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p-Adic Analysis with *q*-Analysis and Its Applications

Guest Editors: CheonSeoung Ryoo, Taekyun Kim, A. Bayad, and Yilmaz Simsek

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Editorial Board

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Editorial

p-Adic Analysis with q-Analysis and Its Applications

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Bernoulli numbers, Bernoulli polynomials, and Euler numbers, Euler polynomials were studied by many authors. Bernoulli numbers, Bernoulli polynomials, Euler numbers, and Euler polynomials possess many interesting properties and arise in many areas of mathematics and physics. These numbers are still in the center of the advanced mathematical research. Especially, in number theory and quantum theory, they have many applications.

p-Adic analysis with q-analysis includes several domains in mathematics and physics, including the number theory, algebraic geometry, algebraic topology, mathematical analysis, mathematical physics, string theory, field theory, stochastic differential equations, quantum groups, and other parts of the natural sciences.

The intent of this special issue was to survey major interesting results and current trends in the theory of p-adic analysis associated with q-analogs of zeta functions, Hurwitz zeta functions, Dirichlet series, L-series, special values, q-analogs of Bernoulli, Euler, and Genocchi numbers and polynomials, q-integers, q-integral, q-identities, q-special functions, q-continued fractions, gamma functions, sums of powers, q-analogs of multiple zeta functions, Barnes multiple zeta functions, multiple L-series, and computational and numerical aspects of q-series and q-analysis.

The Guest Editors and Referees of this special issue are well-known mathematicians that work in this field of interest. Thus, we got the best articles to be included in this issue.

The results and properties of accepted papers are very interesting, well written, and mathematically correct. The work is a relevant contribution in the field of applied mathematics.

CheonSeoung Ryoo
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A. Bayad
Yilmaz Simsek

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Research Article

Arithmetic Identities Involving Bernoulli and Euler Numbers

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The purpose of this paper is to give some arithmatic identities for the Bernoulli and Euler numbers. These identities are derived from the several p-adic integral equations on $\mathbb{Z}_p$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The p-adic norm is normalized so that $|p|_p = 1/p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x) \mu\left(x + p^N \mathbb{Z}_p\right) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}$$

and the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows (see [1–8]):

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.2}$$

The Euler polynomials, $E_n(x)$, are defined by the generating function as follows (see [1–16]):

$$F^E(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.3}$$

In the special case, $x = 0$, $E_n(0) = E_n$ is called the nth Euler number.

By (1.3) and the definition of Euler numbers, we easily see that

$$E_n(x) = \sum_{l=0}^{n} \binom{n}{l} E_l x^{n-l} = (E+x)^n, \tag{1.4}$$

with the usual convention about replacing E^l by E_l (see [10]). Thus, by (1.3) and (1.4), we have

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n}, \tag{1.5}$$

where $\delta_{k,n}$ is the Kronecker symbol (see [9, 10, 17–19]).

From (1.2), we can also derive the following integral equation for the fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \tag{1.6}$$

see [1, 2]. By (1.3) and (1.6), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.7}$$

Thus, by (1.7), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \tag{1.8}$$

see [1–8, 13–16].

The Bernoulli polynomials, $B_n(x)$, are defined by the generating function as follows:

$$F^B(t,x) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.9}$$

see [18]. In the special case, $x = 0$, $B_n(0) = B_n$ is called the nth Bernoulli number. From (1.9) and the definition of Bernoulli numbers, we note that

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l = (B+x)^n, \tag{1.10}$$

see [1–19], with the usual convention about replacing B^l by B_l. By (1.9) and (1.10), we easily see that

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n}, \tag{1.11}$$

see [13].

From (1.1), we can derive the following integral equation on $\mathbb{Z}_p$:

$$I(f_1) = I(f) + f'(0), \tag{1.12}$$

where $f_1(x) = f(x+1)$ and $f'(0) = (df(x)/dx)|_{x=0}$.

By (1.12), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.13}$$

Thus, by (1.13), we can derive the following Witt's formula for the Bernoulli polynomials:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{1.14}$$

In [19], it is known that for $k, m \in \mathbb{Z}_+$,

$$\sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{B_{k+m+1-j}(x)}{k+m+1-j} = x^k (x-1)^m + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}, \tag{1.15}$$

where $\binom{k}{j} = 0$ if $j < 0$ or $j > k$.

The purpose of this paper is to give some arithmetic identities involving Bernoulli and Euler numbers. To derive our identities, we use the properties of p-adic integral equations on $\mathbb{Z}_p$.

2. Arithmetic Identities for Bernoulli and Euler Numbers

Let us take the bosonic p-adic integral on $\mathbb{Z}_p$ in (1.15) as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} \binom{m}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+m-l} d\mu(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} \binom{m}{l} (-1)^l B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.1}$$

On the other hand, we get

$$\begin{aligned}I_1 &= \sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\frac{1}{k+m+1-j}\int_{\mathbb{Z}_p}B_{k+m+1-j}(x)d\mu(x)\\ &= \sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\frac{1}{k+m+1-j}\\ &\quad\times\sum_{l=0}^{k+m+1-j}\binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l.\end{aligned}\tag{2.2}$$

By (2.1) and (2.2), we get

$$\begin{aligned}&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\\ &\qquad\times\binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l\\ &\quad=\sum_{l=0}^{m}(-1)^l\binom{m}{l}B_{k+m-l}+\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}.\end{aligned}\tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. *For $k,m\in\mathbb{Z}_+$, one has*

$$\begin{aligned}&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\\ &\qquad\times\binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l-\frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}\\ &\quad=\sum_{l=0}^{m}(-1)^l\binom{m}{l}B_{k+m-l}.\end{aligned}\tag{2.4}$$

Now we consider the fermionic p-adic integral on $\mathbb{Z}_p$ in (1.15) as follows:

$$\begin{aligned}I_2 &= \sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\frac{1}{k+m+1-j}\sum_{l=0}^{k+m+1-j}\binom{k+m+1-j}{l}\\ &\quad\times B_{k+m+1-j-l}\int_{\mathbb{Z}_p}x^l d\mu_{-1}(x)\end{aligned}$$

$$= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{1}{k+m+1-j} \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} \times B_{k+m+1-j-l} E_l. \tag{2.5}$$

On the other hand, we get

$$\begin{aligned} I_2 &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} \int_{\mathbb{Z}_p} x^{m-l+k} d\mu_{-1}(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we get

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \binom{k+m+1-j}{l} \\ &\qquad \times B_{k+m+1-j-l} E_l \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

Theorem 2.2. *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \binom{k+m+1-j}{l} \\ &\qquad \times B_{k+m+1-j-l} E_l - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l}. \end{aligned} \tag{2.8}$$

Replacing x by $(1-x)$ in (1.15), we have the identity:

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{B_{k+m+1-j}(1-x)}{k+m+1-j} \\ &\qquad = (-1)^{k+m} x^m (1-x)^k + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.9}$$

Let us take the bosonic p-adic integral on $\mathbb{Z}_p$ in (2.9) as follows:

$$\begin{aligned}
I_3 = & \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{1}{k+m+1-j} \\
& \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu(x) \\
= & \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{1}{k+m+1-j} \\
& \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l \\
& + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{1}{k+m+1-j} \\
& \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} l \\
& + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{1}{k+m+1-j} \\
& \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \delta_{1,l} \\
= & \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \\
& \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l \\
& + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \left(2B_{k+m-j} + \delta_{1,(k+m-j)} \right) \\
= & \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \\
& \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l + 2 \sum_{j=1}^{\max(k,m)} \left[\binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \\
& \times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m} \binom{m}{k+m-1}.
\end{aligned} \tag{2.10}$$

On the other hand, we see that

$$I_3 = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \tag{2.11}$$

By (2.10) and (2.11), we get

$$\begin{aligned}
&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad\times\binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l + 2\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad\times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m}\binom{m}{k+m-1} \\
&\quad= (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.3. *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad\times\binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l + 2\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad\times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m}\binom{m}{k+m-1} - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\
&\quad= (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l}.
\end{aligned} \tag{2.13}$$

We consider the fermionic p-adic integral on $\mathbb{Z}_p$ in (2.9) as follows:

$$\begin{aligned}
I_4 = &\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\frac{1}{k+m+1-j} \\
&\times\sum_{l=0}^{k+m+1-j}\binom{k+m+1-j}{l}B_{k+m+1-j-l}\int_{\mathbb{Z}_p}(1-x)^l d\mu_{-1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l \\
&\quad + 2\sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \\
&\quad - 2\sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \delta_{0,l} \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l \\
&\quad + 2\sum_{j=1}^{\max\{k,m\}} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \delta_{1,(k+m+1-j)} \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l + 2\left[\binom{k}{k+m} + (-1)^{k+m+1}\binom{m}{k+m}\right].
\end{aligned}
\tag{2.14}
$$

On the other hand, we get

$$
I_4 = (-1)^{k+m} \sum_{l=0}^{k} (-1)^l \binom{k}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \tag{2.15}
$$

By (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned}\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} & \frac{1}{k+m+1-j}\left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right]\binom{k+m+1-j}{l} \\ & \times B_{k+m+1-j-l}E_l + 2\left[\binom{k}{k+m} + (-1)^{k+m+1}\binom{m}{k+m}\right] \\ & - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}E_{k+m-l}.\end{aligned} \tag{2.16}$$

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Research Article

Generalized (q, w)-Euler Numbers and Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

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We generalize the Euler numbers and polynomials by the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane.

1. Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and polynomials (see [1–15]). The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [14], we introduced that Euler equation $E_n(x) = 0$ has symmetrical roots for $x = 1/2$ (see [14]). It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ in complex plane. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$

$$[x]_q = \frac{1 - q^x}{1 - q}, \qquad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \tag{1.1}$$

Compared with [1, 4, 5]. Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N, we set

$$X = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right),$$
$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \tag{1.2}$$
$$a + dp^N \mathbb{Z}_p = \left\{ x \in X \mid x \equiv a \left(\operatorname{mod} dp^N \right) \right\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N,

$$\mu_q \left(a + dp^N \mathbb{Z}_p \right) = \frac{q^a}{\left[dp^N \right]_q} \tag{1.3}$$

is known to be a distribution on X, compared with [1–10, 14]. For

$$g \in UD(\mathbb{Z}_p) = \{ g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \}. \tag{1.4}$$

Kim defined the fermionic p-adic q-integral on $\mathbb{Z}_p$

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{0\leq x<p^N} g(x)(-q)^x. \tag{1.5}$$

From (1.5), we also obtain

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0), \tag{1.6}$$

where $g_1(x) = g(x+1)$ (see [1–3]).

From (1.6), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.7}$$

where $g_n(x) = g(x+n)$.

As well-known definition, the Euler polynomials are defined by

$$F(t) = \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!},$$
$$F(t,x) = \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.8}$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n-th Euler numbers (cf. [1–15]).

Our aim in this paper is to define the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We investigate some properties which are related to the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. Especially, distribution of roots for $E_{n,q,w}(x : a) = 0$ is different from $E_n(x) = 0^{'}$s. We also derive the existence of a specific interpolation function which interpolate the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$.

2. The Generalized (q, w)-Euler Numbers and Polynomials

Our primary goal of this section is to define the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We also find generating functions of the generalized (q, w)-Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. Let a be strictly positive real number.

The generalized (q, w)-Euler numbers and polynomials $E_{n,q,w}(a)$, $E_{n,q,w}(x : a)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,q,w}(a)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-q}(x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} E_{n,q,w}(x : a)\frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ay}e^{(ay+x)t}d\mu_{-q}(y), \quad \text{for } t, w \in \mathbb{C}, \tag{2.2}$$

respectively.

From above definition, we obtain

$$\begin{aligned} E_{n,q,w}(a) &= \int_{\mathbb{Z}_p} w^{ax}(ax)^n d\mu_{-q}(x), \\ E_{n,q,w}(x : a) &= \int_{\mathbb{Z}_p} w^{ay}(x + ay)^n d\mu_{-q}(y). \end{aligned} \tag{2.3}$$

Let $g(x) = w^{ax}e^{axt}$. By (1.6) and using p-adic q-integral on $\mathbb{Z}_p$, we have

$$\begin{aligned} qI_{-q}(g_1) + I_{-q}(g) &= \int_{\mathbb{Z}_p} w^{a(x+1)}e^{a(x+1)t}d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-q}(x) \\ &= (qw^a e^{at} + 1)\int_{\mathbb{Z}_p} w^{ax}e^{axt}d\mu_{-q}(x) \\ &= [2]_q. \end{aligned} \tag{2.4}$$

Hence, by (2.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(a)\frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at} + 1}. \tag{2.5}$$

By (1.6), (2.2) and $g(y) = w^{ay}e^{(ay+x)t}$, we have

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at}+1}e^{xt}. \tag{2.6}$$

After some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty}(-1)^n q^n w^{an} e^{ant} e^{xt}. \tag{2.7}$$

From (2.6), we have

$$\begin{aligned} E_{n,q,w}(x:a) &= \sum_{k=0}^{n}\binom{n}{k} x^{n-k} E_{k,q,w}(a) \\ &= \left(x + E_{q,w}(a)\right)^n, \end{aligned} \tag{2.8}$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

3. Basic Properties for the Generalized (q, w)-Euler Numbers and Polynomials

By (2.5), we have

$$\begin{aligned} \frac{\partial}{\partial x}\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} &= \frac{\partial}{\partial x}\left(\frac{[2]_q}{qw^a e^{at}+1}e^{xt}\right) \\ &= t\sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} nE_{n-1,q,w}(x:a)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

By (3.1), we have the following differential relation.

Theorem 3.1. *For positive integers n, one has*

$$\frac{\partial}{\partial x}E_{n,q,w}(x:a) = nE_{n-1,q,w}(x:a). \tag{3.2}$$

By Theorem 3.1, we easily obtain the following corollary.

Corollary 3.2 (integral formula). *Consider that*

$$\int_p^q E_{n-1,q,w}(x:a)dx = \frac{1}{n}\left(E_{n,q,w}(q:a) - E_{n,q,w}(p:a)\right). \tag{3.3}$$

By (2.5), *one obtains*

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q,w}(x+y:a)\frac{t^n}{n!} &= \frac{[2]_q}{qw^a e^{at}+1}e^{(x+y)t} \\
&= \sum_{n=0}^{\infty} E_{n,q,w}(x:a)\frac{t^n}{n!}\sum_{k=0}^{\infty} y^k\frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)y^{n-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.4}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following addition theorem.

Theorem 3.3 (addition theorem). *For $n \in \mathbb{Z}_+$,*

$$E_{n,q,w}(x+y:a) = \sum_{k=0}^{n}\binom{n}{k}E_{k,q,w}(x:a)y^{n-k}. \tag{3.5}$$

By (2.5), *for $m \equiv 1(\bmod 2)$, one has*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(m^n\frac{[2]_q}{[2]_{q^m}}\sum_{k=0}^{m-1}(-1)^k q^k w^{ak}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{t^n}{n!} \\
&\quad= \sum_{k=0}^{m-1}(-1)^k q^k w^{ak}\left(\sum_{n=0}^{\infty}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right)\right)\frac{(mt)^n}{n!} \\
&\quad= \sum_{k=0}^{m-1}\left((-1)^k q^k w^{ak}\frac{[2]_q}{q^m w^{ma}e^{mat}+1}e^{(x+ak)t}\right) \\
&\quad= \frac{[2]_q}{1+qw^a e^{at}}e^{xt} \\
&\quad= \sum_{n=0}^{\infty}E_{n,q,w}(x:a)\frac{t^n}{n!}.
\end{aligned} \tag{3.6}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following multiplication theorem.

Theorem 3.4 (multiplication theorem). *For $m, n \in \mathbb{N}$*

$$E_{n,q,w}(x:a) = m^n\frac{[2]_q}{[2]_{q^m}}\sum_{k=0}^{m-1}(-1)^k q^k w^{ak}E_{n,q^m,w^m}\left(\frac{x+ak}{m}:a\right). \tag{3.7}$$

From (1.6), *one notes that*

$$\begin{aligned}
[2]_q &= \int_{\mathbb{Z}_p} q w^{ax+a} e^{(ax+a)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
&= \sum_{n=0}^{\infty} \left(q w^a \int_{\mathbb{Z}_p} w^{ax} (ax+a)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q w^a E_{n,q,w}(a:a) + E_{n,q,w}(a) \right) \frac{t^n}{n!}.
\end{aligned} \tag{3.8}$$

From the above, we obtain the following theorem.

