equality by the spanning argument for linear functionals. Thus $V^*(L) = U(L)$ for all $L \in P^*$ and so $V^* = U$.

We return now to the main stream of this section. The *shift* of a polynomial sequence $p_n(x)$ is the operator θ, mapping P into itself, defined by $\theta p_n(x) = p_{n+1}(x)$. If $p_n(x)$ is of binomial type, we say that θ is an *umbral shift*. By θ_L, we mean the umbral shift defined by the associated sequence for L.

Umbral operators and umbral shifts are related to automorphisms and derivations of the umbral algebra. Recall that a derivation ∂ of the umbral algebra is a linear operator such that $\partial(LM) = (\partial L)M + L(\partial M)$.

In order to exhibit the aforementioned relationship, we require two lemmas.

LEMMA 1. *Any continuous automorphism of the umbral algebra maps delta functionals to delta functionals.*

Proof. Let β be a continuous automorphism of P^*, and let L be a delta functional. By Proposition 3.6, $\langle L \mid 1 \rangle = 0$ implies L^n converges to zero. The continuity of β implies that $\beta(L^n) = \beta(L)^n$ converges to zero, and another application of Proposition 3.6 implies $\langle \beta(L) \mid 1 \rangle = 0$. By Proposition 4.2, the powers of L span P^*, and thus so do the powers of $\beta(L)$. The same proposition implies $\beta(L)$ is a delta functional.

LEMMA 2. *Let ∂ be a derivation of the umbral algebra which is everywhere defined, continuous, and onto. Then there is a delta functional L such that $\partial L = \epsilon$.*

Proof. Since ∂ is onto, there is a linear functional L for which $\partial L = \epsilon$. Since ∂ is a derivation, we infer that $\partial \epsilon = 0$, hence, subtracting from L a constant if necessary, we may assume that $\langle L \mid 1 \rangle = 0$. Now we expand L into a series of powers of the generator $a_1 A + a_2 A^2 + \cdots$, and since ∂ is continuous, we may apply it term by term to the series. Since $\partial A^n = nA^{n-1}\,\partial A$, we have

$$\epsilon = (a_1 + 2a_2 A + \cdots)\,\partial A.$$

Thus the series $(a_1 + 2a_2 A + \cdots)$ is invertible and so $a_1 \neq 0$. That is, $\langle L \mid x \rangle \neq 0$ and L is a delta functional.

We are now ready to prove

THEOREM 5. (a) *An operator α of P onto itself is an umbral operator if and only if its adjoint α^* is a continuous automorphism of the umbral algebra.*

(b) *An operator θ of P into itself is an umbral shift if and only if its adjoint θ^* is a continuous, everywhere defined derivation of the umbral algebra onto itself.*

Proof. (a) It is clear that the adjoint α^* of an umbral operator α is linear, continuous, one-to-one, and onto. Thus all that remains is to show that α^* preserves multiplication. Letting $\alpha x^n = p_n(x)$, this follows from the spanning argument and the following calculations:

$$\begin{aligned}\langle \alpha^*(MN) \mid x^n \rangle &= \langle MN \mid \alpha x^n \rangle = \langle MN \mid p_n(x) \rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle M \mid p_k(x) \rangle \langle N \mid p_{n-k}(x) \rangle \\ &= \sum_{k=0}^{n} \binom{n}{k} \langle \alpha^*(M) \mid x^k \rangle \langle \alpha^*(N) \mid x^{n-k} \rangle \\ &= \langle \alpha^*(M)\, \alpha^*(N) \mid x^n \rangle.\end{aligned}$$

For the converse, suppose β is a continuous automorphism of the umbral algebra. In view of Lemma 1, we may let $p_n(x)$ be the associated sequence for the delta functional $L = \beta^{-1}(A)$. Defining the umbral operator α by $\alpha x^n = p_n(x)$, we have

$$\langle \beta(L)^k \mid x^n \rangle = n!\, \delta_{n,k} = \langle L^k \mid \alpha x^n \rangle,$$

for all k and n. By the Expansion Theorem, the same identity holds for all linear functionals M,

$$\langle \beta(M) \mid x^n \rangle = \langle M \mid \alpha x^n \rangle,$$

and thus by the spanning argument, $\beta(M) = \alpha^*(M)$. Hence part (a) is proved.

(b) Let θ be the umbral shift defined by $\theta p_n(x) = p_{n+1}(x)$, and suppose $p_n(x)$ is the associated sequence for L. We have seen that the adjoint of a linear operator on P is continuous. Moreover,

$$\langle L^k \mid p_{n+1}(x) \rangle = k \langle L^{k-1} \mid p_n(x) \rangle$$

and thus

$$\langle \theta^*(L^k) \mid p_n(x) \rangle = k \langle L^{k-1} \mid p_n(x) \rangle,$$

for all $n, k \geqslant 0$. Therefore, by the spanning argument $\theta^*(L^k) = kL^{k-1}$ and so θ^* is an everywhere defined derivation, and is onto.

Conversely, let ∂ be a continuous derivation of the umbral algebra onto itself. By virtue of Lemma 2, we may let $p_n(x)$ be the associated sequence for the delta functional L, with $\partial(L) = \epsilon$. Then for $k \geqslant 0$, we have

$$\begin{aligned}\langle L^k \mid \partial^* p_n(x) \rangle &= \langle \partial(L^k) \mid p_n(x) \rangle \\ &= k \langle L^{k-1} \mid p_n(x) \rangle = k(k-1)!\, \delta_{k-1,n} \\ &= k!\, \delta_{k,n+1} \,.\end{aligned}$$

By the uniqueness of the associated sequence, it follows that $\partial^* p_n(x) = p_{n+1}(x)$, and part (b) is proved.

Every continuous automorphism β of the umbral algebra is thus associated with a unique delta functional L, namely, the delta functional whose associated polynomials are $p_n(x) = \beta^* x^n$. Similarly, every continuous, everywhere defined derivation β of the umbral algebra onto itself is associated with a unique delta functional L, the one for which ∂^* is the umbral shift of the associated sequence. We shall stress this association by writing $\beta = \beta_L$ and $\partial = \partial_L$. We remark that $\beta_L(L) = A$ and $\partial_L(L) = \epsilon$.

As an example, the simplest umbral operator is the substitution $x^n \to a^n x^n$, for $a \in K$. Its adjoint maps A^k to $a^k A^k$. The simplest shift is the map θ_A: $x^n \to x^{n+1}$, and its adjoint ∂_A is

$$\langle \partial_A L \mid p(x) \rangle = \langle L \mid xp(x) \rangle.$$

We proceed to develop some corollaries of Theorem 5.

COROLLARY 1. (a) *An umbral operator maps sequences of binomial type to sequences of binomial type.*

(b) *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, and if α is an operator defined by $\alpha p_n(x) = q_n(x)$, then α is an umbral operator.*

(c) *If $p_n(x)$ is the associated sequence for L and $q_n(x)$ is the associated sequence for M, then the adjoint of the umbral operator $\alpha p_n(x) = q_n(x)$ satisfies $\alpha^*(M) = L$.*

(d) *If ∂_L is the derivation associated with the delta functional L, then $\partial_L M = 0$ if and only if $M = a\epsilon$, for some $a \in K$.*

Proof. (a) Suppose α is an umbral operator, and $q_n(x)$ is a sequence of binomial type, with associated delta functional M. Then

$$\begin{aligned}\langle (\alpha^{-1})^*(M)^k \mid \alpha q_n(x) \rangle &= \langle M^k \mid \alpha^{-1} \alpha q_n(x) \rangle \\ &= \langle M^k \mid q_n(x) \rangle = k!\, \delta_{n,k}\,.\end{aligned}$$

Thus $\alpha q_n(x)$ is the associated sequence for the delta functional $(\alpha^{-1})^* M$, and is therefore of binomial type.

(b) A slight modification of the calculations in the proof of Theorem 5 will show that, if $\alpha p_n(x) = q_n(x)$, then α^* is a continuous automorphism of the umbral algebra, and thus α is an umbral operator.

(c) This follows by noticing that

$$\begin{aligned}\langle \alpha^*(M) \mid p_n(x) \rangle &= \langle M \mid \alpha p_n(x) \rangle \\ &= \langle M \mid q_n(x) \rangle = \delta_{n,1} = \langle L \mid p_n(x) \rangle\end{aligned}$$

for all $n \geqslant 0$.

(d) Clearly, $\partial_L(a\epsilon) = 0$. The converse follows by observing that, for $p_n(x)$ the associated sequence for L, $0 = \langle \partial_L M \mid p_n(x) \rangle = \langle M \mid p_{n+1}(x) \rangle$, and thus $M = a\epsilon$ for some $a \in K$.

Part (a) of the preceding corollary implies that the composition of two umbral operators is an umbral operator. This allows us to define a group operation on delta functionals, which we call *composition*, as follows. If L and M are two delta functionals with associated umbral operators α_L and α_M, the composition $L \circ M$ is the delta functional associated with the umbral operator $\alpha_L \circ \alpha_M$.

PROPOSITION 6.2. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for L and M, respectively, then $q_n(\mathbf{p}(x))$ is of binomial type, being the associated sequence for $L \circ M$.*

Proof. Since $\alpha_L\colon x^n \to p_n(x)$ and $\alpha_M\colon x^n \to q_n(x)$, it follows that $\alpha_L \circ \alpha_M\colon x^n \to q_n(\mathbf{p}(x))$. Since $\alpha_L \circ \alpha_M$ is an umbral operator, $q_n(\mathbf{p}(x))$ is of binomial type, and is the associated sequence of $L \circ M$ by definition of composition of delta functionals.

Since the umbral operator α_A is the identity, the generator A is the identity under composition of delta functionals, and thus $L \circ A = A \circ L = L$ for all delta functionals L.

Recall that we defined the delta functional M to be reciprocal to the delta functional L whenever the associated sequence for L is the conjugate sequence for M.

PROPOSITION 6.3. *A delta functional M is reciprocal to a delta functional L if and only if $L \circ M = A$.*

Proof. Suppose M is reciprocal to L, and let L have associated sequence $p_n(x)$. Then since $p_n(x)$ is the conjugate sequence for M, we have

$$p_n(x) = \sum_{k=0}^{n} \frac{\langle M^k \mid x^n \rangle}{k!} x^k$$

and, by the spanning argument, for any polynomial $q(x)$,

$$q(\mathbf{p}(x)) = \sum_{k=0}^{\infty} \frac{\langle M^k \mid q(x) \rangle}{k!} x^k.$$

If we take $q(x) = q_n(x)$, an associated polynomial for M, we find

$$q_n(\mathbf{p}(x)) = x^n.$$

Therefore, $L \circ M = A$ by Proposition 6.2. The converse is obvious.

We remark that if $\alpha_L^{-1} = \alpha_M$ then $\alpha_M \circ \alpha_L = I$ and $M \circ L = A$. Thus by the previous proposition $M = \tilde{L}$, and therefore $\alpha_L^{-1} = \alpha_{\tilde{L}}$.

We are now able to give the connection constants for sequences of binomial type.

PROPOSITION 6.4. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for L and M, respectively, and if*

$$q_n(x) = r_n(\mathbf{p}(x))$$

for a polynomial sequence $r_n(x)$, then $r_n(x)$ is of binomial type, and is the associated sequence for the delta functional $\tilde{L} \circ M$.

Proof. The proof is immediate from Proposition 6.2.

We now interpret the composition of delta functionals in terms of their indicators. Recall that, if $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ is any formal power series in $\mathbf{F}$, and $g(t)$ is any formal power series with zero constant term, then the series

$$f(g(t)) = a_0 + a_1 g(t) + a_2 (g(t))^2 + \cdots,$$

called the composition of $f(t)$ with $g(t)$, converges in the topology of $\mathbf{F}$. In particular, if $g(t)$ is any formal power series whose constant term is zero and whose linear term is nonzero, then there exists a unique formal power series $g^{-1}(t)$, called the inverse of $g(t)$, with the property that $g(g^{-1}(t)) = g^{-1}(g(t)) = t$.

Finally, recall that every formal power series $f(t) = a_0 + a_1 t + a_2 t^2 + \cdots$ has a derivative $f'(t)$, obtained by termwise differentiation; that is, $f'(t) = a_1 + 2a_2 t + 3a_3 t^2 + \cdots$.

THEOREM 6. *Let L and M be delta functionals, with indicators $f(t)$ and $g(t)$, respectively. Then the composition $L \circ M$ is a delta functional with indicator $g(f(t))$.*

Proof. Writing β_L, β_M, and $\beta_{L \circ M} = \beta_M \circ \beta_L$ for the automorphisms of P^* associated with the delta functionals L, M and $L \circ M$, respectively, we have $\beta_L f(A) = A$, $\beta_M g(A) = A$ and $\beta_{L \circ M}(L \circ M) = A$. Thus $L \circ M = \beta_{L \circ M}^{-1}(A) = (\beta_M \circ \beta_L)^{-1}(A) = \beta_L^{-1} \circ \beta_M^{-1}(A) = \beta_L^{-1} g(A) = g(\beta_L^{-1} A) = g(f(A))$. Therefore the indicator of $L \circ M$ is $g(f(t))$.

COROLLARY 1. *Two delta functionals L and $\tilde{L}$ are reciprocals if and only if their indicators are inverse formal power series.*

We can now include indicators in our solution of the connection constants problem.

PROPOSITION 6.5. *If $p_n(x)$ and $q_n(x)$ are sequences of binomial type, being the associated sequences for $L = f(A)$ and $M = g(A)$, respectively, and if*

$$q_n(x) = r_n(\mathbf{p}(x))$$

for a polynomial sequence $r_n(x)$, then $r_n(x)$ is the associated sequence for the delta functional $\tilde{L} \circ M = g(f^{-1}(A))$.

We conclude this section with two results on derivations. The chain rule for derivations of the umbral algebra is easily derived:

PROPOSITION 6.6. *Let ∂_L and ∂_M be the derivations of the umbral algebra associated with the delta functionals L and M, respectively. Then*

$$\partial_L = (\partial_L M)\, \partial_M .$$

Proof. Any linear functional N can be expanded into a convergent series of powers of M:

$$N = a_0 + a_1 M + a_2 M^2 + \cdots, \qquad a_i \in K,$$

and since ∂_L and ∂_M are continuous, we have

$$\partial_L N = a_1 + 2a_2 M \partial_L M + 3a_3 M^2 \partial_L M + \cdots = (\partial_L M)\, \partial_M N.$$

Thus

$$\partial_L = (\partial_L M)\, \partial_M .$$

The following proposition is immediate.

PROPOSITION 6.7. *If L is any delta functional and M is any linear functional, the L-indicator of $\partial_L M$ is the derivative of the L-indicator of M.*

7. SHIFT-INVARIANT OPERATORS

On the algebra of all linear operators on P we define a topology by specifying that a sequence T_k of operators converges to an operator T whenever, given a polynomial $p(x)$, there is an index n_0 such that if $n \geqslant n_0$ then $T_n\, p(x) = Tp(x)$. Under this topology, the algebra of all linear operators is a complete topological algebra.

Every linear functional L defines a *multiplication operator* on P^*, mapping the linear functional M to the linear functional $L \cdot M$. We denote this operator by $\mu(L)^*$. Thus, $\mu(L)^* M = L \cdot M$. Every multiplication operator is continuous, hence by Proposition 6.1, its adjoint $\mu(L)$ maps polynomials into polynomials. In symbols,

$$\langle LM \mid p(x) \rangle = \langle \mu(L)^* M \mid p(x) \rangle = \langle M \mid \mu(L)\, p(x) \rangle.$$

We investigate the properties of the map $L \to \mu(L)$, beginning with

PROPOSITION 7.1. *The mapping $L \to \mu(L)$ of linear functionals into linear operators is a continuous algebra monomorphism.*

Proof. Only the continuity need be verified. Let the sequence L_k of linear functionals converge to zero. Given $n \geqslant 0$, we have $\langle L_k \mid x^j \rangle = 0$ for $j = 0, 1, \ldots, n$ and for large k, depending on n. Hence, for all scalars $a \in K$ and for large k, $\langle L_k \mid (x+a)^n \rangle = 0$ and thus $0 = \langle L_k \mid (x+a)^n \rangle = \langle \epsilon_a \mid \mu(L_k) x^n \rangle$. Therefore $\mu(L_k) x^n = 0$ for k large. Q.E.D.

The set of all operators of the form $\mu(L)$, for some linear functional L, is thus a topological algebra. We call an operator of this form a *shift-invariant operator*, and denote the algebra of all shift-invariant operators by Σ. Thus, the umbral algebra and the algebra of shift-invariant operators on P are isomorphic as topological algebras.

COROLLARY 1. *A shift-invariant operator T is invertible if and only if $T1 \neq 0$.*

As an example, consider the shift-invariant operator $E^a = \mu(\epsilon_a)$. From

$$\langle \epsilon_{a+b} \mid p(x) \rangle = \langle \epsilon_a \epsilon_b \mid p(x) \rangle = \langle \epsilon_b \mid E^a p(x) \rangle$$

we conclude that $E^a p(x) = p(x+a)$. We call E^a the *translation operator*. In particular, $E^0 = I$, the identity of Σ. Similarly, it is seen that $D = \mu(A)$ is the ordinary derivative $Dp(x) = p'(x)$.

A characterization of shift-invariant operators is

PROPOSITION 7.2. *A linear operator T is shift-invariant if and only if $TE^a = E^a T$ for all $a \in K$, that is, if and only if it commutes with all translation operators.*

Proof. Suppose $TE^a = E^a T$ for all $a \in K$ and for some operator T on P. We show that $T = \mu(L)$, where L is the linear functional defined by $\langle L \mid p(x) \rangle = \langle \epsilon \mid Tp(x) \rangle$. In fact:

$$\begin{aligned} \langle \epsilon_a \mid \mu(L)\, p(x) \rangle &= \langle L\epsilon_a \mid p(x) \rangle \\ &= \langle L \mid p(x+a) \rangle = \langle \epsilon \mid Tp(x+a) \rangle \\ &= \langle \epsilon \mid E^a T p(x) \rangle = \langle \epsilon_a \mid Tp(x) \rangle \end{aligned}$$

for all $a \in K$ and thus $\mu(L)\, p(x) = Tp(x)$.