Theorem 3.5. *For $n \in \mathbb{Z}_+$, we have*

$$q w^a E_{n,q,w}(a:a) + E_{n,q,w}(a) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{3.9}$$

By (2.8) in the above, we arrive at the following corollary.

Corollary 3.6. *For $n \in \mathbb{Z}_+$, one has*

$$q w^a \left(a + E_{q,w}(a) \right)^n + E_{n,q,w}(a) = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{3.10}$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

From (1.7), one notes that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left([2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l w^{al} (al)^m \right) \frac{t^n}{m!} \\
&\quad = q^n \int_{\mathbb{Z}_p} w^{ax+an} e^{(ax+an)t} d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
&\quad = \sum_{m=0}^{\infty} \left(q^n w^{an} \int_{\mathbb{Z}_p} w^{ax} (ax+an)^m d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} (ax)^m d\mu_{-q}(x) \right) \frac{t^m}{m!} \\
&\quad = \sum_{m=0}^{\infty} \left(q^n w^{an} E_{m,w}(an:a) + (-1)^{n-1} E_{m,w}(a) \right) \frac{t^m}{m!}.
\end{aligned} \tag{3.11}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 3.7. *For $n \in \mathbb{Z}_+$, one has*

$$q^n w^{an} E_{m,w}(na : a) + (-1)^{n-1} E_{m,w}(a) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} w^{al} q^l (al)^m. \tag{3.12}$$

4. The Analogue of the q-Euler Zeta Function

By using the generalized (q,w)-Euler numbers and polynomials, the generalized (q,w)-Euler zeta function and the generalized Hurwitz (q,w)-Euler zeta functions are defined. These functions interpolate the generalized (q,w)-Euler numbers and (q,w)-Euler polynomials, respectively. Let

$$F_{q,w}(x : a)(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} e^{ant} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!}. \tag{4.1}$$

By applying derivative operator, $d^k/dt^k|_{t=0}$ to the above equation, we have

$$\frac{d^k}{dt^k} F_{q,w}(x : a)(t)\bigg|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an + x)^k, \quad (k \in \mathbb{N}), \tag{4.2}$$

$$E_{k,q,w}(x : a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an + x)^k. \tag{4.3}$$

By using the above equation, we are now ready to define the generalized (q,w)-Euler zeta functions.

Definition 4.1. For $s \in \mathbb{C}$, one defines

$$\zeta_{q,w}^{(a)}(x : s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an + x)^s}. \tag{4.4}$$

Note that $\zeta_w^{(a)}(x,s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $w \to 1$, $w \to 1$, and $a = 1$, then $\zeta_{q,w}^{(a)}(x : s) = \zeta(x : s)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_w^{(a)}(x : s)$ and $E_{k,w}(x : a)$ is given by the following theorem.

Theorem 4.2. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(x : -k) = E_{k,w}(x : a). \tag{4.5}$$

By using (4.2), *one notes that*

$$\frac{d^k}{dt^k} F_{q,w}(0 : a)(t)\bigg|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k, \quad (k \in \mathbb{N}). \tag{4.6}$$

Hence, one obtains

$$E_{k,q,w}(a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k. \tag{4.7}$$

By using the above equation, one is now ready to define the generalized Hurwitz (q,w)-Euler zeta functions.

Definition 4.3. Let $s \in \mathbb{C}$. One defines

$$\zeta_{q,w}^{(a)}(s) = 2\sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an)^s}. \tag{4.8}$$

Note that $\zeta_{q,w}^{(a)}(s)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $w \to 1$, $q \to 1$, and $a = 1$, then $\zeta_w^{(a)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_w^{(a)}(s)$ and $E_{k,w}(s)$ is given by the following theorem.

Theorem 4.4. *For $k \in \mathbb{N}$, one has*

$$\zeta_{q,w}^{(a)}(-k) = E_{k,q,w}(a). \tag{4.9}$$

5. Zeros of the Generalized (q,w)-Euler Polynomials $E_{n,q,w}(x : a)$

In this section, we investigate the reflection symmetry of the zeros of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x : a)$.

In the special case, $w = 1$ and $q \to 1$, $E_{n,q,w}(x : a)$ are called generalized Euler polynomials $E_n(x : a)$. Since

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(a-x : a)\frac{(-t)^n}{n!} &= \frac{2}{e^{-at}+1} e^{(a-x)(-t)} \\ &= \frac{2}{e^{at}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x : a)\frac{t^n}{n!}, \end{aligned} \tag{5.1}$$

we have

$$E_n(x : a) = (-1)^n E_n(a-x : a) \quad \text{for } n \in \mathbb{N}. \tag{5.2}$$

We observe that $E_n(x : a), x \in \mathbb{C}$ has $\mathrm{Re}(x) = a/2$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions.

Let

$$F_{q,w}(x:t) = \frac{[2]_q}{qw^a e^{at}+1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!}. \tag{5.3}$$

Then, we have

$$\begin{aligned} F_{q^{-1},w^{-1}}(a-x:-t) &= \frac{[2]_{q^{-1}}}{q^{-1}w^{-a}e^{-at}+1} e^{(a-x)(-t)} \\ &= w^a \frac{[2]_q}{qw^a e^{at}+1} e^{xt} \\ &= w^a \sum_{n=0}^{\infty} E_{n,q,w}(x:a) \frac{t^n}{n!}. \end{aligned} \tag{5.4}$$

Hence, we arrive at the following complement theorem.

Theorem 5.1 (complement theorem). *For $n \in \mathbb{N}$,*

$$E_{n,q^{-1},w^{-1}}(a-x:a) = (-1)^n w^a E_{n,q,w}(x:a). \tag{5.5}$$

Throughout the numerical experiments, we can finally conclude that $E_{n,q,w}(x:a), x \in \mathbb{C}$ has not $\operatorname{Re}(x) = a/2$ reflection symmetry analytic complex functions. However, we observe that $E_{n,q,w}(x:a), x \in \mathbb{C}$ has $\operatorname{Im}(x) = 0$ reflection symmetry (see Figures 1, 2, and 3). The obvious corollary is that the zeros of $E_{n,q,w}(x:a)$ will also inherit these symmetries.

$$\text{If } E_{n,q,w}(x_0:a) = 0, \quad \text{then } E_{n,q,w}\left(x_0^*:a\right) = 0, \tag{5.6}$$

where $*$ denotes complex conjugation (see Figures 1, 2, and 3).

We investigate the beautiful zeros of the generalized (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ by using a computer. We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n = 30, a = 1, 2, 3, 4$, and $x \in \mathbb{C}$ (Figure 1). In Figure 1 (top-left), we choose $n = 30, q = 1/2, w = 1$, and $a = 1$. In Figure 1 (top-right), we choose $n = 30, q = 1/2, w = 2$, and $a = 2$. In Figure 1 (bottom-left), we choose $n = 30, q = 1/2, w = 3$, and $a = 3$. In Figure 1 (bottom-right), we choose $n = 30, q = 1/2, w = 4$, and $a = 4$.

We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x:a)$ for $n = 30, a = 2, w = 2$, and $x \in \mathbb{C}$ (Figure 2).

In Figure 2 (top-left), we choose $n = 30, q = 1/10, w = 2$, and $a = 2$. In Figure 2 (top-right), we choose $n = 30, q = 3/10, w = 2$, and $a = 2$. In Figure 2 (bottom-left), we choose $n = 30, q = 7/10, w = 2$, and $a = 2$. In Figure 2 (bottom-right), we choose $n = 30, q = 9/10, w = 2$ and $a = 2$.

Plots of real zeros of $E_{n,q,w}(x:a)$ for $1 \leq n \leq 25$ structure are presented (Figure 3).

In Figure 3 (top-left), we choose $q = 1/2, w = 1$, and $a = 2$. In Figure 3 (top-right), we choose $q = 1/2, w = 2$, and $a = 2$. In Figure 3 (bottom-left), we choose $q = 1/2, w = 3$, and $a = 2$. In Figure 3 (bottom-right), we choose $q = 1/2, w = 4$, and $a = 2$.