Another characterization of shift-invariant operators is

PROPOSITION 7.3. *Let M be a delta functional. Then a linear operator T is shift-invariant if and only if $T\mu(M) = \mu(M)T$.*

The proof is omitted.

In view of the isomorphism between the algebras P^* and Σ, we may expect operator analogs of some of the notions introduced for the study of the umbral algebra.

A *delta operator* is an operator of the form $Q = \mu(L)$, where L is a delta functional. Delta operators are characterized by the following property of immediate verification:

PROPOSITION 7.4. *A shift-invariant operator Q is a delta operator if and only if $Q1 = 0$ and Qx is a nonzero constant.*

If $Q = \mu(L)$ is a delta operator, the *associated sequence for Q* is defined to be associated sequence for L. The relationship between a delta operator and its associated sequence $p_n(x)$ is a generalization of the relationship between the derivative operator and the sequence $p_n(x) = x^n$.

PROPOSITION 7.5. *The polynomial sequence $p_n(x)$ is the associated sequence for the delta operator Q if and only if it satisfies the following conditions:*

(i) $p_0(x) = 1$,

(ii) $p_n(0) = 0$ *for* $n > 0$,

(iii) $Qp_n(x) = np_{n-1}(x)$.

Proof. Let $Q = \mu(L)$ and suppose first that $p_n(x)$ is the associated sequence for Q, and hence for L. Then

$$\begin{aligned}\langle L^k \mid Qp_n(x)\rangle &= \langle L^{k+1} \mid p_n(x)\rangle \\ &= n!\,\delta_{k+1,n} = \langle L^k \mid np_{n-1}(x)\rangle.\end{aligned}$$

Therefore, by the Expansion Theorem,

$$\langle M \mid Qp_n(x)\rangle = \langle M \mid np_{n-1}(x)\rangle$$

for every linear functional M, and thus $Qp_n(x) = np_{n-1}(x)$.

Conversely, suppose the polynomial sequence $p_n(x)$ satisfies (i), (ii), and (iii). Then

$$\begin{aligned}\langle L^k \mid p_n(x)\rangle &= \langle \epsilon \mid Q^k p_n(x)\rangle \\ &= \langle \epsilon \mid (n)_k\, p_{n-k}(x)\rangle = n!\,\delta_{n,k}\,,\end{aligned}$$

so that $p_n(x)$ is the associated sequence for L.

The Expansion Theorem, stated in terms of shift-invariant operators, leads to another generalization of Taylor's formula:

PROPOSITION 7.6. *Let Q be a delta operator with associated sequence $p_n(x)$, and let T be a shift-invariant operator. Then*

$$T = \sum_{k=0}^{\infty} \frac{\langle \epsilon \mid Tp_k(x) \rangle}{k!} Q^k.$$

COROLLARY 1. *Let Q be a delta operator with associated sequence $p_n(x)$. Then*

$$E^y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} Q^k.$$

COROLLARY 2. *Let Q be a delta operator with associated sequence $p_n(x)$, then if $p(x)$ is any polynomial, we have*

$$p(x+y) = \sum_{k=0}^{\infty} \frac{Q^k p(x)}{k!} p_k(y).$$

For $Q = D$, Corollary 2 is precisely Taylor's formula.

If $Q = \mu(L)$ and $T = \mu(M)$, then the *Q-indicator of T* is the L-indicator of M.

We next consider automorphisms and derivations of the algebra Σ of shift-invariant operators. In view of the isomorphism μ: $P^* \to \Sigma$ of Proposition 7.1, every automorphism γ of Σ is of the form $\gamma = \mu\beta\mu^{-1}$, for some automorphism β of P^*. In fact, every automorphism of Σ is related to a unique delta operator $Q = \mu(L)$ by $\beta_Q = \mu\beta_L\mu^{-1}$.

Similarly, every continuous derivation of Σ is of the form $\partial_Q = \mu\partial_L\mu^{-1}$. These characterizations can be made more explicit as follows:

THEOREM 7. (a) *Every continuous automorphism of the algebra of shift-invariant operators is of the form $T \to \alpha^{-1}T\alpha$, where α is an umbral operator, and conversely.*

(b) *Every continuous derivation of the algebra of shift-invariant operators is of the form $T \to T\theta - \theta T$, where θ is an umbral shift, and conversely.*

Proof. (a) Suppose β_Q is a continuous automorphism of Σ, where $Q = \mu(L)$. For a shift-invariant operator $T = \mu(M)$ we have

$$\beta_Q(T) = \mu\beta_L\mu^{-1}(T) = \mu\beta_L(M).$$

Now if N is a linear functional and $p(x)$ is a polynomial, we may write

$$\begin{aligned}\langle N \mid \mu(\beta_L(M))\, p(x) \rangle &= \langle \beta_L(M)\, N \mid p(x) \rangle \\ &= \langle \beta_L(M\beta_L^{-1}(N)) \mid p(x) \rangle = \langle M\beta_L^{-1}(N) \mid \beta_L{}^* p(x) \rangle \\ &= \langle \beta_L^{-1}(N) \mid \mu(M)\, \beta_L{}^* p(x) \rangle = \langle N \mid (\beta_L^{-1})^*\, T\beta_L{}^* p(x) \rangle,\end{aligned}$$

and thus $\beta_Q(T) = (\beta_L^{-1})^* T\beta_L^*$. The same argument proves the converse assertion.

(b) Let ∂_Q be an continuous derivation of Σ. If $Q = \mu(L)$ and if $T = \mu(M)$ is any shift-invariant operator, we have

$$\partial_Q(T) = \mu\partial_L\mu^{-1}(T) = \mu\partial_L(M).$$

If N is any linear functional, and $p(x)$ any polynomial, then

$$\begin{aligned}\langle N \mid \mu\partial_L(M)\, p(x)\rangle &= \langle \partial_L(M)N \mid p(x)\rangle \\ &= \langle \partial_L(MN) - M\partial_L N \mid p(x)\rangle \\ &= \langle N \mid (T\partial_L{}^* - \partial_L{}^*T)\, p(x)\rangle.\end{aligned}$$

Therefore, $\partial_Q(T) = T\partial_L{}^* - \partial_L{}^*T$. The converse is proved similarly.

As an application, we obtain a representation of umbral shifts.

THEOREM 8. *Let θ_L and θ_M be the umbral shifts associated with the delta functionals L and M, respectively, and let $Q = \mu(L)$ and $P = \mu(M)$. Then*

$$\theta_L = \theta_M(\partial_P Q)^{-1}.$$

Proof. By Proposition 6.5, $\partial_L M = (\partial_M L)^{-1}$ and so $\partial_L = (\partial_M L)^{-1}\, \partial_M$. Observing that $\partial_M L = \mu^{-1}\partial_P\mu(\mu^{-1}(Q)) = \mu^{-1}\partial_P Q$, for a linear functional N and polynomial $p(x)$, we have

$$\begin{aligned}\langle N \mid \theta_L p(x)\rangle &= \langle \partial_L N \mid p(x)\rangle \\ &= \langle (\partial_M L)^{-1}\, \partial_M N \mid p(x)\rangle = \langle \partial_M N \mid \mu[(\partial_M L)^{-1}]\, p(x)\rangle \\ &= \langle \partial_M N \mid [\mu(\partial_M L)]^{-1}\, p(x)\rangle \\ &= \langle \partial_M N \mid (\partial_P Q)^{-1}\, p(x)\rangle \\ &= \langle N \mid \theta_M(\partial_P Q)^{-1}\, p(x)\rangle.\end{aligned}$$

The conclusion follows.

By letting $M = A$ in the preceding theorem, we obtain

COROLLARY 1 (Recurrence Formula). *Let $p_n(x)$ be the associated sequence for the delta operator Q. Then*

$$p_{n+1}(x) = x(\partial_D Q)^{-1}\, p_n(x).$$

COROLLARY 2. *Let θ_Q be an umbral shift, with corresponding delta operator Q. Then*

$$Q\theta_Q - \theta_Q Q = I.$$

For the special case $Q = D$, the associated shift θ_D is the operator X of multiplication by x, and Corollary 2 reduces to the familiar formula $DX - XD = I$. For convenience we denote the operator $\partial_D T = TX - XT$ by T', and if $L = \mu^{-1}(T)$ we denote $\mu^{-1}(T')$ by L'.

As expected, the indicator of the operator $\partial_D Q$ is the derivative of the indicator of Q.

We conclude with some powerful formulas for computing the associated sequence for a delta operator.

THEOREM 9 (Transfer Formula). *If $Q = PD$ is a delta operator, where P is an invertible shift-invariant operator, and if $p_n(x)$ is the associated sequence for Q, then*

$$p_n(x) = Q'P^{-n-1}x^n$$

for all $n \geqslant 0$.

Proof. Letting $q_n(x) = Q'P^{-n-1}x^n$, we see that

$$Qq_n(x) = PDQ'P^{-n-1}x^n = nq_{n-1}(x)$$

and thus by Proposition 7.5 we need only show that $q_0(x) = 1$ and $q_n(0) = 0$ for $n > 0$.

It is clear that $q_0(x)$ is a constant. Furthermore,

$$\begin{aligned}\langle \epsilon \mid q_0(x) \rangle &= \langle \epsilon \mid Q'P^{-1}1 \rangle = \langle \epsilon \mid (P + DP')\, P^{-1}1 \rangle \\ &= \langle \epsilon \mid 1 \rangle = 1,\end{aligned}$$

and we have $q_0(x) = 1$. For $n > 0$,

$$\begin{aligned}\langle \epsilon \mid q_n(x) \rangle &= \langle \epsilon \mid Q'p^{-n-1}x^n \rangle \\ &= \langle \epsilon \mid (P + DP')\, P^{-n-1}x^n \rangle \\ &= \langle \epsilon \mid P^{-n}x^n \rangle + \langle \epsilon \mid nP'P^{-n-1}x^{n-1} \rangle \\ &= \langle \epsilon \mid P^{-n}x^n \rangle - \langle \epsilon \mid (P^{-n})'x^{n-1} \rangle \\ &= \langle \epsilon \mid P^{-n}x^n \rangle - \langle \mu^{-1}(P^{-n}) \mid x^n \rangle \\ &= 0.\end{aligned}$$

Thus $q_n(0) = 0$ for $n > 0$ and the theorem is proved.

COROLLARY 1 (Transfer Formula). *If $Q = PD$ is a delta operator, with associated sequence $p_n(x)$, then*

$$p_n(x) = xP^{-n}x^{n-1}$$

for all $n \geqslant 1$.

Proof. The result follows from Theorem 9 and from the following computation:

$$\begin{aligned} Q'P^{-n-1}x^n &= (P + DP')\,P^{-n-1}x^n \\ &= P^{-n}x^n + nP'P^{-n-1}x^{n-1} \\ &= P^{-n}x^n - (P^{-n})'\,x^{n-1} \\ &= P^{-n}x^n - (P^{-n}X - XP^{-n})\,x^{n-1} \\ &= xP^{-n}x^{n-1}. \end{aligned}$$

COROLLARY 2. *Let Q be a delta operator, with associated sequence $p_n(x)$. Let $R = QT$ be another delta operator, with associated sequence $q_n(x)$, where T is an invertible shift invariant operator. Then*

$$q_n(x) = xT^{-n}x^{-1}p_n(x),$$

for $n \geqslant 1$.

Since any two delta operators Q and R are related by $QT = R$ for some invertible shift-invariant operator, Corollary 2 relates any two associated sequences.

8. EXAMPLES

We are now ready to show how the methods developed so far give an efficient technique for the computation of associated polynomials and connection constants. Specifically, to compute the matrix of constants $c_{n,k}$ in

$$p_n(x) = \sum_{k=0}^{n} c_{n,k}q_k(x),$$

where $p_n(x)$ and $q_n(x)$ are of binomial type, one uses the fact that the sequence $r_n(x) = \sum_k c_{n,k}x^k$ is also of binomial type, and that its indicator is computed by umbral methods in terms of the indicators for $p_n(x)$ and $q_n(x)$. Once the indicator for $r_n(x)$ is known, the coefficients of $r_n(x)$ are computed by one of the explicit formulas given in the previous section.

1.3. We have already remarked that the operator $\mu(A)$ is D, the ordinary derivative. Clearly $D' = I$, and the associated sequence is $p_n(x) = x^n$.

2.3. The *forward difference operator* is $\Delta_a = \mu(\epsilon_a - \epsilon) = E^a - I$, and its derivative is $\Delta_a' = aE^a$. To compute the associated sequence, we use the Recurrence Formula:

$$\begin{aligned} p_n(x) &= x(\Delta_a')^{-1}\,p_{n-1}(x) \\ &= xa^{-1}E^{-a}p_{n-1}(x) \\ &= a^{-1}xp_{n-1}(x - a), \end{aligned}$$

whence

$$\begin{aligned} p_n(x) &= a^{-n}x(x-a)(x-2a)\cdots(x-(n-1)a) \\ &= (x/a)_n\,, \end{aligned}$$

as previously announced.

We can use the Recurrence Formula to compute the conjugate sequence for $\epsilon_a - \epsilon = e^{aA} - \epsilon$ by computing the associated sequence for the conjugate functional $[\log(1 + A)]/a$. Indeed, we have

$$\begin{aligned} q_n(x) &= xa(1 + D)\, q_{n-1}(x) \\ &= \cdots = [ax(1 + D)]^n 1 \\ &= e^{-x}(axD)^n\, e^x. \end{aligned}$$

We know from previous discussions that the $q_n(x)$ are the exponential polynomials. Thus we have proved the Stirling numbers identity

$$e^{-x}(xD)^n\, e^x = \sum_{k=0}^{n} S(n, k)\, x^k,$$

where $S(n, k)$ are the Stirling numbers of the second kind. It is easy to see by Rolle's theorem that these polynomials have real roots.

3.3. The *backward difference operator* $I - E^a$, with derivative aE^{-a}, is similarly treated, giving the associated polynomials $p_n(x) = \langle x/a\rangle_n$.

If Q is a delta operator with associated sequence $p_n(x)$, and if $q_n(x)$ is the associated sequence for the Abelization $R = QE^a$ of Q, then we have

$$\begin{aligned} q_n(x) &= xE^{-an}x^{-1}p_n(x) \\ &= \frac{x}{x - an}\, p_n(x - an). \end{aligned} \tag{$*$}$$

This specializes to a host of polynomial sequences studied in various circumstances.

4.3. The *Abel operator* $\mu(A\epsilon_a)$ is DE^a; hence its derivative is $(DE^a)' = E^a(1 + aD)$. The Transfer Formula computes the Abel polynomials

$$\begin{aligned} p_n(x) &= xE^{-an}x^{n-1} \\ &= x(x - an)^{n-1}. \end{aligned}$$

5.3. The *difference–Abel operator* is $E^a(E^b - I)$ and its derivative is $E^a((a + b) E^b - a)$. From Eq. (*), we compute the Gould polynomials

$$\begin{aligned} p_n(x) &= xE^{-an}x^{-1}(x/b)_n \\ &= \frac{x}{x - an}\left(\frac{x - an}{b}\right)_n . \end{aligned}$$

6.3. The *central difference operator* is $\mu(\delta_a) = E^{a/2} - E^{-a/2}$ and its derivative is $(E^{a/2} + E^{-a/2})/2$. For $a = 1$, Eq. (*) (with a replaced by $-a/2$) gives the Steffensen polynomials

$$\begin{aligned} p_n(x) &= xE^{n/2}x^{-1}(x)_n \\ &= x(x + n/2 - 1)_{n-1} = x^{[n]}. \end{aligned}$$

7.3. The *Laguerre operator* is $L = \mu(l) = D(D - I)^{-1}$. The Laguerre operator satisfies

$$Lp(x) = \int_{-\infty}^{0} e^t p'(x + t)\, dt.$$

To compute the derivative L', we recall that

$$L'p(x) = (LX - XL)\, p(x),$$

whence

$$L' = \int_{-\infty}^{0} te^t p'(x + t)\, dt.$$

Several expansions for the associated sequence can be obtained. By the Transfer Formula we have

$$L_n(x) = x(D - I)^n\, x^{n-1} = xe^x D^n e^{-x} x^{n-1},$$

which is the classical Rodriques formula. By the Transfer Formula,

$$\begin{aligned} L_n(x) &= L'(D - I)^{n+1}\, x^n \\ &= -(D - I)^{n-1}\, x^n \\ &= -e^x D^{n-1} e^{-x} x^n. \end{aligned}$$

Finally, expanding $(D - I)^{n-1}$ we obtain the coefficients explicitly:

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n - 1}{k - 1}(-x)^k.$$

Next we give some examples of computation of connection constants. By way of orientation, we repeat a classical instance:

2.4. Determine the constants $c_{n,k}$ in

$$(x)_n = \sum_{k=0}^{n} c_{n,k} \langle x \rangle_k .$$

Since $(x)_n$ is the associated sequence for $g(A) = e^A - \epsilon$ and $\langle x \rangle_n$ is the associated sequence for $f(A) = \epsilon - e^{-A}$, $g(f^{-1}(A)) = A/(\epsilon - A)$. Therefore, $r_n(x) = \sum_{k=0}^{n} c_{n,k} x^k = L_n(-x)$, where $L_n(x)$ are the (basic) Laguerre polynomials. One can hardly hope for anything simpler.

3.4. Determine the constants $c_{n,k}$ in

$$\langle x \rangle_n = \sum_{k=0}^{n} c_{n,k} \langle x/a \rangle_k .$$

Since $\langle x \rangle_n$ is the associated sequence for $g(A) = \epsilon - e^{-A}$, and $\langle x/a \rangle_n$ is the associated sequence for $f(A) = \epsilon - e^{-aA}$, we have $g(f^{-1}(A)) = \epsilon - (\epsilon - A)^a$. Thus by the Recurrence Formula,

$$\begin{aligned} r_n(x) &= xa(I - D)^{a-1}\, r_{n-1}(x) \\ &= \cdots = a^n(x(I - D)^{a-1})^n\, 1 \\ &= a^n e^x (xD)^n\, e^{-x}. \end{aligned}$$

4.4. Express the Abel polynomials as linear combinations of the Laguerre polynomials. That is, determine the constants $c_{n,k}$ such that

$$A_n(x, a) = \sum_{k=0}^{n} c_{n,k} L_k(x).$$

The sequence $L_n(x)$ is the associated sequence for $f(A) = A/(A - \epsilon)$, and $A_n(x, a)$ is the associated sequence for $g(A) = Ae^{aA}$. Thus $g(f^{-1}(A)) = [A/(A - 1)]\, e^{aA/(A-1)}$. By the Transfer Formula, the associated sequence for $g(f^{-1}(A))$ is

$$\begin{aligned} r_n(x) &= x(D - I)^n\, e^{-anD/(D-I)} x^{n-1} \\ &= xe^x D^n e^{-x} e^{-anD/(D-I)} x^{n-1}. \end{aligned}$$

The coefficients $c_{n,k}$ are now obtained by a routine Taylor expansion.