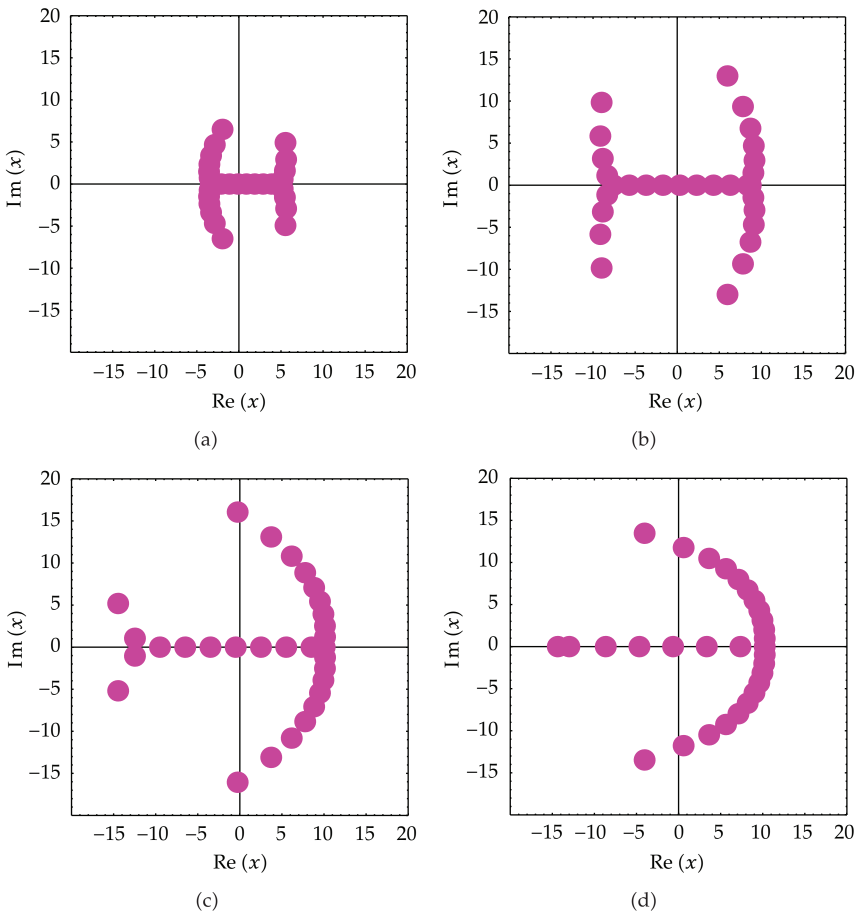

Figure 1: Zeros of $E_{n,q,w}(x : a)$ for $a = 1, 2, 3, 4$.

Stacks of zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 30, q = 1/2, w = 4$, and $a = 4$ from a 3-D structure are presented (Figure 4).

Our numerical results for approximate solutions of real zeros of the generalized $E_{n,q,w}(x : a)$ are displayed (Tables 1 and 2).

We observe a remarkably regular structure of the complex roots of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$. We hope to verify a remarkably regular structure of the complex roots of the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ (Table 1).

Next, we calculated an approximate solution satisfying $E_{n,q,w}(x : a), q = 1/2, w = 2, a = 2, x \in \mathbb{R}$. The results are given in Table 2.

Figure 5 shows the generalized (q, w)-Euler polynomials $E_{n,q,w}(x : a)$ for real $-9/10 \leq q \leq 9/10$ and $-5 \leq x \leq 5$, with the zero contour indicated in black (Figure 5). In Figure 5 (top-left), we choose $n = 1$, $w = 2$, and $a = 2$. In Figure 5 (top-right), we choose $n = 2$, $w = 2$, and $a = 2$. In Figure 5 (bottom-left), we choose $n = 3$, $w = 2$, and $a = 2$. In Figure 5 (bottom-right), we choose $n = 4, w = 2$, and $a = 2$.

Finally, we will consider the more general problems. How many roots does $E_{n,q,w}(x : a)$ have? This is an open problem. Prove or disprove: $E_{n,q,w}(x : a) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x:a)}$ of $E_{n,q,w}(x : a), \mathrm{Im}(x : a) \neq 0$. Since n is

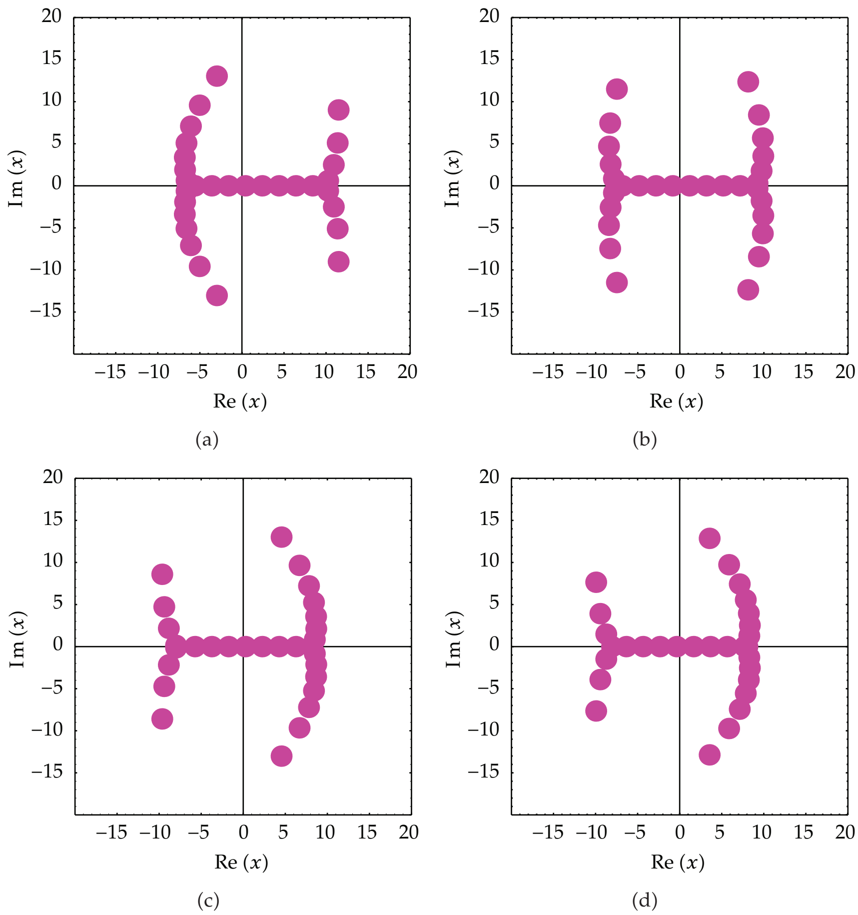

Figure 2: Zeros of $E_{n,q,w}(x : a)$ for $q = 1/10, 3/10, 7/10, 9/10$.

Table 1: Numbers of real and complex zeros of $E_{n,q,w}(x : a)$.

n	$q = 1/2, w = 2, a = 2$		$q = 1/2, w = 4, a = 4$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	1	2
4	2	2	2	2
5	3	2	1	4
6	4	2	2	4
7	3	4	3	4
8	4	4	2	6
9	3	6	3	6
10	4	6	2	8
11	5	6	3	8
12	6	6	4	8
13	5	8	3	10

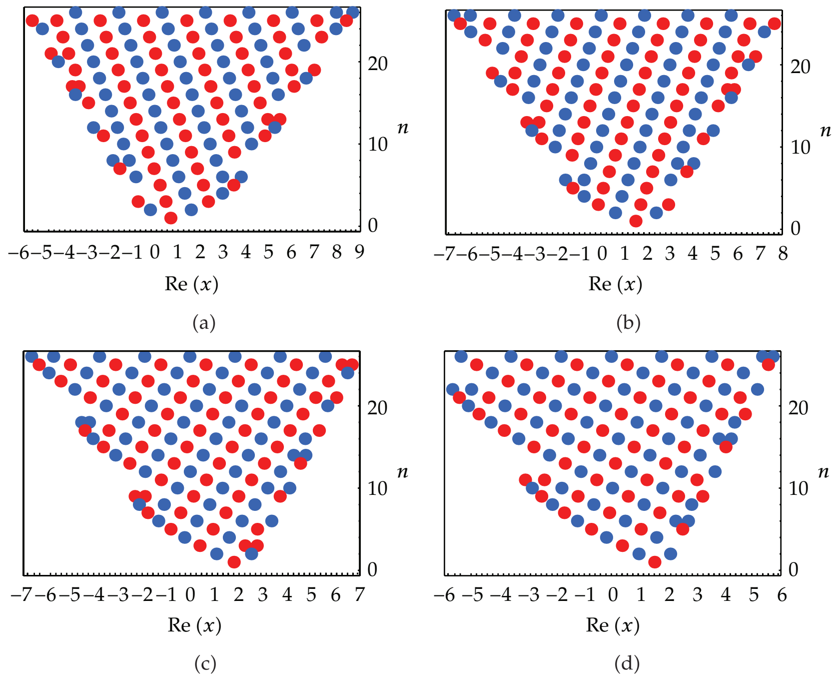

Figure 3: Real zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 25$.