5.4. Determine the connection constants $c_{n,k}$ of the Gould polynomials with the factorial powers:

$$G_n(x, a, -1) = \sum_{k=0}^{n} c_{n,k} \langle x \rangle_k .$$

Again, $G_n(x, a, -1)$ is the associated sequence for $g(A) = e^{aA}(e^{-A} - \epsilon)$ and $\langle x \rangle_n$ is the associated sequence for $f(A) = \epsilon - e^{-A}$, hence, $g(f^{-1}(A)) = -A(\epsilon - A)^{-a}$ By the Transfer Formula

$$\begin{aligned} r_n(x) &= (-1)^n\, x(I - D)^{an}\, x^{n-1} \\ &= (-1)^n \sum_{k \geqslant 0} \binom{an}{k} (-1)^k\, (n-1)_k\, x^{n-k}, \end{aligned}$$

a relative of the Laguerre sequence.

6.4. Determine the connection constants $c_{n,k}$ of the Steffensen polynomials with the factorial powers:

$$x^{[n]} = \sum_{k=0}^{n} c_{n,k}(x)_k\,.$$

The Steffensen polynomials are the associated polynomials for $\delta = \epsilon_{-1/2}(\epsilon_1 - \epsilon)$. In this case $g(A) = e^{-A/2}(e^A - \epsilon)$ and $f(A) = e^A - \epsilon$ Thus $g(f^{-1}(A)) = A(\epsilon + A)^{-1/2}$ and

$$\begin{aligned} r_n(x) &= x(1 + D)^{n/2}\, x^{n-1} \\ &= \sum_{k \geqslant 0} \binom{n/2}{k} (n-1)_k\, x^{n-k}, \end{aligned}$$

again a most explicit answer.

7 4. We derive Erdelyi's duplication formulas for Laguerre polynomials; that is, we determine the $c_{n,k}$ for which

$$L_n(ax) = \sum_{k=0}^{n} c_{n,k} L_k(x).$$

Now $L_n(ax)$ is the associated sequence for $g(A) = a^{-1}A/(a^{-1}A - \epsilon)$ and therefore $r_n(x) = \sum_{k=0}^{n} c_{n,k}x^k$ is the associated sequence for $A/[(\epsilon - a)A + a]$. By the Transfer Formula,

$$\begin{aligned} r_n(x) &= x((1 - a)\, D + aI)^n\, x^{n-1} \\ &= \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (1-a)^{n-k}\, (ax)^k. \end{aligned}$$

Thus:

$$L_n(ax) = \sum_{k \geqslant 0} \frac{n!}{k!} \binom{n-1}{k-1} (1-a)^{n-k}\, a^k L_k(x).$$

2.5. We give some applications of umbral techniques to the Stirling numbers $s(n, k)$ and $S(n, k)$ of the first and second kind. Recall that the exponential polynomials

$$\phi_n(x) = \sum_{k=0}^{n} S(n, k)\, x^k$$

are the associated polynomials for the delta functional $\log(\epsilon + A)$.

The Recurrence Formula gives a recurrence formula for the exponential polynomials:

$$\begin{aligned}\phi_n(x) &= x(I + D)\,\phi_{n-1}(x)\\ &= x(\phi_{n-1}(x) + \phi'_{n-1}(x)).\end{aligned}$$

Dobinsky's formula is practically trivial. Letting $p_n(x) = (x)_n$, we take an umbral composition

$$p_n(\phi(x)) = x^n = e^{-x} \sum_{k\geqslant 0} \frac{p_n(k)}{k!}\, x^k,$$

and thus for any polynomial $p(x)$,

$$p(\phi(x)) = e^{-x} \sum_{k\geqslant 0} \frac{p(k)}{k!}\, x^k.$$

For $p(x) = x^n$, we obtain Dobinsky's formula:

$$\phi_n(x) = e^{-x} \sum_{k\geqslant 0} \frac{k^n x^n}{k!}\,.$$

Consider the polynomials

$$\psi_n(x) = \sum_{k=0}^{n} s(n, k)(x)_k\,.$$

If we define the umbral operators $\alpha\colon x^n \to (x)_n$ and $\beta\colon x^n \to \phi_n(x)$, then Corollary 1 to Theorem 5 gives $\alpha^{-1} = \beta$. Therefore,

$$\psi_n(\phi(x)) = \beta(\psi_n(x)) = \beta\alpha(x)_n = (x)_n\,,$$

or, more explicitly:

$$s(n, k) = \sum_{i,j\geqslant 0} s(n, j)\, s(j, i)\, S(i, k).$$

Similarly, from $\phi_n(\psi(x)) = (x)_n$, we obtain

$$\sum_{k\geqslant 0} S(n, k)\, s(k, i) = \delta_{n,i}\,.$$

We can derive another recurrence for the exponential polynomials as follows. If α is an umbral operator, then

$$\alpha^*(A^k)' = \alpha^*(A^{k'})\,\alpha^*(A)'.$$

Applying to a polynomial $p(x)$ and using the properties of adjoints and derivations:

$$\langle A^k \mid \alpha x p(x)\rangle = \langle A^k \mid x\alpha(\mu(\alpha^*(A)')\,p(x))\rangle.$$

Therefore,

$$\alpha x p(x) = x\alpha(\mu(\alpha^*(A)')\,p(x)).$$

Now if we take α: $x^n \to \phi_n(x)$, then $\alpha^*(A) = e^A - \epsilon$ and so $\mu(\alpha^*(A)') = E^1$. Setting $p(x) = x^n$ gives

$$\phi_{n+1}(x) = x(\phi + 1)^n,$$

which, in terms of coefficients, gives the Stirling numbers recurrence

$$S(n+1, k) = \sum_{i\geqslant 0} \binom{n}{i} S(i, k-1).$$

9. Sheffer Sequences

So far, we have no explicit formula for shift-invariant operators. In obtaining an explicit formula for $\mu(L)$, we are led to a new class of polynomial sequences. A polynomial sequence $s_n(x)$ is a *Sheffer sequence relative to* a sequence $p_n(x)$ of binomial type if it satisfies the functional equation

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\,p_{n-k}(y)$$

for all $n \geqslant 0$ and for all $y \in K$.

Some characterizations of Sheffer sequences follow. The proofs follow a familiar pattern, and are therefore omitted.

PROPOSITION 9.1. *A polynomial sequence $s_n(x)$ is a Sheffer sequence if and only if there exist a sequence of binomial type $p_n(x)$ and an invertible shift-invariant operator P such that*

$$p_n(x) = Ps_n(x)$$

for all $n \geqslant 0$.

PROPOSITION 9.2. *The following are equivalent for a polynomial sequence $s_n(x)$:*

(a) *The sequence $s_n(x)$ is a Sheffer sequence.*

(b) *There exists a delta operator Q such that*

$$Qs_n(x) = ns_{n-1}(x)$$

for all $n \geqslant 1$.

(c) *There exists a delta functional L and an invertible linear functional N such that*

$$\langle NL^k \mid s_n(x)\rangle = n!\,\delta_{n,k}$$

for all $n, k \geqslant 0$.

If Q is a delta operator and $Qs_n(x) = ns_{n-1}(x)$, we say that $s_n(x)$ is *Sheffer for Q*. Moreover, if $s_n(x)$ is a Sheffer sequence with respect to $p_n(x)$, the associated sequence for Q, then $s_n(x)$ is Sheffer for Q, and conversely. If T is an invertible shift-invariant operator, and $s_n(x)$ is Sheffer for Q, then $Ts_n(x)$ is also Sheffer for Q, and $T^n s_n(x)$ is Sheffer for $T^{-1}Q$.

Given a delta functional L and an invertible linear functional N, there exists exactly one polynomial sequence $s_n(x)$ satisfying

$$\langle NL^k \mid s_n(x)\rangle = n!\,\delta_{n,k}\,,$$

namely, the sequence $s_n(x) = \mu(N)^{-1}\,p_n(x)$, where $p_n(x)$ is the associated sequence for L. We say that $s_n(x)$ is *the Sheffer sequence for N with respect to L*, or *the (N, L)-Sheffer sequence.*

A pair (N, L), where N is an invertible linear functional and L is a delta functional, determines a unique Sheffer sequence $s_n(x)$ in this way.

THEOREM 10 (Second Expansion Theorem). *Let $s_n(x)$ be the (N, L)-Sheffer sequence, and let $Q = \mu(L)$, $S = \mu(N)$. Then*

(a) *Every linear functional M can be uniquely expanded into the convergent series*

$$M = \sum_{k=0}^{\infty} \frac{\langle M \mid s_k(x)\rangle}{k!} L^k N.$$

(b) *Every shift-invariant operator T can be uniquely expanded into the convergent series*

$$T = \sum_{k=0}^{\infty} \frac{\langle \epsilon \mid Ts_k(x)\rangle}{k!} Q^k S.$$

(c) *Every polynomial $p(x)$ can be uniquely expanded into the finite sum*

$$p(x) = \sum_{k\geqslant 0} \frac{\langle NL^k \mid p(x)\rangle}{k!} s_k(x).$$

We can now give explicit formulas for shift-invariant operators:

THEOREM 11. (a) *Let $s_n(x)$ be a Sheffer sequence relative to the sequence $p_n(x)$ of binomial type. Every shift-invariant operator $\mu(L)$ can be represented by*

$$\mu(L)\, s_n(x) = \sum_{k=0}^{n} \binom{n}{k} \langle L \mid p_k(x)\rangle\, s_{n-k}(x).$$

(b) *Conversely, suppose that for a delta operator $Q = \mu(L)$ there is a sequence of constants a_k such that*

$$Qs_n(x) = \sum_{k=0}^{n} \binom{n}{k} a_k s_{n-k}(x).$$

Then $s_n(x)$ is a Sheffer sequence relative to a sequence $p_n(x)$ of binomial type, and $a_n = \langle L \mid p_n(x)\rangle$ for all $n \geqslant 0$.

Proof. (a) Suppose $p_n(x)$ is the associated sequence for the delta functional M. Then

$$\begin{aligned}\mu(M^j)\, s_n(x) &= (n)_j\, s_{n-j}(x)\\ &= \sum_{k=0}^{n} \binom{n}{k} \langle M^j \mid p_k(x)\rangle\, s_{n-k}(x).\end{aligned}$$

By a closure argument, we may replace M^j by any linear functional. Q.E.D.

(b) Define the operator T by $Ts_n(x) = ns_{n-1}(x)$ for $n \geqslant 1$ and $Ts_0(x) = 0$. Then $s_n(x)$ will be a Sheffer sequence if T is shift-invariant. But

$$\begin{aligned}TQs_n(x) &= \sum_{k=0}^{n} \binom{n}{k} a_k T s_{n-k}(x)\\ &= \sum_{k=0}^{n-1} \binom{n}{k} a_k (n-k)\, s_{n-k-1}(x)\\ &= n \sum_{k=0}^{n-1} \binom{n-1}{k} a_k s_{n-k-1}(x)\\ &= nQs_{n-1}(x) = QTs_n(x),\end{aligned}$$

and then Proposition 7.4 implies T is shift-invariant. Since $s_n(x)$ is a Sheffer sequence, part (a) implies $a_k = \langle L \mid p_k(x)\rangle$.

We define the *conjugate Sheffer sequence* of the pair (N, L) as the polynomial sequence

$$r_n(x) = \sum_{k\geqslant 0} \frac{\langle NL^k \mid x^n\rangle}{k!}\, x^k.$$

Not unexpectedly, it turns out that every conjugate Sheffer sequence is Sheffer, and conversely. The proofs of Proposition 9.3 and Theorem 12 below are similar to those of Proposition 4.4 and Theorem 4.

PROPOSITION 9.3. *A polynomial sequence*

$$s_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$$

is a Sheffer sequence with respect to the sequence of binomial type

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

if and only if

$$\binom{i+j}{i} c_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} c_{k,i} a_{n-k,j}\,. \tag{***}$$

THEOREM 12. (a) *Every conjugate Sheffer sequence is a Sheffer sequence.*

(b) *Every Sheffer sequence is a conjugate Sheffer sequence.*

Every pair (N, L) is associated with two Sheffer sequences, its Sheffer sequence $s_n(x)$ and its conjugate Sheffer sequence $r_n(x)$. We say that $r_n(x)$ is *reciprocal Sheffer* to $s_n(x)$.

Similarly, a Sheffer sequence $s_n(x)$ is associated with two pairs, namely, the pair (M, L) for which $s_n(x)$ is the Sheffer sequence, and the pair $(\widetilde{M, L})$, for which $s_n(x)$ is the conjugate sequence. We say that $(\widetilde{M, L})$ is *the reciprocal pair* to (M, L).

Our goal now is to give a solution to the connection constants problem for Sheffer sequences. We proceed in a manner analogous to that for sequences of binomial type.

A *Sheffer operator* is a linear operator λ defined by λ: $x^n \to s_n(x)$, where $s_n(x)$ is a Sheffer sequence. If $Ps_n(x) = p_n(x)$, where $p_n(x)$ is of binomial type, then

$$\lambda = P^{-1} \circ \alpha$$

where α is the umbral operator α: $x^n \to p_n(x)$. We infer

THEOREM 13. *An operator λ is a Sheffer operator if and only if its adjoint is of the form $\beta \circ \mu(M^{-1})^*$, where β is a continuous automorphism of the umbral algebra, and $\mu(M^{-1})^*$ is multiplication by an invertible linear functional M^{-1}.*

Proof. If λ is a Sheffer operator then $\lambda = P^{-1} \circ \alpha$, where $P = \mu(M)$. Taking adjoints and applying Theorem 5 gives the result. The converse is obvious.

PROPOSITION 9.4. (a) *A Sheffer operator maps Sheffer sequences into Sheffer sequences.*

(b) *If* $\lambda\colon s_n(x) \to r_n(x)$, *where* $r_n(x)$ *is* (N, R)-*Sheffer and* $s_n(x)$ *is* (M, L)-*Sheffer, then* λ *is a Sheffer operator and* $\lambda^*(NR^k) = ML^k$, $k = 0, 1, 2,\ldots$.

We come now to the principal question for Sheffer sequences. Given two Sheffer sequences $r_n(x)$ and $s_n(x)$, determine the connection constants $c_{n,k}$ in

$$r_n(x) = \sum_{k=0}^{n} c_{n,k} s_k(x).$$

We know that the polynomial sequence

$$t_n(x) = \sum_{k=0}^{n} c_{n,k} x^k$$

is also Sheffer. Thus the problem of computing the connection constants reduces to the problem of determining the pair of linear functionals which determine the sequence $t_n(x)$. Stated in other terms, the problem is to determine the pair of linear functionals corresponding to the umbral composition of two Sheffer sequences. We shall state the solution in terms of indicators.

PROPOSITION 9.5. *If the pair* (M, L) *with Sheffer sequence* $s_n(x)$ *has indicators* $(f(t), g(t))$, *and the pair* (N, R) *with Sheffer sequence* $t_n(x)$ *has indicators* $(h(t), k(t))$, *then the pair of the Sheffer sequence* $s_n(\mathbf{t}(x))$ *has indicators*

$$(f(t)\, h(g(t)),\, k(g(t))).$$

Proof. Let (X, Y) be the desired pair of linear functionals. Then clearly

$$\lambda^*_{(M,L)}\colon f(A)\, g(A)^k \to A^k,$$
$$\lambda^*_{(N,R)}\colon h(A)\, k(A)^k \to A^k,$$

and

$$\lambda^*_{(X,Y)}\colon XY^k \to A^k.$$

Therefore,

$$\begin{aligned}
XY^k &= (\lambda^*_{(X,Y)})^{-1}(A^k) = (\lambda^*_{(M,L)})^{-1}(\lambda^*_{(N,R)})^{-1}(A^k)\\
&= (\lambda^*_{(M,L)})^{-1}(h(A)\, k(A)^k)\\
&= \mu(M)^*\, (\alpha_L{}^*)^{-1}\, (h(A)\, k(A)^k)\\
&= Mh((\alpha_L{}^*)^{-1} A)\, k((\alpha_L{}^*)^{-1} A)^k\\
&= Mh(L)\, k(L)^k\\
&= f(t)\, h(g(t))\, k(g(t))^k.
\end{aligned}$$

Q.E.D.

COROLLARY 1. *If the pair (M, L) has indicators $(f(t), g(t))$, then the reciprocal pair $(\widetilde{M, L})$ has indicators*

$$\left(\frac{1}{f(g^{-1}(t))}, g^{-1}(t)\right).$$

COROLLARY 2. *Suppose $s_n(x)$ is Sheffer for (M, L), with indicators $(f(t), g(t))$ and $r_n(x)$ is Sheffer for (N, R), with indicators $(h(t), k(t))$. If*

$$r_n(x) = t_n(\mathbf{s}(x))$$

for a polynomial sequence $t_n(x)$, then $t_n(x)$ is Sheffer for the pair with indicators

$$\left(\frac{h(g^{-1}(t))}{f(g^{-1}(t))}, k(g^{-1}(t))\right).$$

The following theorem is a recurrence formula for Sheffer sequences.

THEOREM 14. *Let $s_n(x)$ be a Sheffer sequence relative to the associated sequence $p_n(x)$ for $Q = \mu(L)$, and let $Ps_n(x) = p_n(x)$. Then*

$$s_{n+1}(x) = (P\partial_Q(P^{-1}) + \theta_L)\, s_n(x).$$

Proof. First notice that

$$s_{n+1}(x) = P^{-1}p_{n+1}(x) = P^{-1}\theta_L p_n(x) = P^{-1}\theta_L P s_n(x).$$

From part (b) of Theorem 7, we have

$$P^{-1}\theta_L P = (P^{-1}\theta_L - \theta_L P^{-1})P + \theta_L = P\partial_Q(P^{-1}) + \theta_L,$$

hence the conclusion.

A wide variety of polynomial sequences studied since Euler turned out to be Sheffer sequences, and no computer list can be drawn here. We shall only give a few examples to illustrate how the seemingly endless variety of identities is in fact the repetition of a few general formulas.

Sheffer sequences relative to the sequence x^n are called *Appell sequences.* Some of the best-known instances are:

The *Bernoulli polynomials*, defined by the functional

$$\langle \gamma \mid p(x) \rangle = \int_0^1 p(x)\, dx.$$

Thus,

$$\langle \gamma A^k \mid B_n(x) \rangle = n!\, \delta_{kn},$$

or, setting $J = \mu(\gamma)$, in operator notation $B_n(x) = J^{-1}x^n$. All identities for Bernoulli polynomials follow from the definition and from the above theory.

For example, an application of the Second Expansion Theorem gives the Euler–MacLaurin expansion formula

$$\epsilon_a = \sum_{k\geqslant 0} \frac{\langle \epsilon_a \mid B_k(x)\rangle}{k!} \gamma A^k,$$

or more explicitly

$$p(x) = \sum_{k\geqslant 0} \frac{B_k(x)}{k!} \int_0^1 p^{(k)}(t)\, dt.$$

Similarly, the *Euler functional* e defined by

$$\langle e \mid p(x)\rangle = \frac{p(1) + p(0)}{2}$$

gives the *Euler polynomials* $e_n(x) = \mu(e)^{-1} x^n$ and again the Second Expansion Theorem delivers *Boole's summation formula*

$$p(x) = \sum_{k\geqslant 0} \frac{e_k(x)}{k!} \langle \epsilon \mid p^{(k)}(x)\rangle.$$

Along the same lines, the *Boole polynomials* are the Sheffer set $\zeta_n(x) = \mu(e)^{-1}(x)_n$, and the corresponding expansion goes by the name of *Boole's second summation formula*

$$p(x) = \sum_{k\geqslant 0} \frac{\zeta_k(x)}{k!} \langle e \mid \Delta^k p(x)\rangle.$$

The *Bernoulli polynomials of the second kind* are the Sheffer sequence defined by $b_n(x) = J(x)_n$, so that, for example, the identity

$$b_n(0) = \sum_{k=0}^{n} s(n, k)/(k + 1)$$

is trivial in the present context. The corresponding expansion gives a variant of the Euler–MacLaurin formula where derivatives are replaced by differences.

The umbral composition of Appell sequences reduces to the following simple rule: The Appell sequence $r_n(x) = t_n(\mathbf{s}(x))$ is the sequence TSx^n, where $t_n(x) = Tx^n$ and $s_n(x) = Sx^n$.

10. Factor Sequences

An *inverse formal power series*—or inverse series for short—is a formal power series of the form

$$f(x) = \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots = \sum_{k=1}^{\infty} a_k x^{-k}.$$

The family Γ of all such formal power series is an algebra under ordinary addition, formal multiplication and multiplication by scalars; the algebra does not have an identity. The series $f(x)$ is said to be of degree $-n$ if $a_1 = a_2 = \cdots = a_{n-1} = 0$, but $a_n \neq 0$.

In a *sequence* $f_{-n}(x)$ of inverse formal power series it is tacitly understood that $f_{-n}(x)$ is of degree $-n$, for $n = 1, 2, \ldots$.

We indicate sequences of inverse formal power series by the notation $f_{-n}(x)$, $n = 1, 2, \ldots$, in contrast to polynomial sequences $p_n(x)$. We endow Γ with a topology which stipulates that a sequence $f_{-n}(x) = \sum_{k=1}^{\infty} a_{n,k} x^{-k}$ converges to $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$ if, for each k, there exists an index n_k such that if $n > n_k$ then $a_{n,k} = a_k$. Under this topology, Γ becomes a topological algebra, and every sequence $f_{-n}(x)$ spans; that is, every inverse formal power series $f(x)$ can be uniquely expressed as a convergent series $f(x) = \sum_{k \geq 1} a_k f_{-k}(x)$ for suitable constants a_k.

Recalling that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}; \qquad \binom{0}{k} = \delta_{0,k},$$

for a scalar a, we set

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k x^{-n-k},$$

the right-hand series being convergent. One easily verifies that

$$(x+a)^{-m}(x+a)^{-n} = (x+a)^{-m-n}.$$

The symmetry in x and a of the left side is deceptive. The variable a ranges over all scalars, but x is not a variable at all, unlike the case of polynomials. Unlike polynomials, one may not "evaluate" an inverse formal power series by giving x a constant value.

For any inverse formal power series $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$, we may define $f(x+a)$ as

$$E^a : f(x) = \sum_{k=1}^{\infty} a_k (x+a)^{-k},$$

since the series on the right converges. The resulting operator E^a is again called the *translation operator*.

The *derivative* operator D on the algebra Γ is defined by setting $Dx^{-n} = -nx^{-n-1}$ and extending to all of Γ by closure.

We introduce the notion of *factor sequence*, which is in some ways analogous to a Sheffer sequence. Let $f_{-n}(x)$, $n = 1, 2, \ldots$, be a sequence of inverse formal power series, where $f_{-n}(x)$ is of degree $-n$. We say $f_{-n}(x)$ is a *factor sequence relative to* the sequence $p_n(x)$ of binomial type if it satisfies:

$$f_{-n}(x+a) = \sum_{k=0}^{\infty} \binom{-n}{k} p_k(a) f_{-n-k}(x), \tag{$*$}$$

for all $n = 1, 2, \ldots$ and for all scalars $a \in K$. The identity $(*)$ is called the *factor (binomial) identity*. If $p_n(x)$ is the associated sequence for the delta functional L, we say that $f_{-n}(x)$ is the *factor sequence associated to* the delta functional L. Caution: again the symbols x and a cannot be interchanged in $(*)$.

The simplest factor sequence is the sequence x^{-n}, $n = 1, 2, \ldots$, which satisfies the factor (binomial) identity:

$$(x+a)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k x^{-n-k}.$$

Our first goal is to establish an algebra isomorphism from the umbral algebra P^* into the algebra of linear operators on Γ. For any linear functional $L \in P^*$, we define the linear operator $\sigma(L)$, mapping Γ into Γ by

$$\sigma(L)\, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k}. \tag{$**$}$$

We must show that $\sigma(L)$ can be defined on all inverse formal power series. To this end, if $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$, set $\sigma(L) f(x) = \sum_{k=1}^{\infty} a_k \sigma(L)\, x^{-k}$. Since the degree of $\sigma(L)\, x^{-k}$ is at most $-k$, this series is convergent. In other words, we may extend definition $(**)$ by closure to all of Γ. Thus, $\sigma(L)$ is a continuous operator on Γ.

The dual space Γ^* to Γ, that is, the vector space of all continuous linear functionals on Γ, is easily described. It consists of all linear functionals M on Γ such that $\langle M \mid x^{-n} \rangle = 0$ for all nonnegative integers n, except for a finite number.

Now consider $\sigma(L)^*$, the adjoint of the linear operator $\sigma(L)$, acting on the dual space Γ^*.

For any continuous linear functional M in Γ^*, we have

$$\begin{aligned}
\langle \sigma(L)^* M \mid x^{-n} \rangle &= \langle M \mid \sigma(L)\, x^{-n} \rangle \\
&= \left\langle M \,\middle|\, \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k} \right\rangle \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle \langle M \mid x^{-n-k} \rangle.
\end{aligned} \tag{$***$}$$

Moreover, in (***), the sequence x^{-n} can be replaced by an arbitrary factor sequence $f_{-n}(x)$:

THEOREM 15. *Let L be a linear functional in P^* and let $f_{-n}(x)$ be a factor sequence relative to the sequence $p_n(x)$. Then, for any continuous linear functional M in Γ^*, we have*

$$\langle \sigma(L)^* M \mid f_{-n}(x)\rangle = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle\langle M \mid f_{-n-k}(x)\rangle.$$

Proof. Let $\Gamma(x, y)$ be the topological algebra of all inverse formal power series in the variable x, whose coefficients are polynomials in the variable y. Define the map $L_y M_x$ of $\Gamma(x, y)$ into the field K by

$$L_y M_x(y^j x^{-i}) = \langle L \mid x^j\rangle\langle M \mid x^{-i}\rangle.$$

Since any element $f(x, y)$ in $\Gamma(x, y)$ is of the form $f(x, y) = \sum_{k=1}^{\infty} p_k(y)\, x^{-k}$, where $p_k(y)$ is a polynomial in y, and since $\langle M \mid x^{-k}\rangle = 0$ for all but a finite number of x^{-k}, we may define

$$L_y M_x f(x, y) = \sum_{k \geqslant 0} \langle L \mid p_k(x)\rangle\langle M \mid x^{-k}\rangle,$$

the sum on the right being finite. This makes $L_y M_x$ a continuous linear functional on $\Gamma(x, y)$. Thus equation (***) becomes

$$\langle \sigma(L)^* M \mid x^{-n}\rangle = L_y M_x (x + y)^{-n}.$$

Since $\sigma(L)^* M$ is in Γ^*, it follows that $\langle \sigma(L)^* M \mid x^{-n}\rangle = 0$ except for a finite number of integers $n \geqslant 0$. For $f(x) = \sum_{k=1}^{\infty} a_k x^{-k}$ we have

$$\begin{aligned}
\langle \sigma(L)^* M \mid f(x)\rangle &= \left\langle \sigma(L)^* M \,\middle|\, \sum_{k=1}^{\infty} a_k x^{-k} \right\rangle \\
&= \sum_{k=1}^{\infty} a_k \langle \sigma(L)^* M \mid x^{-k}\rangle \\
&= \sum_{k=1}^{\infty} a_k L_y M_x (x + y)^{-k} \\
&= L_y M_x \sum_{k=1}^{\infty} a_k (x + y)^{-k} \\
&= L_y M_x f(x + y).
\end{aligned}$$

Finally, for the factor sequence $f_{-n}(x)$, we have

$$\begin{aligned}\langle \sigma(L)^* M \mid f_{-n}(x)\rangle &= L_y M_x f_{-n}(x+y)\\ &= L_y M_x \sum_{k=0}^{\infty} \binom{-n}{k} p_k(y) f_{-n-k}(x)\\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle\langle M \mid f_{-n-k}(x)\rangle. \qquad \text{Q.E.D.}\end{aligned}$$

An immediate corollary is a characterization of the shift invariant operators.

COROLLARY 1. *Let L be a linear functional in P^* and let $f_{-n}(x)$ be a factor sequence relative to the sequence $p_n(x)$. Then*

$$\sigma(L) f_{-n}(x) = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle f_{-n-k}(x).$$

PROPOSITION 10.1. *Let L and M be linear functionals in P^* and let N be a continuous linear functional in Γ^*. Then*

$$\sigma(L)^*(\sigma(M)^*N) = \sigma(LM)^*N.$$

Proof. On the one hand,

$$\begin{aligned}\langle \sigma(LM)^* N \mid x^{-n}\rangle &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle LM \mid x^k\rangle\langle N \mid x^{-n-k}\rangle\\ &= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{-n}{k}\binom{k}{j} \langle L \mid x^j\rangle\langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\langle \sigma(L)^* (\sigma(M)^* N) \mid x^{-n}\rangle\\ &= \sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle\langle \sigma(M)^* N \mid x^{-n-j}\rangle\\ &= \sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle \sum_{i=0}^{\infty} \binom{-n-j}{i} \langle M \mid x^i\rangle\langle N \mid x^{-n-j-i}\rangle,\end{aligned}$$

and letting $k = i + j$, this equals

$$\begin{aligned}&\sum_{j=0}^{\infty} \binom{-n}{j} \langle L \mid x^j\rangle \sum_{k=j}^{\infty} \binom{-n-j}{k-j} \langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle\\ &\qquad = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{-n}{j}\binom{-n-j}{k-j} \langle L \mid x^j\rangle\langle M \mid x^{k-j}\rangle\langle N \mid x^{-n-k}\rangle.\end{aligned}$$

We can now prove

PROPOSITION 10.2. *The map $L \to \sigma(L)$ is an algebra monomorphism from the umbral algebra P^* into the algebra of all continuous linear operators on Γ.*

Proof. We have already seen that $\sigma(L)$ is continuous. If $\sigma(L) = 0$, then

$$0 = \sigma(L)\, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid x^k \rangle\, x^{-n-k}$$

and thus $\langle L \mid x^k \rangle = 0$ for all $k \geqslant 0$, so that $L = 0$. Therefore σ is one-to-one. Finally, for any continuous N in Γ^* and any inverse formal power series $f(x)$ in Γ, we have

$$\begin{aligned}\langle N \mid \sigma(LM) f(x) \rangle &= \langle \sigma(LM)^* N \mid f(x) \rangle \\ &= \langle \sigma(L)^*(\sigma(M)^* N) \mid f(x) \rangle = \langle \sigma(M)^* N \mid \sigma(L) f(x) \rangle \\ &= \langle N \mid \sigma(M)\, \sigma(L) f(x) \rangle.\end{aligned}$$

Thus, $\sigma(LM) f(x) = \sigma(M)\, \sigma(L) f(x)$ and so $\sigma(ML) = \sigma(LM) = \sigma(M)\, \sigma(L)$ Therefore σ preserves multiplication and the proposition is proved.

We call the image of σ the algebra of *shift-invariant operators* on Γ, and denote this algebra by Ω.

COROLLARY 1. *A shift-invariant operator T in Ω is invertible if and only if $Tf(x)$ is of degree -1 whenever $f(x)$ is of degree -1.*

Let $p_n(x)$ be a sequence of binomial type, and let $f_{-1}(x)$ be the first member of a factor sequence, and thus of degree -1. If we choose an arbitrary sequence of constants c_k, $k = 0, 1, 2, \ldots$, and set

$$Tf_{-1}(x) = \sum_{k \geqslant 0} \binom{-1}{k} c_k f_{-1-k}(x).$$

then there exists a unique linear functional L in P^* such that $\langle L \mid p_k(x) \rangle = c_k$. Thus, setting

$$Tf_{-n}(x) = \sum_{k \geqslant 0} \binom{-n}{k} c_k f_{-n-k}(x)$$

we obtain a shift-invariant operator. In summary:

PROPOSITION 10.3. *Given a factor sequence $f_{-n}(x)$ and an inverse series $g(x)$, there is a unique shift-invariant operator T such that $Tf_{-1}(x) = g(x)$.*

We are now able to give the following characterization of the shift-invariant operators on Γ.

PROPOSITION 10.4. *A linear operator T on Γ is shift-invariant if and only if it is continuous and $TE^a = E^aT$, for all constants $a \in K$.*

Suppose T is a continuous operator on Γ with $TE^a = E^aT$ for all $a \in K$. Define constants c_k by

$$Tx^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} c_k x^{-1-k}.$$

By the previous proposition, there is a unique shift-invariant operator S for which Sx^{-1} and Tx^{-1}. Thus the operator $S - T$ is continuous, and satisfies $(S - T)E^a = E^a(S - T)$ and $(S - T)x^{-1} = 0$. Therefore we have

$$\begin{aligned} 0 = E^a(S - T)x^{-1} &= (S - T)E^a x^{-1} \\ &= (S - T)\sum_{k=0}^{\infty} \binom{-1}{k} a^k x^{-1-k} \\ &= \sum_{k=0}^{\infty} \binom{-1}{k} a^k (S - T)x^{-1-k} \end{aligned}$$

for all $a \in K$. By alternatingly setting $a = 0$ and dividing by a we conclude that $(S - T)x^{-1-k} = 0$ for all $k \geqslant 0$. Thus $S = T$. Q.E.D.

We define a topology on the algebra of shift-invariant operators Ω as follows. A sequence T_m of operators converges to the operator T if given any inverse formal power series $f(x)$, and any continuous linear functional P in Γ^*, there exists an index m_0 such that if $m > m_0$ then $\langle P \mid T_m f(x)\rangle = \langle P \mid Tf(x)\rangle$. Under this topology, Ω is a topological algebra. Moreover, we have

PROPOSITION 10.5. *The isomorphism σ, mapping P^* onto Ω, is continuous.*

Proof. Let L_m be a sequence of linear functionals in P^*, with L_m converging to the zero functional. Then if P is a continuous linear functional in Γ^*, we have $\langle P \mid x^{-k}\rangle = 0$ for all but a finite number of exponents $k \geqslant 0$. Thus we may choose m_0 such that $m > m_0$ implies $\langle L_m \mid x^k\rangle = 0$ whenever $\langle P \mid x^{-k}\rangle \neq 0$. Then, if $m > m_0$,

$$\begin{aligned} \langle P \mid \sigma(L_m)\, x^{-n}\rangle &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L_m \mid x^k\rangle\langle P \mid x^{-n-k}\rangle \\ &= 0 \end{aligned}$$

for all $n \geqslant 1$. Thus $\langle P \mid \sigma(L_m) f(x)\rangle = 0$ for all inverse formal power series $f(x)$.

We can now prove the Expansion Theorem for shift-invariant operators on Γ.

THEOREM 16 (Expansion Theorem). *Let $T = \sigma(M)$ be a shift-invariant operator, and let $Q = \sigma(L)$ be a delta operator, with associated sequence $p_n(x)$. Then*

$$T = \sum_{k=0}^{\infty} \frac{\langle M \mid p_k(x)\rangle}{k!} Q^k.$$

Proof. The conclusion follows after applying the (continuous) isomorphism σ to the corresponding expansion of the linear functional M in powers of the delta functional L.

We call a shift-invariant operator Q a *delta operator* if $Q = \sigma(L)$ for some delta functional L.