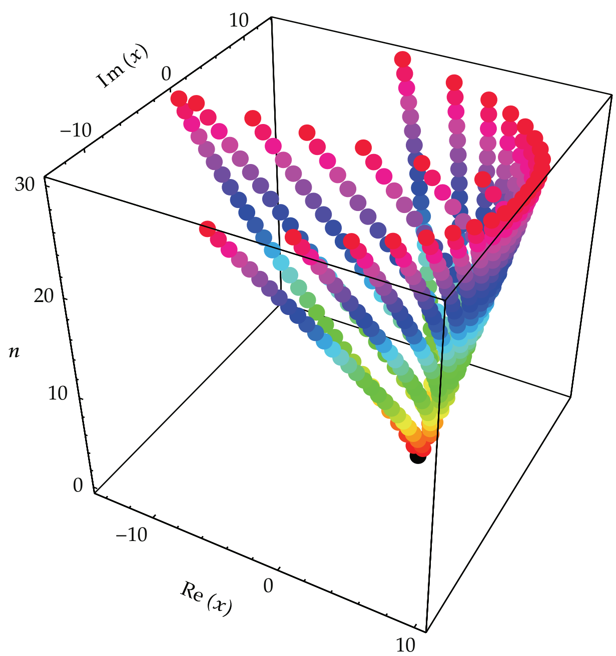

Figure 4: Stacks of zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 30$.

Table 2: Approximate solutions of $E_{n,q,w}(x : a) = 0, x \in \mathbb{R}$.

n	x
1	1.3333
2	0.3905, 2.2761
3	−0.4011, 1.560, 2.841
4	−1.0546, 0.6907
5	−1.5732, −0.17085, 1.829
6	−1.9151, −1.0557, 0.9680, 2.94
7	0.10585, 2.106, 3.68
8	−0.7557, 1.2442, 3.26, 4.00
9	−1.6091, 0.3825, 2.382
10	−2.392, −0.4793, 1.521, 3.52
11	−3.013, −1.3411, 0.6590, 2.66, 4.4

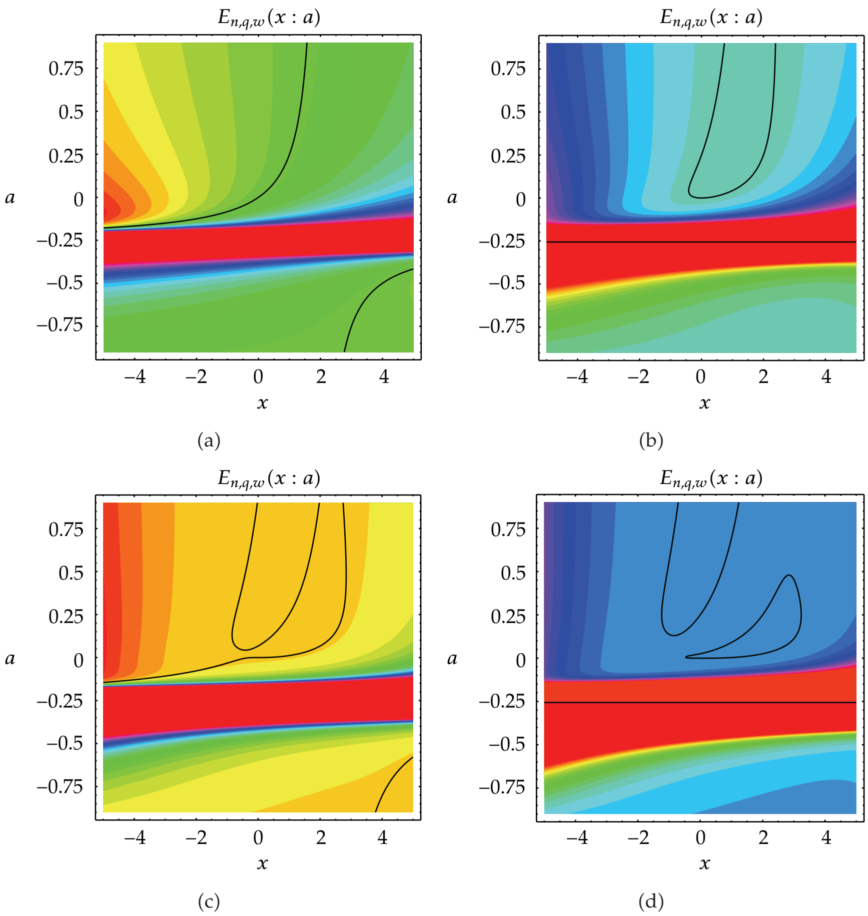

Figure 5: Zero contour of $E_{n,q,w}(x : a)$.

the degree of the polynomial $E_{n,q,w}(x:a)$, the number of real zeros $R_{E_{n,q,w}(x:a)}$ lying on the real plane $\mathrm{Im}(x:a)=0$ is then $R_{E_{n,q,w}(x:a)} = n - C_{E_{n,q,w}(x:a)}$, where $C_{E_{n,q,w}(x:a)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}(x:a)}$ and $C_{E_{n,q,w}(x:a)}$. We plot the zeros of $E_{n,q,w}(x:a)$, respectively (Figures 1–5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,q,w}(x:a)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of (q,w)-Euler polynomials $E_{n,q,w}(x:a)$ to appear in mathematics and physics.

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Research Article

Some Properties of Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

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The present paper deals with the various q-Genocchi numbers and polynomials. We define a new type of multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β by applying the method of p-adic q-integral. We will find a link between their numbers and polynomials with weight α and weak weight β. Also we will obtain the interesting properties of their numbers and polynomials with weight α and weak weight β. Moreover, we construct a Hurwitz-type zeta function which interpolates multiple generalized q-Genocchi polynomials with weight α and weak weight β and find some combinatorial relations.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$ (see [1–21]). When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < 1$.

Throughout this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Hence $\lim_{q\to 1}[x]_q = x$ for all $x \in \mathbb{Z}_p$ (see [1–14, 16, 18, 20, 21]).

We say that $g : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and we write $g \in \mathrm{UD}(\mathbb{Z}_p)$ if the difference quotients $\Phi_g : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that

$$\Phi_g(x,y) = \frac{g(x) - g(y)}{x - y} \tag{1.2}$$

have a limit $g'(a)$ as $(x,y) \to (a,a)$.

Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N, we set

$$\begin{gathered}
\mathrm{X} = \mathrm{X}_d = \varprojlim_{N}\left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad \mathrm{X}_1 = \mathbb{Z}_p, \\
\mathrm{X}^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\
a + dp^N\mathbb{Z}_p = \left\{x \in \mathrm{X} \mid x \equiv a\left(\bmod dp^N\right)\right\},
\end{gathered} \tag{1.3}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q} \tag{1.4}$$

is known to be a distribution on X.

For $g \in \mathrm{UD}(\mathbb{Z}_p)$, Kim defined the q-deformed fermionic p-adic integral on $\mathbb{Z}_p$:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{\left[p^N\right]_{-q}}\sum_{x=0}^{p^N-1} g(x)(-q)^x. \tag{1.5}$$

(see [1–13]), and note that

$$\int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \int_{\mathrm{X}} g(x)d\mu_{-q}(x). \tag{1.6}$$

We consider the case $q \in (-1,0)$ corresponding to q-deformed fermionic certain and annihilation operators and the literature given there in [9, 13, 14].

In [9, 12, 14, 19], we introduced multiple generalized Genocchi number and polynomials. Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. We assume that f

is odd. Then the multiple generalized Genocchi numbers, $G_{n,\chi}^{(r)}$, and the multiple generalized Genocchi polynomials, $G_{n,\chi}^{(r)}(x)$, associated with χ, are defined by

$$F_{\chi}^{(r)}(t) = \left(\frac{2t \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft}+1} \right)^r = \sum_{n=0}^{\infty} G_{n,\chi}^{(r)} \frac{t^n}{n!},$$
$$F_{\chi}^{(r)}(t,x) = \left(\frac{2t \sum_{a=0}^{f-1} \chi(a)(-1)^a e^{at}}{e^{ft}+1} e^{tx} \right)^r = \sum_{n=0}^{\infty} G_{n,\chi}^{(r)}(x) \frac{t^n}{n!}. \tag{1.7}$$

In the special case $x = 0$, $G_{n,\chi}^{(r)} = G_{n,\chi}^{(r)}(0)$ are called the nth multiple generalized Genocchi numbers attached to χ.