PROPOSITION 10.6. *The sequence of inverse formal power series $f_{-n}(x)$, where the degree of $f_{-n}(x)$ is $-n$, is a factor sequence if and only if there exists a delta operator Q such that*

$$Qf_{-n}(x) = -nf_{-n-1}(x).$$

Proof. If $f_{-n}(x)$ is a factor sequence relative to the associated sequence $p_n(x)$ for the delta functional L, then

$$\begin{aligned} \sigma(L) f_{-n}(x) &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle L \mid p_k(x)\rangle f_{n-k}(x) \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \delta_{k,1} f_{-n-k}(x) \\ &= -nf_{-n-1}(x). \end{aligned}$$

Conversely, if $Qf_{-n}(x) = -nf_{-n-1}(x)$, for some delta operator Q, then if $p_n(x)$ is the associated sequence for Q, by the Expansion Theorem,

$$E^a = \sum_{k=0}^{\infty} \frac{p_k(a)}{k!} Q^k$$

and hence

$$\begin{aligned} f_{-n}(x + a) &= \sum_{k=0}^{\infty} \frac{p_k(a)}{k!} Q^k f_{-n}(x) \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} p_k(a) f_{-n-k}(x). \end{aligned}$$

Thus $f_{-n}(x)$ is a factor sequence.

COROLLARY 1. *Given any inverse formal power series $f_{-1}(x)$ of degree -1, and a sequence $p_n(x)$ of binomial type, there is a unique factor sequence $f_{-n}(x)$ for which $f_{-1}(x)$ is the first member.*

To preserve the analogy with Sheffer sequences, if Q is a delta operator, and $Qf_{-n}(x) = -nf_{-n-1}(x)$ we say that $f_{-n}(x)$ is a *factor sequence for Q*. Moreover, if $f_{-n}(x)$ is a factor sequence relative to $p_n(x)$, the associated sequence for $\sigma^{-1}(Q)$, then $f_{-n}(x)$ is a factor sequence for Q, and conversely. By the previous proposition, if T is an invertible shift-invariant operator and $f_{-n}(x)$ is a factor sequence for Q, then $Tf_{-n}(x)$ is also a factor sequence for Q, and $T^nf_{-n}(x)$ is a factor sequence for TQ.

Suppose $f_{-n}(x)$ and $h_{-n}(x)$ are factor sequences relative to the sequence $p_n(x)$ of binomial type. Then by Proposition 10.3, there exists a shift-invariant operator T for which $Tf_{-1}(x) = h_{-1}(x)$. But since $Tf_{-n}(x)$ is a factor sequence relative to $p_n(x)$, the previous corollary implies $Tf_{-n}(x) = h_{-n}(x)$. Thus any two factor sequences relative to the same sequence of binomial type are related by a shift-invariant operator.

The correspondence between linear functionals in the umbral algebra and the shift-invariant operators on Γ can be recast in a suggestive form as follows. Again we consider the algebra $\Gamma(x, y)$ of inverse formal power series in the variable x whose coefficients are polynomials in the variable y. If $T = \sigma(L)$ is a shift-invariant operator on Γ, we denote by the same letter T the operator $\mu(L)$, operating on the vector space of polynomials in the variable y. Then the identity

$$\begin{aligned} Tf_{-n}(x + a) &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon_a \mid p_k(y)\rangle \, Tf_{-n-k}(x) \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon_a L \mid p_k(y)\rangle f_{-k-n}(x) \end{aligned}$$

can be suggestively rewritten in the form

$$\sum_{k=0}^{\infty} \binom{-n}{k} p_k(y) \, Tf_{-n-k}(x) = \sum_{k=0}^{\infty} \binom{-n}{k} \, Tp_k(y) f_{-n-k}(x).$$

In other words, the action of a shift-invariant operator on a factor sequence can be "transferred" to the corresponding sequence of binomial type.

Proposition 10.6 shows that there is a strong analogy between factor sequences and Sheffer sequences. It is natural to single out those factor sequences which are the analogs of sequences of binomial type. We are led to define the *associated factor sequence* for a delta operator Q as the unique factor sequence $f_{-n}(x)$ for $Q = \sigma(L)$ whose first term is

$$f_{-1}(x) = \sum_{k=0}^{\infty} (-1)^k \langle L \mid x^{1+k}\rangle \, x^{-1-k}.$$

If we define the *derivative* of $Q = \sigma(L)$ to be $Q' = \sigma(\partial_A L)$, then since $\partial_A L$ is invertible, so is Q', and we have

$$f_{-1}(x) = Q'x^{-1}.$$

We come now to the explicit computation of associated factor sequences:

THEOREM 17 (Transfer Formulas). *Let $Q = DS$ be a delta operator on Γ, Then if $f_{-n}(x)$ is the associated factor sequence for Γ, we have*

(1) $f_{-n}(x) = Q'S^{n-1}x^{-n}$,

(2) $f_{-n}(x) = xS^n x^{-n-1}$.

Proof. (1) Let $g_{-n}(x) = Q'S^{n-1}x^{-n}$. Then $Qg_{-n}(x) = -ng_{-n-1}(x)$ and so by Proposition 10.6, $g_{-n}(x)$ is a factor sequence, relative to the associated sequence for $\sigma^{-1}(Q)$. Moreover $g_{-1}(x) = Q'x^{-1} = f_{-1}(x)$ and so by Corollary 1 to Proposition 10.6, $g_{-n}(x) = f_{-n}(x)$.

(2) Letting $\sigma(M) = S$, the following string of identities verifies the equivalence of the right-hand sides of (1) and (2), thus proving part (2),

$$\begin{aligned}
Q'S^{n-1}x^{-n} &= \sigma(\partial_A(AM)\,M^{n-1})\,x^{-n} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \partial_A(AM)\,M^{n-1} \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \langle (M + A\partial_A M)\,M^{n-1} \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n}{k} \frac{n+k}{n} \langle M^n \mid x^k \rangle\, x^{-n-k} \\
&= \sum_{k=0}^{\infty} \binom{-n-1}{k} \langle M^n \mid x^k \rangle\, x^{-n-k} \\
&= x\sigma(M^n)\,x^{-n-1} = xS^n x^{-n-1}.
\end{aligned}$$

COROLLARY 1. *Let $f_{-n}(x)$ be the associated factor sequence for the delta operator Q and let $g_{-n}(x)$ be the associated factor sequence for the delta operator $R = QP$, where P is invertible. Then*

(1) $g_{-n}(x) = R'P^{n-1}(Q')^{-1}f_{-n}(x)$,

(2) $g_{-n}(x) = xP^n x^{-1} f_{-n}(x)$.

Proof. Let $Q = DS$ and $R = DT$, where S and T are invertible operators, and $P = S^{-1}T$. To prove part (1), we observe that part (1) of Theorem 17 gives

$$S^{-n+1}(Q')^{-1}f_{-n}(x) = x^{-n} = T^{-n+1}(R')^{-1}g_{-n}(x).$$

The result follows by solving for $g_{-n}(x)$. Part (2) is proved in the same manner using part (2) of Theorem 17.

Since any two delta operators Q and R are related by $QP = R$ for some invertible shift-invariant operators, Corollary 1 relates any two associated factor sequences.

The following corollary is immediate from Theorem 17.

COROLLARY 2. *If $f_{-n}(x)$ is the associated factor sequence for the delta operator $Q = \sigma(L)$, and if $L = AM$, we have*

$$f_{-n}(x) = \sum_{k=0}^{\infty} \binom{-n-1}{k} \langle M^n \mid x^k \rangle \, x^{-n-k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!} \, x^{-n-k}.$$

The Transfer Formula allows us to compute explicitly the coefficients of a factor sequence.

COROLLARY 3. *Let $g_{-n}(x)$ be a factor sequence relative to the delta functional $L = AM$, and let $g_{-n}(x) = Tf_{-n}(x)$, where $f_{-n}(x)$ is the associated factor sequence for L and $T = \sigma(N)$. Then*

$$g_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle N(L^n)' \mid x^{n+k-1} \rangle}{n!} \, x^{-n-k}.$$

Proof. The Transfer Formula gives:

$$g_{-n}(x) = Tf_{-n}(x) = T\sigma(L'M^{n-1}) \, x^{-n}$$

$$= \sigma(NL'M^{n-1}) \, x^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} \langle NL'M^{n-1} \mid x^k \rangle \, x^{-n-k}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\langle N(L^n)' \mid x^{n+k-1} \rangle}{n!} \, x^{-n-k}.$$

We next derive a recurrence formula for the associated factor sequence:

COROLLARY 4 (Recurrence Formula). *If $f_{-n}(x)$ is the associated factor sequence for the delta operator Q, then*

$$f_{-n-1}(x) = Q' x^{-1} f_{-n}(x).$$

Proof. By the second formula in the preceding theorem we have

$$f_{-n}(x) = xS^n x^{-n-1}$$

and so

$$Q' x^{-1} f_{-n}(x) = Q' S^n x^{-n-1},$$

which, by the first formula, equals $f_{-n-1}(x)$.

Given an invertible shift-invariant operator T and a delta operator Q, with associated factor sequence $f_{-n}(x)$, we say that the factor sequence $g_{-n}(x) = Tf_{-n}(x)$ is the *(T, Q)-factor sequence*. Clearly, any such pair (T, Q) determines a unique factor sequence, and conversely. Notice that, in the theory of Sheffer sequences, the role of T is played by T^{-1}.

Now we derive a recurrence formula for factor sequences, which is the analog of Theorem 14.

If $f_{-n}(x)$ is the associated factor sequence for the delta functional $M = \sigma^{-1}(Q)$, the *shift* θ_Q, is the linear operator on Γ defined by $\theta_Q f_{-n}(x) = f_{-n+1}(x)$, for $n \geqslant 2$. Notice that θ_Q is not everywhere defined on Γ. Now if T is a shift-invariant operator on Γ, by the Expansion Theorem we may expand T in powers of Q, say $T = g(Q)$. It is straightforward to verify that on the algebra Γ' of inverse formal power series of degree at most -2, the operator $T\theta_Q - \theta_Q T$ satisfies

$$T\theta_Q - \theta_Q T = g'(Q).$$

Thus, on Γ', $T\theta_Q - \theta_Q T$ is shift-invariant and we denote it by $\partial_Q T$.

PROPOSITION 10.7. *Let $g_{-n}(x)$ be a (T, Q)-factor sequence, and let $f_{-n}(x)$ be the associated factor sequence for Q. Then*

$$g_{-n+1}(x) = (T^{-1}\partial_Q T + \theta_Q)\, g_{-n}(x),$$

for $n \geqslant 2$.

Proof. The result follows from

$$\begin{aligned} g_{-n+1}(x) &= Tf_{-n+1}(x) = T\theta_Q f_{-n}(x) \\ &= T\theta_Q T^{-1} g_{-n}(x) = ((T\theta_Q - \theta_Q T)T^{-1} + \theta_Q)\, g_{-n}(x) \\ &= (T^{-1}\partial_Q T + \theta_Q)\, g_{-n}(x). \end{aligned}$$

We can now study the *umbral composition* of two factor sequences, say $f_{-n}(x) = \sum_{k=n}^{\infty} c_{n,k} x^{-k}$ and $g_{-n}(x)$. The umbral composition $f_{-n}(\mathbf{g}(x))$ is the sequence

$$f_{-n}(\mathbf{g}(x)) = \sum_{k=n}^{\infty} c_{n,k} g_{-k}(x).$$

LEMMA 1. *If L and M are delta functionals in P^*, then*

$$\langle (M \circ L)^n \mid x^{n+i} \rangle = \sum_{k=0}^{i} \frac{1}{(n+k)!} \langle L^n \mid x^{n+k} \rangle \langle M^{n+k} \mid x^{n+i} \rangle .$$

Proof. Let $p_n(x)$ and $q_n(x)$ be the conjugate sequences for L and M, respectively. Then by Proposition 6.2, $M \circ L = \widetilde{\tilde{L} \circ \tilde{M}}$ is the conjugate sequence for $q_n(\mathbf{p}(x))$. This yields the following string of identities:

$$\begin{aligned}
\sum_{l=0}^{n+i} \frac{\langle (M \circ L)^l \mid x^{n+i} \rangle}{l!} x^l &= q_{n+i}(\mathbf{p}(x)) \\
&= \sum_{j=0}^{n+i} \frac{\langle M^j \mid x^{n+i} \rangle}{j!} \sum_{l=0}^{j} \frac{\langle L^l \mid x^j \rangle}{l!} x^l \\
&= \sum_{l=0}^{n+i} \frac{1}{l!} \sum_{j=l}^{n+i} \frac{\langle M^j \mid x^{n+i} \rangle \langle L^l \mid x^j \rangle}{j!} x^l \\
&= \sum_{l=0}^{n+i} \frac{1}{l!} \sum_{k=0}^{n+i-l} \frac{\langle M^{k+l} \mid x^{n+i} \rangle \langle L^l \mid x^{k+l} \rangle}{(k+l)!} x^l .
\end{aligned}$$

Comparing the coefficients of x^n in the first and last formula gives the result.

We can now prove

THEOREM 18. *If $f_{-n}(x)$ is the associated factor sequence for the delta operator $Q = \sigma(L)$ and if $g_{-n}(x)$ is the associated factor sequence for the delta operator $R = \sigma(M)$, then the umbral composition $f_{-n}(\mathbf{g}(x))$ is the associated factor sequence for the delta operator $\sigma(M \circ L)$.*

Proof. By Corollary 2 of Theorem 17 we have

$$f_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!} x^{-n-k}$$

and

$$g_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle M^n \mid x^{n+k} \rangle}{n!} x^{-n-k} .$$

Thus

$$\begin{aligned}
f_{-n}(\mathbf{g}(x)) &= \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!} \sum_{j=0}^{\infty} (-1)^j \frac{\langle M^{n+k} \mid x^{n+k+j} \rangle}{(n+k)!} x^{-n-k-j} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{k+j} \frac{1}{n!\,(n+k)!} \langle L^n \mid x^{n+k} \rangle \langle M^{n+k} \mid x^{n+k+j} \rangle x^{-n-k-j} \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{n!} \sum_{k=0}^{i} \frac{1}{(n+k)!} \langle L^n \mid x^{n+k} \rangle \langle M^{n+k} \mid x^{n+i} \rangle x^{-n-i} .
\end{aligned}$$

By Lemma 1, the last member simplifies to

$$\sum_{i=0}^{\infty} (-1)^i \frac{\langle (M \circ L)^n \mid x^{n+i} \rangle}{n!} x^{-n-i},$$

which is the associated factor sequence for the delta functional $M \circ L$.

We may carry the analogy a step further with

THEOREM 19. *If $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence and if $g_{-n}(x)$ is the $(h(D), k(D))$-factor sequence, then the umbral composition $g_{-n}(\mathbf{f}(x))$ is the factor sequence with pair*

$$(f(D)\, h(g(D)),\, k(g(D))).$$

Proof. Suppose $g(D)$ has associated sequence $p_n(x)$ and associated factor sequence $h_{-n}(x)$, and suppose $k(D)$ has associated sequence $q_n(x)$ and associated factor sequence $k_{-n}(x)$. Then if we let $T = f(D)$ and $S = h(D)$,

$$\begin{aligned} g_{-n}(\mathbf{f}(x)) = Sk_{-n}(\mathbf{f}(x)) &= \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon \mid Sq_k(x) \rangle\, k_{-n-k}(\mathbf{f}(x)) \\ &= T \sum_{k=0}^{\infty} \binom{-n}{k} \langle \epsilon \mid Sq_k(x) \rangle\, k_{-n-k}(\mathbf{h}(x)). \end{aligned}$$

Defining the umbral operator $\alpha\colon x^n \to p_n(x)$, then $(\alpha^{-1})^* A = g(A)$ and so the above equals

$$\begin{aligned} T &\sum_{k=0}^{\infty} \binom{-n}{k} \langle (\alpha^{-1})^*\, \sigma^{-1}(S) \mid q_k(\mathbf{p}(x)) \rangle\, k_{-n-k}(\mathbf{h}(x)) \\ &= T\sigma((\alpha^{-1})^*\, \sigma^{-1}(S))\, k_{-n}(\mathbf{h}(x)) \\ &= f(D)\, h(g(D))\, k_n(\mathbf{h}(x)). \end{aligned}$$

The result follows.

We can now give a solution to the connection constants problem for factor sequences.

COROLLARY 1. *Suppose $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence and $g_{-n}(x)$ is the $(h(D), k(D))$-factor sequence, and suppose*

$$g_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} f_{-k}(x)$$

for constants $c_{-n,k}$. Then the sequence $r_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} x^{-k}$ is a factor sequence for the pair

$$\left(\frac{h(g^{-1}(D))}{f(g^{-1}(D))},\, k(g^{-1}(D))\right).$$

COROLLARY 2. *Suppose $f_{-n}(x)$ is the associated factor sequence for $f(D)$ and $g_{-n}(x)$ is the associated factor sequence for $g(D)$, and suppose*

$$g_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} f_{-k}(x),$$

for constants $c_{-n,k}$. Then the sequence $r_{-n}(x) = \sum_{k=0}^{\infty} c_{-n,k} x^{-k}$ is the associated factor sequence for $g(f^{-1}(D))$.

11. APPLICATIONS TO FORMAL POWER SERIES

Given a formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k,$$

we can define a linear functional L in P^* by $\langle L \mid x^k \rangle = a_k$. We call L the *generating functional* of the sequence a_k. The series $f(t)$ is the indicator of the linear functional L and $L = f(A)$.

When $a_0 = 0$ and $a_1 \neq 0$ we call $f(t)$ a *delta series*. We have seen that the composition $f(g(t))$ is well defined when the constant coefficient of $g(t)$ vanishes, in particular when $g(t)$ is a delta series, and that

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle f(g(A)) \mid x^k \rangle}{k!} t^k. \qquad (*)$$

If $f(t)$ is the indicator of the delta functional L, we have seen (Corollary 1 to Theorem 6) that

$$f^{-1}(t) = \sum_{k=0}^{\infty} \frac{\langle \tilde{L} \mid x^k \rangle}{k!} t^k.$$

That is, the reciprocal series $f^{-1}(t)$ is the indicator of $\tilde{L}$, the reciprocal functional to L.

If $f(t)$ and $g(t)$ are the indicators of the delta functionals L and M, then Theorem 6 tells us that $f(g(t))$ is the indicator of the delta functional $f(g(A)) = M \circ L$, and $(*)$ becomes

$$f(g(t)) = \sum_{k=0}^{\infty} \frac{\langle M \circ L \mid x^k \rangle}{k!} t^k. \qquad (**)$$

The problem of determining the composition of formal power series is thus equivalent to the problem of determining the composition of delta functionals. It turns out that the latter can often be explicitly computed by the present methods, as we shall see.

We can relate the composition of delta functionals to the umbral composition of sequences of binomial type. Suppose $p_n(x)$ and $q_n(x)$ are the conjugate sequences for the delta functionals $L = f(A)$ and $M = g(A)$, hence the associated sequences for $\tilde{L}$ and $\tilde{M}$. Then $q_n(\mathbf{p}(x))$ is the associated sequence for $\widetilde{\tilde{L} \circ \tilde{M}}$, and thus the conjugate sequence for $\widetilde{\tilde{L} \circ \tilde{M}} = (g^{-1} \circ f^{-1})^{-1}(A) = (f \circ g)(A) = M \circ L$. By definition therefore,

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} \frac{\langle (M \circ L)^k \mid x^n \rangle}{k!} x^k. \qquad (***)$$

Comparing $(**)$ and $(***)$, we see that the coefficient of $t^n/n!$ in $f(g(t))$ is the linear coefficient in $q_n(\mathbf{p}(x))$.

By definition of umbral composition, we have

$$q_n(\mathbf{p}(x)) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L^k \mid x^j \rangle x^k$$

and so $(***)$ gives

$$\langle (M \circ L)^k \mid x^n \rangle = \sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L^k \mid x^j \rangle.$$

Thus the coefficient of $t^n/n!$ in $f(g(t))$ is

$$\sum_{j=0}^{n} \frac{1}{j!} \langle M^j \mid x^n \rangle \langle L \mid x^j \rangle. \qquad (****)$$

As an example, we compute the power series $(1 + g(t))^r - 1$, where r is a real number, and $g(t)$ is a delta series. Here $f(t)$ is the delta series $f(t) = (1 + t)^r - 1$. Expanding $L = f(A)$ in powers of A by means of the binomial series, we find that $\langle L \mid x^j \rangle = (r)_j - \delta_{j,0}$ and thus the coefficient of $t^n/n!$ in $(1 + g(t))^r - 1$ is

$$\sum_{j=1}^{n} \binom{r}{j} \langle M^j \mid x^n \rangle. \qquad (*****)$$

Formula $(****)$ yields at once Faà di Bruno's formula for the composition of two formal power series. The special cases of this formula to be found in the literature are obtained by explicitly computing a sequence of binomial type. For example, setting $g(t) = \log(1 + t)$, we find immediately from $(****)$ that the coefficients of $f(\log(1 + t))$ are given by umbral composition of $\phi_n(\mathbf{a})$, when $\phi_n(x)$ are the Stirling polynomials, and $\mathbf{a}$ is the umbral sequence a_n of coefficients of $f(t)$. Similarly, the coefficients of $f(e^t - 1)$ are given by $\phi_n(\mathbf{a})$, when $\phi_n(x)$ are the exponential polynomials.

8.2. We now compute the reciprocal polynomials to the Bell polynomials; that is, the associated sequence $p_n(x)$ for the delta functional $L = AN = x_1A + x_2A^2/2! + x_3A_3/3! + \cdots$. We may take $x_1 = 1$. We wish to use the Transfer Formula $p_n(x) = xP^{-n}x^{n-1}$, where $P = \mu(N)$, so we compute P^{-n}.

The indicator of P^{-n} is $(1 + g(t))^{-n}$, where $g(t)$ is the indicator of $M = N - \epsilon$. Hence, the coefficient of $D^k/k!$ in the expansion of P^{-n} is given by (*****), with $r = -n$ and $n = k$. The computation of (*****) is straightforward by binomial expansion and by the identity

$$\langle N^i \mid x^k \rangle = \frac{\langle L^i \mid x^{k+i} \rangle}{(k+i)_i} = \frac{i!\,k!}{(k+i)!} B_{k+i,i}\,.$$

We obtain

$$P^{-n} = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^i \frac{(n+j-1)_j}{(j-i)!\,(k+i)!} B_{k+i,i} D^k$$

and then

$$b_n(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^i \frac{(n+j-1)_{k+j}}{(j-i)!\,(k+i)!} B_{k+i,i} x^{n-k}.$$

Similarly, the associated factor sequence to the Bell polynomials is

$$f_{-n}(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{k} \sum_{i=0}^{j} (-1)^{k+j-i} \frac{(n+k)_{k+j}}{(j-i)!\,(k+i)!} B_{k+i,i} x^{-n-k}.$$

Umbral techniques can be used in several ways to compute power series expansions.

Consider the function $[\log(1+t)]^r$. If we take $f(t) = t^r + t$ and $g(t) = \log(1+t)$, then both $f(t)$ and $g(t)$ are delta series, and the expansion of $f(g(t))$ differs from the desired one only by the addition of $\log(1+t)$. To find the coefficients of $f(g(t))$ we compute the umbral composition $q_n(\mathbf{p}(x))$, where $q_n(x)$ is the conjugate sequence for $g(A)$ and $p_n(x)$ is the conjugate sequence for $f(A)$. The sequence $q_n(x)$ is the associated sequence for $g^{-1}(A) = e^A - \epsilon = \epsilon_1 - \epsilon$, and we have seen that $q_n(x) = (x)_n = \sum_{k=0}^{n} s(n,k)x^k$, where $s(n,k)$ are Stirling numbers of the first kind. It is even easier to compute the polynomials $p_n(x)$. For

$$p_n(x) = \sum_{k=0}^{n} \frac{\langle (A^r + A)^k \mid x^n \rangle}{k!} x^k$$

$$= \sum_{k=0}^{n} \binom{k}{(n-k)/(n-1)} \frac{n!}{k!} x^k.$$

Thus the linear coefficient in $q_n(\mathrm{p}(x))$ is $r!\, s(n, r) + s(n, 1) = r!\, s(n, r) + (-1)^{n-1}(n-1)!$, and so

$$[\log(1+t)]^r = \sum_{k=1}^{\infty} \frac{r!\, s(k, r)}{k!} x^k.$$

Consider next the function $\log(1 + \sin t)$. We have $f(t) = \log(1 + t)$ and $g(t) = \sin t$. Since $M = g(A) = \sin A$, by expansion we have

$$M^j = \frac{1}{(2i)^j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} e^{iA(2k-j)}$$

But $\langle e^{mA} \mid x^n \rangle = \langle (mA)^n/n! \mid x^n \rangle = m^n$ and so

$$\langle M^j \mid x^n \rangle = \frac{1}{(2i)^j} \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} i^n (2k-j)^n.$$

Now if $L = f(A) = \log(1 + A)$, then

$$\langle L \mid x^j \rangle = \langle [(-1)^{j+1}/j]\, A^j \mid x^j \rangle = (-1)^{j+1}(j-1)!$$

Thus the coefficient of $t^n/n!$ in $\log(1 + \sin t)$ is

$$\sum_{j=0}^{n} \sum_{k=0}^{j} \frac{(-1)^{k+1}}{j 2^j} i^{n-j} \binom{j}{k} (2k-j)^n.$$

Next we give the generating functions of associated and Sheffer sequences.

If $p_n(x)$ is the associated sequence for the delta functional $L = f(A)$, then by the Expansion Theorem

$$\epsilon_y = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} f(A)^k.$$

Passing to indicators gives

$$e^{yt} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} f(t)^k.$$

Finally, replacing $f(t)$ by t gives

$$e^{yf^{-1}(t)} = \sum_{k=0}^{\infty} \frac{p_k(y)}{k!} t^k.$$

Thus, if $f(t)$ is a delta series its reciprocal is the series

$$f^{-1}(t) = \sum_{k=0}^{\infty} \frac{p_k{}'(0)}{k!} t^k,$$

where $p_n(x)$ are the associated polynomials to $f(A)$. Our recipe for finding the reciprocal of a formal power series is thus the following: Compute the associated polynomials, possibly by using the Recurrence Formula or the Transfer Formula, and then take the coefficients of x in these polynomials. It turns out that computing the whole polynomial sequence is often speedier than computing a single coefficient.

We turn to some more examples.

2.6. The exponential polynomials $\phi_n(x) = \sum_{k=0}^{n} S(n, k)x^k$ are the associated polynomials for $f^{-1}(A) = \log(1 + A)$. Thus

$$e^{x(e^t-1)} = \sum_{k=0}^{\infty} \frac{\phi_k(x)}{k!} t^k.$$

4.5. The Abel polynomials $p_n(x) = x(x - an)^{n-1}$ have the generating function

$$e^{xf^{-1}(t)} = \sum_{k=0}^{\infty} \frac{x(x-ak)^{k-1}}{k!} t^k,$$

where $f(t) = te^{at}$.

7.5. The basic Laguerre polynomials

$$L_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \binom{n-1}{k-1} (-x)^k$$

have the generating function

$$e^{xt/(t-1)} = \sum_{k=0}^{\infty} \frac{L_k(x)}{k!} t^k.$$

If $s_n(x)$ is the Sheffer sequence for $N = f(t)$ with respect to the delta functional $L = g(t)$, then by Theorem 10

$$\epsilon_y N^{-1} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} L^k.$$

Taking indicators and simplifying as before, we find the generating function

$$\frac{1}{f(g^{-1}(t))} e^{yg^{-1}(t)} = \sum_{k=0}^{\infty} \frac{s_k(y)}{k!} t^k.$$

For the higher Laguerre polynomials $L_n^{(\alpha)}(x)$ we have $f(t) = (1 - t)^{-\alpha-1}$ and $g(t) = t(t - 1)^{-1}$, hence

$$(1-t)^{-\alpha-1} e^{xt/(t-1)} = \sum_{k=0}^{\infty} \frac{L_k^{(\alpha)}(x)}{k!} t^k.$$

For the Hermite polynomials of variance v, $H_n^{(v)}(x)$, we have $f(t) = e^{vt^2/2}$ and $g(t) = t$. Thus

$$e^{-vt^2/2}e^{xt} = \sum_{k=0}^{\infty} \frac{H_k^{(v)}(x)}{k!} t^k.$$

Lagrange's inversion formula is immediate in the present notation. It states that if $f(t)$ is a delta series, then the nth coefficient in $f^{-1}(t)^k$ equals the $(n-k)$th coefficient in $(f(t)/t)^{-n}$, multiplied by k/n. In our notation, this reads

$$\frac{\langle \tilde{L}^k \mid x^n \rangle}{n!} = \frac{k}{n} \frac{\langle M^{-n} \mid x^{n-k} \rangle}{(n-k)!},$$

where the indicator of $L = AM$ is $f(t)$. The verification of this fact is now a trivial computation with adjoints. If $p_n(x)$ is the associated sequence for L, then using the Transfer Formula we find

$$\begin{aligned}\langle \tilde{L}^k \mid x^n \rangle &= \langle A^k \mid p_n(x) \rangle = \langle A^k \mid x\mu(M)^{-n} x^{n-1} \rangle \\ &= \langle kA^{k-1} \mid \mu(M)^{-n} x^{n-1} \rangle = k\langle A^{k-1}M^{-n} \mid x^{n-1} \rangle \\ &= k\langle M^{-n} \mid D^{k-1}x^{n-1} \rangle = \frac{k}{n} (n)_k \langle M^{-n} \mid x^{n-k} \rangle,\end{aligned}$$

as desired.

We can just as easily prove the variants of the Lagrange inversion formulas, for example: Given two delta series $f(t)$ and $g(t)$, the nth coefficient in $g(f^{-1}(t))$, multiplied by n, equals the $(n-1)$st coefficient in $g'(t)(f(t)/t)^{-n}$. In symbols:

$$\langle g(f^{-1}(A)) \mid x^n \rangle = \left\langle g'(A) \left(\frac{f(A)}{A}\right)^{-n} \middle| \, x^{n-1} \right\rangle.$$

But this is also an immediate consequence of adjointness. Indeed, the right side can be written as

$$\langle g(A) \mid xP^{-n}x^{n-1} \rangle,$$

where $P = \mu(f(A)/A)$. We recognize an instance of the Transfer Formula: $\langle g(A) \mid p_n(x) \rangle$, where $p_n(x)$ are the associated polynomials for $f(A)$. Letting α be the umbral operator mapping x^n to $p_n(x)$, and recalling that the automorphism α^* maps $f(A)$ to A, we have

$$\begin{aligned}\langle g(A) \mid p_n(x) \rangle &= \langle g(A) \mid \alpha x^n \rangle \\ &= \langle \alpha^* g(A) \mid x^n \rangle = \langle g(f^{-1}(A)) \mid x^n \rangle.\end{aligned}$$

It is hard to imagine a simpler proof.

A variant of the same reasoning gives Hermite's version of the Lagrange inversion formula, namely,

$$\left\langle \frac{Ag(f^{-1}(A))}{f^{-1}(A)\,f'(f^{-1}(A))} \,\middle|\, x^n \right\rangle = \left\langle g(A) \left(\frac{f(A)}{A}\right)^{-n} \,\middle|\, x^n \right\rangle.$$

The generating functions of factor sequences cannot be expressed by ordinary generating functions, and lead us to introduce an analogous formal device. Let $f(t)$ be a delta series and let $g(t)$ be a formal power series. We define the *Cigler transform* of the pair (f, g), in symbols

$$F(x) = \int_{-\infty}^{0} g(t)\, e^{xf(t)}\, dt$$

to be the formal power series obtained after term-by-term integration of

$$\int_{-\infty}^{0} g(f^{-1}(s))(f^{-1})'(s)\, e^{sx}\, ds.$$

The point is that one can compute with the Cigler transform in much the same way as with an ordinary integral, for example,

$$\int_{-\infty}^{0} g(t)\, e^{xf(t)}\, dt + \int_{-\infty}^{0} g(t)\, e^{xh(t)}\, dt = \int_{-\infty}^{0} g(t)\, e^{x(f(t)+h(t))}\, dt;$$

thus the Cigler transform is an "integral" analog of a formal power series.

If $f_{-n}(x)$ is the associated factor sequence of the delta operator $Q = f(D)$, then

$$\begin{aligned} f_{-1}(x) &= Q' \frac{1}{x} = \int_{-\infty}^{0} f'(t)\, e^{xt}\, dt \\ &= \int_{-\infty}^{0} e^{xf^{-1}(t)}\, dt, \end{aligned}$$

and more generally, applying Q successively,

$$f_{-n}(x) = \frac{(-1)^{n-1}}{(n-1)!} \int_{-\infty}^{0} t^{n-1} e^{xf^{-1}(t)}\, dt;$$

thus the generating function of $f_{-n}(x)$ can be expressed by the Cigler transform:

$$\sum_{k \geqslant 1} f_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xf^{-1}(t)}\, dt.$$

Similarly, if $g_n(x) = Tf_{-n}(x)$ is the factor sequence obtained from the associated factor sequence by applying the invertible shift-invariant operator

$T = g(D)$, then by a Cigler transform one can express the generating function of $g_{-n}(x)$ in the form

$$g_{-1}(x) = TQ' \frac{1}{x} = \int_{-\infty}^{0} g(t) f'(t)\, e^{xt}\, dt$$

$$= \int_{-\infty}^{0} g(f^{-1}(t))\, e^{xf^{-1}(t)}\, dt,$$

whence

$$g_{-n}(x) = Q^{n-1} T Q' \frac{1}{x} = \int_{-\infty}^{0} \frac{(-1)^{n-1}\, t^{n-1}}{(n-1)!} g(f^{-1}(t))\, e^{xf^{-1}(t)}\, dt$$

and again

$$\sum_{k \geqslant 1} g_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} g(f^{-1}(t))\, e^{-st+xf^{-1}(t)}\, dt.$$

12. Examples of Factor Sequences

2.7. The *negative factorial powers*

$$(x)_{-n} = \frac{1}{(x+1)(x+2)\cdots(x+n)}$$

are the associated factor sequence for the operator $\Delta = \sigma(\epsilon_1 - \epsilon)$. This follows from the Recurrence Formula:

$$\Delta' x^{-1}(x)_{-n} = E^1 x^{-1}(x)_{-n} = (x)_{-n-1}\,.$$

Thus we immediately have

$$(x+y)_{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (y)_k\, (x)_{-n-k}\,,$$

as well as

$$\Delta^m (x)_{-n} = (-n)_m (x)_{-n-m}\,.$$

Corollary 2 to Theorem 17 gives

$$(x)_{-n} = \sum_{k=0}^{\infty} (-1)^k \frac{\langle L^n \mid x^{n+k} \rangle}{n!}\, x^{-n-k},$$

where $L = \epsilon_1 - \epsilon$. But $q_n(x) = \sum_{k=0}^{n} (\langle L^k \mid x^n \rangle / k!)\, x^k$ is the conjugate sequence for L. Thus

$$\langle L^n \mid x^{n+k} \rangle / n! = S(n+k, n),$$

where $S(n, k)$ are the Stirling numbers of the second kind. Therefore, we obtain the identity

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=0}^{\infty} (-1)^k S(n+k, n)\, x^{-n-k}.$$

3.5. A similar treatment may be given to the associated factor sequence for $\nabla = \sigma(\epsilon - \epsilon_{-1})$, which is

$$\langle x\rangle_{-n} = \frac{(-1)^n}{(x-1)(x-2)\cdots(x-n)}.$$

We state as a sample:

$$\langle x\rangle_{-n} = \sum_{k=0}^{\infty} S(n+k, n)\, x^{-n-k}.$$

4.6. The associated factor sequence for the Abel operator De^{aD} is given most easily by the Transfer Formula:

$$\begin{aligned} A_{-n}(x, a) &= xE^{an}x^{-n-1} \\ &= x(x+an)^{-n-1}. \end{aligned}$$

Thus the identity

$$(x+y)(x+y+an)^{-n-1} = \sum_{k=0}^{\infty} \binom{-n}{k} xy(y+ak)^{k-1} (x+a(n-k))^{-n-k-1}$$

is immediate.

Corollary 2 of Theorem 17 yields

$$A_{-n}(x, a) = \sum_{k=0}^{\infty} \binom{-n-1}{k} (an)^k\, x^{-n-k}.$$

6.6. The *negative Steffensen polynomials* are the associated factor sequence for $e^{-D/2}(e^D - 1)$, and thus by Corollary 1 to Theorem 17,

$$\begin{aligned} x^{[-n]} &= xE^{-n/2}x^{-1}(x)_{-n} \\ &= x(x-n/2-1)_{-n-1} \\ &= \frac{x}{(x-n/2)(x-n/2+1)\cdots(x+n/2)}. \end{aligned}$$

7.6. The associated factor sequence for the Laguerre operator $D/(D-I)$ is

$$\begin{aligned} L_{-n}(x) &= x(D-I)^{-n}\, x^{-n-1} \\ &= \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^{-n-k} (-n-1)_k\, x^{-n-k}, \end{aligned}$$

this by the Transfer Formula.