Now, having discussed the multiple generalized Genocchi numbers and polynomials, we were ready to multiple-generalize them to their q-analogues. In generalizing the generating functions of the Genocchi numbers and polynomials to their respective q-analogues; it is more useful than defining the generating function for the Genocchi numbers and polynomials (see [12]).

Our aim in this paper is to define multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β. We investigate some properties which are related to multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β. We also derive the existence of a specific interpolation function which interpolate multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β at negative integers.

2. The Generating Functions of Multiple Generalized q-Genocchi Numbers and Polynomials with Weight α and Weak Weight β

Many mathematicians constructed various kinds of generating functions of the q-Gnocchi numbers and polynomials by using p-adic q-Vokenborn integral. First we introduce multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β.

Let us define the generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta)}(x)$ with weight α and weak weight β, respectively,

$$F_{\chi,q}^{(\alpha,\beta)}(t) = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)} \frac{t^n}{n!} = \int_X t\chi(x) e^{[x]_{q^\alpha} t} d\mu_{-q^\beta}(x),$$
$$F_{\chi,q}^{(\alpha,\beta)}(t,x) = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)}(x) \frac{t^n}{n!} = \int_X t\chi(y) e^{[x+y]_{q^\alpha} t} d\mu_{-q^\beta}(y). \tag{2.1}$$

By using the Taylor expansion of $e^{[x]_{q^\alpha} t}$, we have

$$\sum_{n=0}^{\infty} \int_X \chi(x) [x]_{q^\alpha}^n d\mu_{-q^\beta}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta)} \frac{t^{n-1}}{n!} = G_{0,\chi,q}^{(\alpha,\beta)} + \sum_{n=0}^{\infty} \frac{G_{n+1,\chi,q}^{(\alpha,\beta)}}{n+1} \frac{t^n}{n!}. \tag{2.2}$$

By comparing the coefficient of both sides of $t^n/n!$ in (2.2), we get

$$\frac{G_{n+1,\chi,q}^{(\alpha,\beta)}}{n+1} = \frac{[2]_{q^\beta}}{(1-q^\alpha)^n}\sum_{a=0}^{f-1}(-1)^a q^{\beta a}\chi(a)\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha a l}\frac{1}{1+q^{f(\alpha l+\beta)}}. \tag{2.3}$$

From (2.2) and (2.3), we can easily obtain that

$$\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(t\int_X \chi(x)[x]_{q^\alpha}^n d\mu_{-q^\beta}(x)\right)\frac{t^n}{n!} = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[l]_{q^\alpha}t}. \tag{2.4}$$

Therefore, we obtain

$$F_{\chi,q}^{(\alpha,\beta)}(t) = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[l]_{q^\alpha}t} = \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}\frac{t^n}{n!}. \tag{2.5}$$

Similarly, we find the generating function of generalized q-Genocchi polynomials with weight α and weak weight β:

$$G_{0,\chi,q}^{(\alpha,\beta)}(x) = 0, \qquad \frac{G_{n+1,\chi,q}^{(\alpha,\beta)}(x)}{n+1} = \int_X \chi(y)[x+y]_{q^\alpha}^n d\mu_{-q^\beta}(y) = [2]_{q^\beta}\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)[x+l]_{q^\alpha}^n. \tag{2.6}$$

From (2.6), we have

$$F_{\chi,q}^{(\alpha,\beta)}(t,x) = [2]_{q^\beta}t\sum_{l=0}^{\infty}(-1)^l q^{\beta l}\chi(l)e^{[x+l]_{q^\alpha}t} = \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta)}(x)\frac{t^n}{n!}. \tag{2.7}$$

Observe that $F_{\chi,q}^{(\alpha,\beta)}(t) = F_{\chi,q}^{(\alpha,\beta)}(t,0)$. Hence we have $G_{n,\chi,q}^{(\alpha,\beta)} = G_{n,\chi,q}^{(\alpha,\beta)}(0)$. If $q \to 1$ into (2.7), then we easily obtain $F_\chi(t,x)$.

First, we define the multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ with weight α and weak weight β:

$$\begin{aligned}
F_{\chi,q}^{(\alpha,\beta,r)}(t) &= [2]_{q^\beta}^r t^r \sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i} q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^{r}\chi(k_i)\right)e^{[\sum_{i=1}^r k_i]_{q^\alpha}t} \\
&= t^r \underbrace{\int_X\cdots\int_X}_{r\text{-times}} \chi(x_1)\cdots\chi(x_r)e^{[x_1+\cdots+x_r]_{q^\alpha}t}d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r) \\
&= \sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(x_1)\cdots\chi(x_r)[x_1+\cdots+x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r)\frac{t^n}{n!} \\
&\quad=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^{n-r}}{n!}=\sum_{n=0}^{r-1}G_{n,\chi,q}^{(\alpha,\beta,r)}\frac{t^{n-r}}{n!}+\sum_{n=0}^{\infty}\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r}r!}\frac{t^n}{n!},
\end{aligned} \tag{2.9}$$

where $\binom{n+r}{r}=(n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.9), we obtain the following theorem.

Theorem 2.1. *Let $q\in\mathbb{C}_p$ with $|1-q|_p<1$ and $n\in\mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&G_{0,\chi,q}^{(\alpha,\beta,r)}=G_{1,\chi,q}^{(\alpha,\beta,r)}=\cdots=G_{r-1,\chi,q}^{(\alpha,\beta,r)}=0, \\
&\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}}{\binom{n+r}{r}r!}=\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(x_1)\cdots\chi(x_r)[x_1+\cdots+x_r]_{q^\alpha}^n d\mu_{-q^\beta}(x_1)\cdots d\mu_{-q^\beta}(x_r) \\
&\quad=\frac{[2]_{q^\beta}^r}{(1-q^\alpha)^n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}\left(\prod_{i=1}^r\chi(a_i)\right)\frac{(-1)^{l+\sum_{i=1}^r a_i}q^{(\alpha l+\beta)\sum_{i=1}^r a_i}}{\left(1+q^{f(\alpha l+\beta)}\right)^r} \\
&\quad=[2]_{q^\beta}^r\sum_{m=0}^{\infty}\sum_{a_1,\ldots,a_r=0}^{f-1}\binom{m+r-1}{m}(-1)^{\sum_{i=1}^r a_i+m}q^{\beta(\sum_{i=1}^r a_i+fm)}\times\left(\prod_{i=1}^r\chi(a_i)\right)\left[\sum_{i=1}^r a_i+fm\right]_{q^\alpha}^n.
\end{aligned} \tag{2.10}$$

From now on, we define the multiple generalized q-Genocchi polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β.

$$\begin{aligned}
F_{\chi,q}^{(\alpha,\beta,r)}(t,x)&=[2]_{q^\beta}^r t^r\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r\chi(k_i)\right)e^{[\sum_{i=1}^r k_i+x]_{q^\alpha}t} \\
&=t^r\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)e^{[x+y_1+\cdots+y_r]_{q^\alpha}t}d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r) \\
&=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)\left[x+y_1+\cdots+y_r\right]_{q^\alpha}^n d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r)\frac{t^n}{n!}\\
&\quad=\sum_{n=0}^{\infty}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^{n-r}}{n!}=\sum_{n=0}^{r-1}G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^{n-r}}{n!}+\sum_{n=0}^{\infty}\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!}\frac{t^n}{n!},
\end{aligned}\tag{2.12}$$

where $\binom{n+r}{r}=(n+r)!/n!r!$.

By comparing the coefficients on the both sides of (2.12), we have the following theorem.

Theorem 2.2. *Let $q\in\mathbb{C}_p$ with $|1-q|_p<1$ and $n\in\mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&G_{0,\chi,q}^{(\alpha,\beta,r)}(x)=G_{1,\chi,q}^{(\alpha,\beta,r)}(x)=\cdots=G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x)=0,\\
\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!}&=\underbrace{\int_X\cdots\int_X}_{r\text{-times}}\chi(y_1)\cdots\chi(y_r)\left[x+y_1+\cdots+y_r\right]_{q^\alpha}^n d\mu_{-q^\beta}(y_1)\cdots d\mu_{-q^\beta}(y_r)\\
&=\frac{[2]_{q^\beta}^r}{(1-q^\alpha)^n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{l=0}^{n}\binom{n}{l}\left(\prod_{i=1}^{r}\chi(a_i)\right)\frac{(-1)^{l+\sum_{i=1}^r a_i}q^{\alpha lx+(\alpha l+\beta)\sum_{i=1}^r a_i}}{\left(1+q^{f(\alpha l+\beta)}\right)^r}\\
&=[2]_{q^\beta}^r\sum_{m=0}^{\infty}\sum_{a_1,\ldots,a_r=0}^{f-1}\binom{m+r-1}{m}(-1)^{\sum_{i=1}^r a_i+m}q^{\beta(\sum_{i=1}^r a_i+fm)}\\
&\quad\times\left(\prod_{i=1}^{r}\chi(a_i)\right)\left[\sum_{i=1}^{r}a_i+fm+x\right]_{q^\alpha}^n.
\end{aligned}\tag{2.13}$$

In (2.11), we simply identify that

$$\begin{aligned}
\lim_{q\to1}F_{\chi,q}^{(\alpha,\beta,r)}(t,x)&=2^rt^r\sum_{k_1,\ldots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}\left(\prod_{i=1}^{r}\chi(k_i)\right)e^{(\sum_{i=1}^r k_i+x)t}\\
&=\left(\frac{2t\sum_{a=0}^{f-1}(-1)^a\chi(a)e^{at}}{1+e^{ft}}\right)^r e^{tx}=F_\chi^{(r)}(t,x).
\end{aligned}\tag{2.14}$$

So far, we have studied the generating functions of the multiple generalized q-Genocchi numbers $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and polynomials $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β.

3. Modified Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

In this section, we will investigate about modified multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β. Also, we will find their relations in multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β.

Firstly, we modify generating functions of $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We access some relations connected to these numbers and polynomials with weight α and weak weight β. For this reason, we assign generating function of modified multiple generalized q-Genocchi numbers and polynomials with weight α and weak weight β which are implied by $G_{n,\chi,q}^{(\alpha,\beta,r)}$ and $G_{n,\chi,q}^{(\alpha,\beta,r)}(x)$. We give relations between these numbers and polynomials with weight α and weak weight β.

We modify (2.11) as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = F_{\chi,q}^{(\alpha,\beta,r)}\left(q^{-\alpha x}t, x\right), \tag{3.1}$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t,x)$ is defined in (2.11).

From the above we know that

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = \sum_{n=0}^{\infty} q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}. \tag{3.2}$$

After some elementary calculations, we attain

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = q^{-\alpha r x} e^{(q^{-\alpha x}[x]_{q^\alpha} t)} F_{\chi,q}^{(\alpha,\beta,r)}(t), \tag{3.3}$$

where $F_{\chi,q}^{(\alpha,\beta,r)}(t)$ is defined in (2.8).

From the above, we can assign the modified multiple generalized q-Genocchi polynomials $\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x)$ with weight α and weak weight β as follows:

$$\mathfrak{F}_{\chi,q}^{(\alpha,\beta,r)}(t,x) = \sum_{n=0}^{\infty} \varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}. \tag{3.4}$$

Then we have

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} G_{n,\chi,q}^{(\alpha,\beta,r)}(x). \tag{3.5}$$

Theorem 3.1. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, one has*

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{i=0}^{n} \binom{n}{i} q^{\alpha i x} [x]_{q^\alpha}^{n-i} G_{i,\chi,q}^{(\alpha,\beta,r)}. \tag{3.6}$$

Corollary 3.2. *For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$, by using* (3.7), *one easily obtains*

$$\varepsilon_{n,\chi,q}^{(\alpha,\beta,r)}(x) = q^{-(n+r)\alpha x} \sum_{m=0}^{\infty} \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j,l,n-j-l} \binom{n-j+m-1}{m} (-1)^l q^{\alpha\{(j+l)x+m\}} G_{j,\chi,q}^{(\alpha,\beta,r)}. \quad (3.7)$$

Secandly, by using generating function of the multiple generalized q-Genocchi polynomials with weight α and weak weight β, which is defined by (2.11), we obtain the following identities.

By using (2.13), we find that

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{m=0}^{\infty} \sum_{a_1,\ldots,a_r=0}^{f-1} \binom{m+r-1}{m} (-1)^{\sum_{i=1}^r a_i + m} \\ &\quad \times q^{\beta(\sum_{i=1}^r a_i + fm)} \left(\prod_{i=1}^r \chi(a_i) \right) \left[\sum_{i=1}^r a_i + fm + x \right]_{q^\alpha}^n \\ &= [2]_{q^\beta}^r \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{l=0}^{n} \sum_{a=0}^{l} \binom{n}{a, l-a, n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{f\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.8)$$

Thus we have the following theorem.

Theorem 3.3. *Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$ and $r \in \mathbb{N}$. Then one has*

$$\begin{aligned} \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!} &= [2]_{q^\beta}^r \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{l=0}^{n} \sum_{a=0}^{l} \binom{n}{a, l-a, n-l} (-1)^{a+\sum_{i=1}^r a_i} q^{\{\alpha(a+n-l)+\beta\} \sum_{i=1}^r a_i} \\ &\quad \times \left(\prod_{i=1}^r \chi(a_i) \right) \frac{[x]_{q^\alpha}^{n-l}}{(1-q^\alpha)^l (1+q^{f\{\alpha(a+n-l)+\beta\}})^r}. \end{aligned} \quad (3.9)$$

By using (2.13), we have

$$\begin{aligned} F_{\chi,q}^{(\alpha,\beta,r)}(t,x) &= [2]_{q^\beta}^r t^r \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l q^{\alpha l x}}{(1-q)^n} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} \\ &\quad \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)} \left(\prod_{i=1}^r \chi(a_i) \right) \sum_{m=0}^{\infty} \binom{m+r-1}{m} \left(-q^{f(\alpha l+\beta)} \right)^m \frac{t^n}{n!}. \end{aligned} \quad (3.10)$$

Thus we have

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[2]_{q^\beta}^r t^r \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}(1-q^\alpha)^{-n}\sum_{a_1,\dots,a_r=0}^{f-1}(-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(1+q^{f(\alpha l+\beta)}\right)^{-r}\frac{t^n}{n!}. \tag{3.11}$$

By comparing the coefficients of both sides of $(n+r)!/t^{n+r}$ in the above, we arrive at the following theorem.

Theorem 3.4. *Let $q \in \mathbb{C}_p$ with $|1-q|_p < 1$, $r \in \mathbb{N}$. Then one has*

$$\frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r}r!} = [2]_{q^\beta}^r \sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{\alpha l x}(1-q^\alpha)^{-n}\sum_{a_1,\dots,a_r=0}^{f-1}(-1)^{\sum_{i=1}^r a_i} \times q^{(\alpha l+\beta)(\sum_{i=1}^r a_i)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(1+q^{f(\alpha l+\beta)}\right)^{-r}. \tag{3.12}$$

From (2.12), we easily know that

$$\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[2]_{q^\beta}^r\binom{n+r}{r}r!\sum_{k_1,\dots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r \chi(k_i)\right) \times \left[x+\sum_{i=1}^r k_i\right]_{q^\alpha}^n \frac{t^{n+r}}{(n+r)!}. \tag{3.13}$$

From the above, we get the following theorem.