In view of Theorem 18, the factor sequence $L_{-n}(x)$ is self-reciprocal, as expected.

If $f_{-n}(x)$ is the associated factor sequence for the delta functional L, then the *conjugate factor sequence* $g_{-n}(x)$ to $f_{-n}(x)$ is the associated factor sequence for $\tilde{L}$, the reciprocal to L. By Theorem 18, we have $f_{-n}(g(x)) = g_{-n}(\mathbf{f}(x)) = x^{-n}$.

2.8. The *negative exponential polynomials* $\phi_{-n}(x)$ are the conjugate factor sequence to $(x)_{-n}$, and are therefore the associated factor polynomials for $\log(I + D)$.

Corollary 2 to Theorem 17 gives

$$\phi_{-n}(x) = \sum_{k=0}^{\infty} (-1)^k s(n + k, n)\, x^{-n-k},$$

where $s(n, k)$ are the Stirling numbers of the first kind.

We have by Theorem 18 the umbral substitutions,

$$\begin{aligned} x^{-n} &= \sum_{k=0}^{\infty} (-1)^k s(n + k, n)(x)_{-n-k} \\ &= \sum_{k=0}^{\infty} (-1)^k S(n + k, n)\, \phi_{-n-k}(x), \end{aligned}$$

which are equivalent to the Stirling number identities

$$\sum_{k=0}^{j} s(n + k, n)\, S(n + j, n + k) = \delta_{j,0}$$

and

$$\sum_{k=0}^{j} S(n + k, n)\, s(n + j, n + k) = \delta_{j,0}.$$

By the Recurrence Formula,

$$\phi_{-n-1}(x) = (I + D)^{-1}\, x^{-1} \phi_{-n}(x)$$

and so

$$\begin{aligned} \phi_{-n}(x) &= x(I + D)\, \phi_{-n-1}(x) \\ &= \cdots = [x(I + D)]^k\, \phi_{-n-k}(x). \end{aligned}$$

Taking $n = 1$ and $k = n$ gives

$$\begin{aligned} \phi_{-1}(x) &= [x(I + D)]^n\, \phi_{-1-n}(x) \\ &= e^{-x} (xD)^n\, e^x \phi_{-1-n}(x). \end{aligned}$$

3.6. The conjugate factor sequence to the sequence $A_{-n}(x, a)$ is computed by Corollary 2 to Theorem 17:

$$\mu_{-n}(x, a) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle \tilde{L}^n \mid x^{n+k} \rangle}{n!} x^{-n-k},$$

where $\tilde{L}$ is the reciprocal to $\epsilon_a A$. However,

$$\frac{\langle \tilde{L}^n \mid x^{n+k} \rangle}{n!} = \binom{n+k-1}{n-1} \frac{[-a(n+k)]^k}{n!}$$

and thus

$$\mu_{-n}(x, a) = \sum_{k=0}^{\infty} \binom{-n}{k} \frac{[-a(n+k)]^k}{n!} x^{-n-k}.$$

The umbral identity $x^{-n} = \mu_{-n}(\mathbf{A}(x, a), a)$ gives the elegant power series identity:

$$x^{-n-1} = \sum_{k=0}^{\infty} \binom{-n}{k} \frac{[-a(n+k)]^k}{n!} [x + a(n+k)]^{-n-k-1}.$$

We turn now to some connection-constant problems.

2.9. Determine the connection constants $c_{-n,k}$ in

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=1}^{\infty} \frac{c_{-n,k}(-1)^k}{(x-1)(x-2)\cdots(x-k)}.$$

Since $(x)_{-n}$ is the associated factor sequence for $g(D) = e^D - I$ and $\langle x \rangle_{-k}$ is the associated factor sequence for $f(D) = I - e^{-D}$, we have $g(f^{-1}(D)) = D/(I - D)$ and so

$$\sum_{k=1}^{\infty} c_{-n,k} x^{-k} = L_{-n}(-x).$$

Thus

$$\frac{1}{(x+1)(x+2)\cdots(x+n)} = \sum_{k=1}^{\infty} \binom{-n}{k-n} \frac{(-n-1)_{k-n}}{(x-1)(x-2)\cdots(x-k)}.$$

6.7. Determine the constants $c_{-n,k}$ in

$$\frac{x}{(x-n/2)(x-n/2+1)\cdots(x+n/2)} = \sum_{k=1}^{\infty} \frac{c_{-n,k}}{(x+1)(x+2)\cdots(x+k)}.$$

Since $x^{[-n]}$ is the associated factor sequence for $g(D) = e^{-D/2}(e^D - I)$ and

$(x)_{-n}$ is the associated factor sequence for $f(D) = e^D - I$, we have $g(f^{-1}(D)) = D(I + D)^{-1/2}$. Therefore,

$$\sum_{k=1}^{\infty} c_{-n,k}x^{-k} = x(I + D)^{-n/2}\, x^{-n-1}$$

$$= \sum_{k=1}^{\infty} \binom{-n/2}{k} (-n-1)_k\, x^{-n-k},$$

and so

$$c_{-n,k} = \binom{-n/2}{k-n} (-n-1)_{k-n}\,.$$

7.8. Determine the constants $c_{-n,k}$ relating the Laguerre polynomials to the exponential polynomials:

$$L_{-n}(x) = \sum_{k=1}^{\infty} c_{-n,k}\phi_{-k}(-x).$$

Since $L_{-n}(x)$ is the associated factor sequence for $g(D) = D/(D - I)$ and $\phi_{-n}(-x)$ is the associated factor sequence for $f(D) = \log(I - D)$, we have $g(f^{-1}(D)) = I - e^{-D}$ and so

$$\sum_{k=1}^{\infty} c_{-n,k}x^{-k} = \langle x \rangle_{-n}\,.$$

Thus

$$c_{-n,k} = S(k, n)$$

and

$$L_{-n}(x) = \sum_{k=1}^{\infty} S(k, n)\, \phi_{-k}(-x).$$

We postpone discussion of Hermite and higher-order Laguerre factor sequences until Section 13.

We conclude with some examples of Cigler transforms.

2.10. For the factor sequence $(x)_{-n}$, we obtain as a special case of the Cigler transform Nielsen's factorial expansion of the incomplete gamma function:

$$\sum_{k\geqslant 1} (x)_{-k}\, s^{k-1} = \int_{-\infty}^{0} e^{-st}(1 + t)^x\, dt.$$

4.9. For the Abel sequence $A_{-n}(x, a)$ we obtain

$$\sum_{k\geqslant 1} A_{-k}(x, a)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xf^{-1}(t)}\, dt$$
$$= \int_{-\infty}^{0} e^{-st} \left(\frac{t}{f^{-1}(t)}\right)^{x/a} dt,$$

where

$$f(t) = te^{at}.$$

7.9. For the Laguerre sequence $L_{-n}(x)$ we obtain

$$\sum_{k\geqslant 1} L_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xt/(t-1)}\, dt.$$

2.11. For the negative exponential polynomials:

$$\sum_{k\geqslant 1} \phi_{-k}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+x(e^{t}-1)}\, dt.$$

4.10. For the sequence $\mu_{-n}(x, a)$, reciprocal to the Abel factor sequence:

$$\sum_{k\geqslant 1} \mu_{-k}(x, a)\, s^{k-1} = \int_{-\infty}^{0} e^{-st+xte^{at}}\, dt.$$

13. Hermite and Laguerre Polynomials

Theories of special functions often present those functions that are of frequent occurrence as special cases of some general concept, and the present development is no exception. In actual fact, however, those special sequences of polynomials that have actually occurred are best defined by their own structural conditions. Such axiomatic descriptions remain largely undiscovered, partially because of a deficiency of notational suppleness in the theory of special functions which it is the avowed purpose of the present work to remedy.

Hermite Polynomials

As an instance of such a structural characterization, we consider the following problem: Find all Appell sequences $s_n(x)$ with the property that

$$\langle s_j(A) \mid s_n(x)\rangle = -(1/v)\langle s_{j-1}(A) \mid s_{n+1}(x)\rangle,$$

for some constant v.

By the Recurrence Formula,

$$\begin{aligned} s_{n+1}(x) &= (x + T(T^{-1})')\, s_n(x) \\ &= (x + S)\, s_n(x), \end{aligned}$$

where $s_n(x) = T^{-1}x^n$. But this gives $s_j(A) = -(1/v)(\partial_A + S^*)\, s_{j-1}(A)$, whence S^* is multiplication by $-vA$ and $S = -vD$, and thus $T = e^{-vD^2/2}$. The resulting polynomials are the Hermite polynomials $H_n^{(v)}(x) = e^{-vD^2/2}x^n$ of variance v. For $v = 1$, we obtain the classical Hermite polynomials.

The elementary properties of the Hermite polynomials have been derived in "Finite Operator Calculus." We shall give a sampling of applications of the present methods. From the operational formula

$$(x - vD)\, p(x) = -e^{x^2/2v}(vD)\, e^{-x^2/2v}p(x),$$

one infers the recurrence

$$H_{n+j}^{(v)}(x) = (x - vD)^j\, H_n(x) = (-1)^j\, e^{x^2/2v}(vD)^j\, e^{-x^2/2v}H_n^{(v)}(x), \qquad (*)$$

and for $n = 0$ the Rodrigues formula

$$H_n^{(v)}(x) = (-1)^n\, e^{x^2/2v}(vD)^n\, e^{-x^2/2v}.$$

Expanding $(*)$ by the Leibnitz formula gives

$$H_{n+j}^{(v)}(x) = \sum_{k=0}^{j} (-v)^{j-k} \binom{j}{k} (n)_{j-k}\, H_k^{(v)}(x)\, H_{n-j+k}^{(v)}(x). \qquad (**)$$

Replacing n by $n - 2m$ and setting $j = n$, Eq. $(**)$ becomes

$$H_{2n-2m}^{(v)}(x) = \sum_{k=0}^{n} (-v)^{n-k} \binom{n}{k} (n - 2m)_{n-k}\, H_k^{(v)}(x)\, H_{k-2m}^{(v)}(x).$$

We recognize an umbral composition with the Laguerre polynomials

$$L_n^{(-2m)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n - 2m)_{n-k}\, x^k.$$

Using the fact that the Laguerre polynomials are self-reciprocal (proved later in this section, or see "Finite Operator Calculus"), we obtain

$$v^{2n}H_n^{(v)}(x)\, H_{n-2m}^{(v)}(x) = \sum_{k=0}^{n} \binom{n}{k} (n - 2m)_{n-k}\, H_{2k-2m}^{(v)}(x).$$

Changing n to $n + m$ gives

$$v^{2n+2m}H_{n+m}^{(v)}(x)\, H_{n-m}^{(v)}(x) = \sum_{k=0}^{n+m} \binom{n+m}{k} (n-m)_{n+m-k}\, H_{2k-2m}^{(v)}(x),$$

and finally letting $j = k - m$ we have

$$v^{2n+2m}H_{n+m}^{(v)}(x)\, H_{n-m}^{(v)}(x) = \sum_{j=m}^{n} \binom{n+m}{j+m} (n-m)_{n-j}\, H_{2j}^{(v)}(x).$$

From this formula we obtain

$$v^{2n}[H_n^{(v)}(x)]^2 - v^{n+m}H_{n-m}^{(v)}(x)\, H_{n+m}^{(v)}(x)$$
$$= \sum_{j=0}^{n} \frac{(n-m)!}{(j-m)!} \binom{n}{j}\Big[\binom{n}{m}\binom{j+m}{m} - \binom{j}{m}\binom{n+m}{m}\Big]\Big/\binom{j}{m}\binom{j+m}{m}\, H_{2j}^{(v)}(x).$$

Now for negative variance v, we know (Finite Operator Calculus) that $H_{2j}^{(v)}(x)$ is nonnegative, and since the above coefficients are nonnegative, we obtain

$$v^{2n}[H_n^{(v)}(x)]^2 - v^{n+m}H_{n-m}^{(v)}(x)\, H_{n+m}^{(v)}(x) \geqslant 0.$$

For m even, we obtain a Turán-type inequality.

We give now a duplication formula for Hermite polynomials. That is, we determine the connection constants $c_{n,k}$ in

$$H_n^{(v)}(ax) = \sum_{k=0}^{n} c_{n,k} H_k^{(w)}(x).$$

Since $H_n^{(w)}(x)$ is the Sheffer sequence for the pair $(e^{wA^2/2}, A)$ and $H_n^{(v)}(ax)$ is Sheffer for the pair $(e^{va^{-2}A^2/2}, a^{-1}A)$, we have by Corollary 2 to Proposition 9.5 that $t_n(x) = \sum_{k=0}^{n} c_{n,k}x^k$ is Sheffer for the pair $(e^{(va^{-2}-w)A^2/2}, a^{-1}A)$ and so

$$\begin{aligned} t_n(x) &= a^n e^{(w-va^{-2})D^2/2} x^n \\ &= \sum_{k=0}^{n} a^n \frac{(w-va^{-2})^k}{2^k} (n)_{2k}\, x^{n-2k}. \end{aligned}$$

We next determine the connection constants $c_{n,k}$ connecting the Hermite polynomials to the Bernoulli polynomials:

$$H_n^{(v)}(x) = \sum_{k=0}^{n} c_{n,k} B_k^{(\alpha)}(x).$$

Since $H_n^{(v)}(x)$ is Sheffer for the pair $(e^{vA^2/2}, A)$ and $B_n^{(\alpha)}(x)$ is Sheffer for the pair

$(((e^A - \epsilon)/A)^\alpha, A)$, the sequence $t_n(x) = \sum_{k=0}^n c_{n,k}x^k$ is Sheffer for the pair $(((e^A - \epsilon)/A)^{-\alpha} e^{vA^2/2}, A)$ and thus

$$t_n(x) = e^{-vD^2/2}\left(\frac{e^D - I}{D}\right)^\alpha x^n.$$

The constants $c_{n,k}$ are then determined by a routine Taylor's expansion.

Laguerre Polynomials

We have seen that the basic Laguerre polynomials arise in computing the connection constants between $(x)_n$ and $\langle x\rangle_n$. We now consider the more general problem of computing the connection constants between $E^{-\alpha-1}(x)_n$ and $\langle x\rangle_n$, for α a real number. More explicitly, we determine the constants $c_{n,k}$ in

$$(x - \alpha - 1)(x - \alpha - 2) \cdots (x - \alpha - n) = \sum_{k=0}^n c_{n,k}x(x + 1) \cdots (x + n - 1).$$

The sequence $E^{-\alpha-1}(x)_n$ is Sheffer for the pair $(e^{(\alpha+1)A}, e^A - \epsilon)$ and the sequence $\langle x\rangle_n$ is Sheffer for the pair $(\epsilon, \epsilon - e^{-A})$. Thus Corollary 2 to Proposition 9.5 tells us that $t_n(x) = \sum_{k=0}^n c_{n,k}x^k$ is Sheffer for the pair $((\epsilon - A)^{-\alpha-1}, A/(\epsilon - A))$. Thus

$$t_n(x) = (I - D)^{\alpha+1} L_n(-x).$$

The Laguerre polynomials of order α are

$$L_n^{(\alpha)}(x) = (I - D)^{\alpha+1} L_n(x).$$

The ubiquitous presence of these polynomials can be traced to the fact that they give this important set of connection constants. The reader is referred to "Finite Operator Calculus" for the elementary properties of the Laguerre polynomials. We cite only

$$L_n^{(\alpha+\beta+1)}(x + y) = \sum_{k\geqslant 0} \binom{n}{k} L_k^{(\alpha)}(x)\, L_{n-k}^{(\beta)}(y) \qquad (*)$$

and the formula due to Kahaner, Odlyzko, and Rota:

$$L_n^{(\alpha_1)}(\mathbf{L}^{(\alpha_2)}(\mathbf{L}^{(\alpha_3)}(\cdots \mathbf{L}^{(\alpha_k)}(x) \cdots))) = \begin{cases} L_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), & k \text{ odd}, \\ M_n^{(\alpha_1-\alpha_2+\alpha_3-\cdots+\alpha_k)}(x), & k \text{ even}, \end{cases} \qquad (**)$$

where $M_n^{(\lambda)}(x) = (I - D)^{-\lambda} x^n$.

Equation $(*)$ gives the connection constants between Laguerre polynomials of different orders. Equation $(**)$, for $k = 2$, gives $L_n^{(\alpha_1)}(\mathbf{L}^{(\alpha_2)}(x)) = (I - D)^{\alpha_2-\alpha_1} x^n = (-1)^n L_n^{(\alpha_2-\alpha_1-n)}(x)$. For $\alpha_1 = \alpha_2 = \alpha$, we obtain $L_n^{(\alpha)}(\mathbf{L}^{(\alpha)}(x)) = x^n$ showing that all the Laguerre polynomials are self-reciprocal.

Various representations of the Laguerre polynomials of Rodrigues type follow from our methods. As an example, we prove Carlitz's beautiful:

$$L_n^{(\alpha)}(x) = \langle XD - X + \alpha + 1\rangle_n\, 1.$$

This formula is a consequence of Theorem 14. In the notation of the theorem, we have $(Q')^{-1} = -(D-I)^2$, whence the corresponding shift operator is $\theta = -X(D-I)^2$. Similarly, $P = (I-Q)^{\alpha+1}$, so that $P\partial_Q P^{-1} = (\alpha+1)(I-D)$. Thus

$$\begin{aligned} L_n^{(\alpha)}(x) &= (-(\alpha+1)(D-I) - X(D-I)^2)\, L_{n-1}^{(\alpha)}(x) \\ &= (XD - X + \alpha + 1)(I-D)\, L_{n-1}^{(\alpha)}(x) \\ &= (XD - X + \alpha + 1)\, L_{n-1}^{(\alpha+1)}(x) \\ &= \cdots = \langle XD - X + \alpha + 1\rangle_n\, 1. \end{aligned}$$

More generally, we have proved

$$L_{m+n}^{(\alpha)}(x) = \langle XD - X + \alpha + 1\rangle_n\, L_m^{(\alpha+m)}(x).$$

We derive the Erdelyi duplication formula for Laguerre polynomials:

$$L_n^{(\alpha)}(ax) = \sum_{k=0}^{n} c_{n,k} L_k^{(\alpha)}(x).$$

Since $L_n^{(\alpha)}(x)$ is Sheffer for the pair $((\epsilon - A)^{-\alpha-1}, A/(A-\epsilon))$ and $L_n^{(\alpha)}(ax)$ is Sheffer for the pair $((\epsilon - a^{-1}A)^{-\alpha-1}, A/(A-a))$ the sequence $t_n(x) = \sum_{k=0}^{n} c_{n,k}x^k$ is Sheffer for the pair $(a^{\alpha+1}(a+(1-a)A)^{-\alpha-1}, A/(a+(1-a)A))$. By the Transfer Formula,

$$\begin{aligned} t_n(x) &= a^{-\alpha}(a + (1-a)\, D)^{\alpha+n}\, x^n \\ &= \sum_{k=0}^{n} \binom{\alpha+n}{n-k} a^k(1-a)^{n-k}\, (n)_{n-k}\, x^k. \end{aligned}$$

Factor Hermite Sequences

The *factor Hermite sequence* of variance v is the factor sequence for the pair $(e^{-vD^2/2}, D)$. Thus

$$H_{-n}^{(v)}(x) = e^{-vD^2/2}x^{-n}.$$

We have

$$H_{-n}^{(v)}(x+y) = \sum_{k=0}^{\infty} \binom{-n}{k} y^k H_{-n-k}^{(v)}(x).$$

Corollary 3 to Theorem 17 gives

$$H_{-n}^{(v)}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\langle e^{-vA^2/2} nA^{n-1} \mid x^{n+k-1} \rangle}{n!} x^{-n-k}$$

$$= \sum_{k=0}^{\infty} \left(-\frac{v}{2}\right)^k (-n)_{2k}\, x^{-n-2k}.$$

Theorem 19 easily establishes the umbral composition

$$H_{-n}^{(v)}(\mathbf{H}_{-}^{(w)}(x)) = H_{-n}^{(v+w)}(x).$$

The Cigler transform for $H_{-n}^{(v)}(x)$ gives

$$\sum_{k\geqslant 1} H_{-k}^{(v)}(x)\, s^{k-1} = \int_{-\infty}^{0} e^{-vt^2/2-(s-x)t}\, dt.$$

We establish the duplication formula for factor Hermite sequences

$$H_{-n}^{(v)}(ax) = \sum_{k=1}^{\infty} c_{-n,k} H_{-k}^{(w)}(x).$$

It is not hard to see that if $f_{-n}(x)$ is the $(f(D), g(D))$-factor sequence, then $f_{-n}(ax)$ is the $(f(a^{-1}D), g(a^{-1}D))$-factor sequence. Thus since $H_{-n}^{(v)}(ax)$ is the factor sequence for the pair $(e^{-va^{-2}D^2/2}, D)$ and $H_{-n}^{(w)}(x)$ is the factor sequence for the pair $(e^{-wD^2/2}, D)$, the sequence $r_{-n}(x) = \sum_{k=1}^{\infty} c_{-n,k} x^{-k}$ is the factor sequence for the pair $(e^{(w-va^{-2})D^2/2}, a^{-1}D)$ and thus

$$r_{-n}(x) = a^{-n} e^{(w-va^{-2})D^2/2} x^{-n}$$

$$= \sum_{k=0}^{\infty} a^{-n} \frac{(w-va^{-2})^k}{2^k} (-n)_{2k}\, x^{-n-2k}.$$

Factor Laguerre Sequences

The *factor Laguerre sequence* of order α is the factor sequence for the pair $((I-D)^{\alpha+1}, D/(D-I))$. Thus, by analogy with $L_n^{(\alpha)}(x)$, we have

$$L_{-n}^{(\alpha)}(x) = (I-D)^{\alpha+1} L_{-n}(x)$$

$$= (-1)^n (I-D)^{\alpha-n}\, x^{-n}$$

$$= \sum_{k=0}^{\infty} \binom{\alpha-n}{n-k} (-1)^k (-n)_{n-k}\, x^{-k}.$$

We have

$$L_{-n}^{(\alpha)}(x+y) = \sum_{k=0}^{\infty} \binom{-n}{k} L_k(y)\, L_{-n-k}^{(\alpha)}(x),$$

and applying the operator $(I - D)^{\beta+1} = \sigma(\epsilon - A)^{\beta+1}$ we obtain the composition law

$$L_{-n}^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{\infty} \binom{-n}{k} L_k^{(\beta)}(y) L_{-n-k}^{(\alpha)}(x).$$

Theorem 19 implies

$$\begin{aligned} L_{-n}^{(\alpha)}(\mathbf{L}_-^{(\beta)}(x)) &= (I - D)^{\alpha-\beta} x^{-n} \\ &= (-1)^n L_{-n}^{(\alpha-\beta+n)}(x). \end{aligned}$$

For $\alpha = \beta$, we obtain the identity

$$L_{-n}^{(\alpha)}(\mathbf{L}_-^{(\alpha)}(x)) = x^{-n},$$

showing that the factor Laguerre sequence $L_{-n}^{(\alpha)}(x)$ is self-reciprocal. Explicitly we have

$$x^{-n} = \sum_{k=0}^{\infty} \binom{\alpha - n}{n - k} (-1)^k (-n)_{n-k} L_{-k}^{(\alpha)}(x).$$

The Cigler transform for the sequence $L_{-n}^{(\alpha)}(x)$ gives

$$\sum_{k=1}^{\infty} L_{-k}^{(\alpha)}(x)\, s^{k-1} = \int_{-\infty}^{0} (1 - t)^{-\alpha-1} e^{-st+xt/(t-1)}\, dt.$$

We determine the connection constants $c_{-n,k}$ in

$$\frac{1}{(x-\alpha)(x-\alpha+1)\cdots(x-\alpha-1+n)} = \sum_{k=0}^{\infty} \frac{(-1)^k c_{-n,k}}{(x-1)(x-2)\cdots(x-k)}.$$

Just as before, $E^{-\alpha-1}(x)_{-n}$ is the factor sequence for the pair $(e^{-(\alpha+1)D}, e^D - I)$ and $\langle x \rangle_n$ is the factor sequence for the pair $(I, I - e^{-D})$. Thus Corollary 1 to Theorem 19 implies that $r_{-n}(x) = \sum_{k=1}^{\infty} c_{n,k} x^{-k}$ is the factor sequence for the pair $((I - D)^{\alpha+1}, D/(I - D))$. Hence

$$\begin{aligned} r_{-n}(x) &= L_{-n}^{(\alpha)}(-x) \\ &= \sum_{k=1}^{\infty} \binom{\alpha - n}{n - k} (-n)_{n-k}\, x^{-k}, \end{aligned}$$

as expected.

Finally, we derive the duplication formula for factor Laguerre sequences. Namely, we determine the constants $c_{-n,k}$ in

$$L_{-n}^{(\alpha)}(ax) = \sum_{k=1}^{\infty} c_{-n,k} L_{-k}^{(\alpha)}(x).$$

Since $L_{-n}^{(\alpha)}(x)$ is the $((I-D)^{\alpha+1}, D/(D-I))$-factor sequence, Corollary 1 to Theorem 19 implies that $r_{-n}(x)=\sum_{k=1}^{\infty} c_{-n,k}x^{-k}$ is the factor sequence for the pair $(a^{-\alpha-1}(a+(1-a)D)^{\alpha+1}, D/(a+(1-a)D))$. Thus

$$\begin{aligned} r_{-n}(x) &= a^{-\alpha}(a+(1-a)\,D)^{\alpha-n}\,x^{-n} \\ &= \sum_{k=1}^{\infty} \binom{\alpha-n}{n-k} a^{-k}(1-a)^{n-k}\,(-n)_{n-k}\,x^{-k}. \end{aligned}$$

14. Applications to Combinatorics

We define a *store* σ as a set, in general infinite, together with a map d which assigns to every element of σ a positive integer, called its *degree*. The subset of σ consisting of all elements of a given degree is assumed to be finite. In practice, the elements of σ are sets endowed with some structure, and the problem is to count σ; that is, to determine the number a_n of elements of σ of degree n. We call a_n the *counting sequence* of σ, and we assume that $a_1 > 0$.

We define the *generating functional* of σ as the delta functional L satisfying

$$\langle L \mid x^n \rangle = a_n .$$

The counting sequence is thus the sequence of coefficients of the indicator of L.

The *partitional* of a store σ (a translation of Foata's "compose partitionnel") is a second store *part* (σ) defined as follows. An element p of part (σ) is a set (not a sequence) of pairs $\{(B_1, s_1),..., (B_k, s_k)\}$, where

(i) the B_i are the blocks of a partition of the set $\{1, 2,..., n\}$, for some n (hence B_i is nonempty);

(ii) the s_i are elements of the store σ;

(iii) the degree of s_i equals the number of elements in B_i.

To every such element p, called a *part* of part (σ), we associate two integers; the *degree* $d(p)$ of p is the sum of the degrees of s_i and the *part number* of p is the number of blocks.

The partitional part (σ) is obtained by letting n range over all positive integers. We let $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, and call it the *counting sequence* of the partitional. We set $b_{0,0}=1$. Since $a_1 > 0$, we have $b_{n,n} > 0$ for all n.

The following proposition motivates this definition.

Proposition 14.1. *Let $b_{n,k}$ be the counting sequence of the partitional part (σ) of a store σ having generating functional L, for the degree n and the part number k. Then*

$$b_{n,k} = \langle L^k \mid x^n \rangle / k!.$$

Proof. Evidently the counting sequence satisfies the identity

$$\binom{i+j}{i} b_{n,i+j} = \sum_{k=0}^{n} \binom{n}{k} b_{k,i} b_{n-k,j}.$$

Therefore, by Proposition 4.3, there exists a delta functional M such that

$$b_{n,k} = \langle M^k \mid x^n \rangle / k!.$$

But it is immediate from the definition of partitional that $b_{n,1} = a_n$. Thus $M = L$. Q.E.D.

We remark that $\sum_{k=0}^{n} b_{n,k} x^k$ is the conjugate sequence for the delta functional L.

COROLLARY 1. *Let* $c_n = \sum_{k>0} b_{n,k}$ *be the number of elements of degree* n *in part* (σ). *Then*

$$c_n = \langle e^L \mid x^n \rangle.$$

COROLLARY 2 (Foata). *The exponential generating function of* c_n *is the indicator of the exponential of the generating functional of* σ.

We illustrate these notions with some elementary examples.

EXAMPLE 1. Find the number of partitions of an n-set.

Solution. Let σ be the store having exactly one element of each degree. Then part (σ) is the set of all partitions of finite sets. An element of part (σ) of degree n and part number k is a partition of an n-set into k blocks. Now since $\langle L \mid x^n \rangle = 1$ for all $n > 0$, we conclude that $L = e^A - \epsilon = \epsilon_1 - \epsilon$, the forward difference functional. Thus Proposition 14.1 implies that the number $b_{n,k}$ of partitions of an n-set into k blocks is $S(n, k)$, the Stirling numbers of the second kind, defined by

$$S(n, k) = \frac{(-1)^k}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} i^n.$$

Corollary 1 tells us that the Bell numbers B_n of partitions of an n-set satisfy

$$B_n = \langle e^L \mid x^n \rangle,$$

where $L = e^A - \epsilon$. Corollary 2 gives the exponential generating function for the Bell numbers as $\exp(e^t - 1)$.

EXAMPLE 2. Let S be an n-set. Then a *k-partition of S with selected subsets* is a partition of S into k blocks $B_1, B_2, \ldots, B_k$ together with a nonempty subset C_i of each block B_i.

PROBLEM. Count the number of partitions of an n-set with selected subsets.

Solution. Let σ be the store consisting of all pairs B_i, C_i of nonempty sets, such that $B_i \supseteq C_i$. Then part (σ) is the set of all partitions of finite sets with selected subsets. Since $\langle L \mid x^n \rangle = 2^n - 1$, we see that

$$L = \sum_{k \geqslant 1} (2^k - 1)\, A^k/k! = e^A(e^A - 1),$$

the difference–Abel functional. Applying Proposition 14.1, we find that the number $b_{n,k}$ of k-partitions of an n-set with selected subsets is the kth coefficient of the nth conjugate Gould polynomial:

$$b_{n,k} = \sum_{i=0}^{n} \binom{n}{i} k^i S(n-i, k).$$

By Corollary 1, the number P_n of partitions with selected subsets is

$$P_n = \langle e^L \mid x^n \rangle,$$

where $L = e^A(e^A - 1)$.

EXAMPLE 3. Find the number of rooted, labeled trees on n vertices, where each vertex is of degree 1, except the root.

Solution. Let σ be the store whose elements of degree n are rooted, labeled trees on n vertices in which each vertex has degree one, except the root. Then part (σ) consists of all forests with the specified degree requirements. Since $\langle L \mid x^n \rangle = n$, we conclude that $L = Ae^A$, the Abel functional. Proposition 14.1 implies that the number of forests on n vertices with k components satisfying the above degree requirements is

$$b_{n,k} = \binom{n}{k} k^{n-k}.$$

Corollary 2 implies that the exponential generating function for the number c_n of forests on n vertices with the above degree requirements is $\exp(te^t)$.

EXAMPLE 4. Find the number of permutations of an n-set all of whose cycles have odd cardinality.

Solution. Let σ be the store whose elements of degree $2n + 1$ are all cyclic permutations of the set $\{1, 2, \ldots, 2n + 1\}$, and having no elements of even degree. Then part (σ) is the set of permutations of finite sets whose cycles have odd cardinality, the elements of degree $2n + 1$ and part number k being permutations of the set $\{1, 2, \ldots, 2n+1\}$ with k cycles, all of odd cardinality. Thus $\langle L \mid x^{2n+1}\rangle = (2n)!$ and $\langle L \mid x^{2n}\rangle = 0$. Hence

$$L = \sum_{k\geq 0} \frac{(2k)!}{(2k+1)!} A^{2k+1} = \sum_{k\geq 0} \frac{1}{2k+1} A^{2k+1}$$

$$= \tfrac{1}{2}\log((\epsilon + A)/(\epsilon - A)) = \operatorname{arc\,tanh} A$$

Now let $T_n(x)$ be the conjugate polynomials for L. The recurrence

$$T_{n+1}(x) = xT_n(x) + n(n-1)\, T_{n-1}(x)$$

is established by applying $\tilde{L}^k$ (where $\tilde{L} = \tanh A$) to both sides and using the fact that $\langle \tilde{L}^k \mid T_n(x)\rangle = n!\, \delta_{n,k}$ and

$$\langle \tilde{L}^k \mid xT_n(x)\rangle = \langle \partial_A(\tilde{L}^k) \mid T_n(x)\rangle = \langle k\tilde{L}^{k-1}(\epsilon - \tilde{L}^2) \mid T^n(x)\rangle.$$

Putting $x = 1$ in the above recurrence, we have

$$T_{n+1}(1) = T_n(1) + n(n-1)\, T_{n-1}(1)$$

so that the required number is

$$T_n(1) = 1^2 \cdot 3^2 \cdot 5^2 \cdots (2n-1)^2.$$

EXAMPLE 5 (Cayley). Find all rooted, labeled trees with n vertices.

Solution. Let σ be the store whose elements of degree n are all rooted labeled trees on n vertices. Then part (σ) is the set of forests. Letting $b_{n,k}$ be the number of elements of part (σ) of degree n and part number k, we have the obvious recursion, obtained by removing the root of a tree and counting the resulting forest:

$$b_{n,1} = n \sum_k b_{n-1,k}\,.$$

In terms of the generating functional, this becomes

$$\langle L \mid x^n\rangle = n \sum_k \frac{\langle L^k \mid x^{n-1}\rangle}{k!} = \langle e^L \mid Dx^n\rangle$$

$$= \langle Ae^L \mid x^n\rangle,$$

and thus $L = Ae^L$. We seek the conjugate sequence for L. But $A = Le^{-L} = f(L)$ and so $L = f^{-1}(A)$ and $\tilde{L} = f(A) = Ae^{-A}$. Thus we see that $\tilde{L}$ is the Abel functional and

$$\sum_k b_{n,k} x^k = x(x+n)^{n-1}.$$

Therefore,

$$b_{n,k} = \binom{n-1}{k-1} n^{n-k},$$

and the numbers of rooted labeled trees on n vertices is n^{n-1}.

Recall that a binary tree is a tree in which each vertex has degree one or three, except the root, which has degree 2.

EXAMPLE 6. Find all rooted, labeled binary trees with n vertices.

Solution. Let σ be the store whose elements of degree n are binary trees with n vertices. Then part (σ) is the set of forests of such trees. We have as in Example 3:

$$\langle L \mid x^n \rangle = n\langle L^2 \mid x^{n-1} \rangle + \langle A \mid x^n \rangle,$$

so $L = A(L^2 + 2\epsilon)/2$ and $\tilde{L} = 2A/(A^2 + 2\epsilon)$. By the Transfer Formula,

$$\begin{aligned}\sum_k b_{n,k} x^k &= x 2^{-n}(D^2 + 2)^n x^{n-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} 2^{-k}(n-1)_{2k}\, x^{n-2k}.\end{aligned}$$

Thus,

$$b_{n,k} = \begin{cases} 0, & n + k \text{ odd.} \\ \binom{n}{(n-k)/2} (n-1)_{n-k}\, 2^{(k-n)/2}, & n + k \text{ even.} \end{cases}$$

A linearly ordered tree is one in which all but two vertices are of degree 2.

EXAMPLE 7. Find all rooted, labeled forests on n vertices in which each tree is linearly ordered.

Solution. Let σ be the store in which the elements of degree n are rooted, labeled linearly ordered trees on n vertices. As in Example 3, we see by removing the root that

$$\langle L \mid x^n \rangle = n\langle L \mid x^{n-1} \rangle + \langle A \mid x^n \rangle.$$

Thus $L = LA + A$, and $\tilde{L} = A/(A + \epsilon)$. By the Transfer Formula,

$$\begin{aligned}\sum_k b_{n,k} x^k &= x(D + I)^n x^{n-1} \\ &= \sum_{k=0}^{n} \binom{n}{k} (n-1)_{k-1}\, x^k \\ &= L_n(-x),\end{aligned}$$

where $L_n(x)$ are the Laguerre polynomials.

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