Theorem 3.5. *Let $r \in \mathbb{N}$, $k \in \mathbb{Z}_+$. Then one has*

$$G_{0,\chi,q}^{(\alpha,\beta,r)}(x) = G_{1,\chi,q}^{(\alpha,\beta,r)}(x) = \cdots = G_{r-1,\chi,q}^{(\alpha,\beta,r)}(x) = 0,$$
$$G_{l+r,\chi,q}^{(\alpha,\beta,r)}(x) = [2]_{q^\beta}^r\binom{l+r}{r}r!\sum_{k_1,\dots,k_r=0}^{\infty}(-1)^{\sum_{i=1}^r k_i}q^{\beta\sum_{i=1}^r k_i}\left(\prod_{i=1}^r \chi(k_i)\right)\left[x+\sum_{i=1}^r k_i\right]_{q^\alpha}^l. \tag{3.14}$$

From (2.13), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,r)}(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} G_{n,\chi,q}^{(\alpha,\beta,s)}(x)\frac{t^n}{n!}\\
&\quad= [2]_{q^\beta}^{r+s} t^{r+s}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{m=0}^{\infty}\binom{m+r-1}{m}(-1)^{\sum_{i=1}^r a_i+m} q^{\beta(\sum_{i=1}^r a_i+fm)}\\
&\qquad\times\left(\prod_{i=1}^r \chi(a_i)\right) e^{[\sum_{i=1}^r a_i+fm+x]_{q^\alpha} t}\sum_{b_1,\ldots,b_s=0}^{f-1}\sum_{k=0}^{\infty}\binom{k+s-1}{k}(-1)^{\sum_{i=1}^s b_i+k}\\
&\qquad\times q^{\beta(\sum_{i=1}^s b_i+fk)}\left(\prod_{i=1}^s \chi(b_i)\right) e^{[\sum_{i=1}^r b_i+fk+x]_{q^\alpha} t}.
\end{aligned} \tag{3.15}$$

By using Cauchy product in (3.15), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{n-j,\chi,q}^{(\alpha,\beta,s)}(x)\frac{t^n}{n!}\\
&\quad= [2]_{q^\beta}^{r+s} t^{r+s}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{b_1,\ldots,b_s=0}^{f-1}\binom{j+r-1}{j}\binom{n-j+s-1}{n-j}\\
&\qquad\times(-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(\prod_{i=1}^s \chi(b_i)\right)\\
&\qquad\times e^{[\sum_{i=1}^r a_i+fj+x]_{q^\alpha} t} e^{[\sum_{i=1}^s b_i+f(n-j)+x]_{q^\alpha} t}.
\end{aligned} \tag{3.16}$$

From (3.16), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{m-j,\chi,q}^{(\alpha,\beta,s)}(x)\right)\frac{t^m}{m!}\\
&\quad= \sum_{m=0}^{\infty}[2]_{q^\beta}^{r+s} t^{r+s}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{a_1,\ldots,a_r=0}^{f-1}\sum_{b_1,\ldots,b_s=0}^{f-1}\binom{j+r-1}{j}\binom{n-j+s-1}{n-j}\\
&\qquad\times(-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)}\left(\prod_{i=1}^r \chi(a_i)\right)\left(\prod_{i=1}^s \chi(b_i)\right)\\
&\qquad\times\left(\left[\sum_{i=1}^r a_i+fj+x\right]_{q^\alpha}+\left[\sum_{i=1}^s b_i+f(n-j)+x\right]_{q^\alpha}\right)^m\frac{t^m}{m!}.
\end{aligned} \tag{3.17}$$

By comparing the coefficients of both sides of $t^{m+r+s}/(m+r+s)!$ in (3.17), we have the following theorem.

Theorem 3.6. *Let $r \in \mathbb{N}$and $s \in \mathbb{Z}_+$. Then one has*

$$\begin{aligned}
&\frac{\sum_{j=0}^{l+r+s}\binom{l+r+s}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+s-j,\chi,q}^{(\alpha,\beta,s)}(x)}{\binom{l+r+s}{l}(r+s)!} \\
&\quad = [2]_{q^\beta}^{r+s} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{b_1,\ldots,b_s=0}^{f-1} \binom{j+r-1}{j}\binom{n-j+s-1}{n-j} \\
&\quad\quad \times (-1)^{\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+n} q^{\beta(\sum_{i=1}^r a_i+\sum_{i=1}^s b_i+fn)} \left(\prod_{i=1}^r \chi(a_i)\right)\left(\prod_{i=1}^s \chi(b_i)\right) \\
&\quad\quad \times \left(\left[\sum_{i=1}^r a_i+fj+x\right]_{q^\alpha}+\left[\sum_{i=1}^s b_i+f(n-j)+x\right]_{q^\alpha}\right)^l.
\end{aligned} \tag{3.18}$$

Corollary 3.7. *In* (3.18) *setting $s = 1$, one has*

$$\begin{aligned}
&\frac{\sum_{j=0}^{l+r+1}\binom{l+r+1}{j} G_{j,\chi,q}^{(\alpha,\beta,r)}(x) G_{l+r+1-j,\chi,q}^{(\alpha,\beta,1)}(x)}{\binom{l+r+1}{l}(r+1)!} \\
&\quad = [2]_{q^\beta}^{r+1} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{a_1,\ldots,a_r=0}^{f-1} \sum_{b_1=0}^{f-1} \binom{j+r-1}{j} (-1)^{\sum_{i=1}^r a_i+b_1+n} q^{\beta(\sum_{i=1}^r a_i+b_1+fn)} \\
&\quad\quad \times \left(\chi(b_1)\prod_{i=1}^r \chi(a_i)\right)\left(\left[\sum_{i=1}^r a_i+fj+x\right]_{q^\alpha}+\left[b_1+f(n-j)+x\right]_{q^\alpha}\right)^l.
\end{aligned} \tag{3.19}$$

By using (2.13) we have the following theorem.

Theorem 3.8. *Distribution theorem is as follows:*

$$\begin{aligned}
G_{n+r,\chi,q}^{(\alpha,\beta,r)} &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta\sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i)\right) G_{n+r,q^f}^{(\alpha,\beta,r)}\left(\frac{a_1+\cdots+a_r}{f}\right), \\
G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x) &= \frac{[f]_{q^\alpha}^n}{[f]_{-q^\beta}^r} \sum_{a_1,\ldots,a_r=0}^{f-1} (-1)^{\sum_{i=1}^r a_i} q^{\beta\sum_{i=1}^r a_i} \left(\prod_{i=1}^r \chi(a_i)\right) G_{n+r,q^f}^{(\alpha,\beta,r)}\left(\frac{x+a_1+\cdots+a_r}{f}\right).
\end{aligned} \tag{3.20}$$

4. Interpolation Function of Multiple Generalized q-Genocchi Polynomials with Weight α and Weak Weight β

In this section, we see interpolation function of multiple generalized q-Genocchi polynomials with weak weight α and find some relations.

Let us define interpolation function of the $G_{k+r,q}^{(\alpha,\beta,r)}(x)$ as follows.

Definition 4.1. Let $q, s \in \mathbb{C}$ with $|q| < 1$ and $0 < x \leq 1$. Then one defines

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(s,x) = [2]_{q^\beta}^r \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right)}{\left[x + \sum_{i=1}^r k_i\right]_{q^\alpha}^s}. \tag{4.1}$$

We call $\zeta_q^{(\alpha,\beta,r)}(s,x)$ the multiple generalized Hurwitz type q-zeta funtion.

In (4.1), setting $r = 1$, we have

$$\zeta_{\chi,q}^{(\alpha,\beta,1)}(s,x) = [2]_{q^\beta} \sum_{l=0}^{\infty} \frac{(-1)^l q^{\beta l} \chi(l)}{[x+l]_{q^\alpha}^s} = \zeta_{\chi,q}^{(\alpha,\beta)}(s,x). \tag{4.2}$$

Remark 4.2. It holds that

$$\lim_{q \to 1} \zeta_{\chi,q}^{(\alpha,\beta,r)}(s,x) = 2^r \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(-1)^{\sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right)}{\left(x + \sum_{i=1}^r k_i\right)^s}. \tag{4.3}$$

Substituting $s = -n, n \in \mathbb{Z}_+$ into (4.1), then we have,

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n,x) = [2]_{q^\beta}^r \sum_{k_1,\ldots,k_r=0}^{\infty} (-1)^{\sum_{i=1}^r k_i} q^{\beta \sum_{i=1}^r k_i} \left(\prod_{i=1}^r \chi(k_i)\right) \left[x + \sum_{i=1}^r k_i\right]_{q^\alpha}^n. \tag{4.4}$$

Setting (3.14) into the above, we easily get the following theorem.

Theorem 4.3. *Let $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$. Then one has*

$$\zeta_{\chi,q}^{(\alpha,\beta,r)}(-n,x) = \frac{G_{n+r,\chi,q}^{(\alpha,\beta,r)}(x)}{\binom{n+r}{r} r!}. \tag{4.5}$$

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Research Article

Calculating Zeros of the q-Genocchi Polynomials Associated with p-Adic q-Integral on $\mathbb{Z}_p$

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In this paper we construct the new analogues of Genocchi the numbers and polynomials. We also observe the behavior of complex roots of the q-Genocchi polynomials $G_{n,q}(x)$, using numerical investigation. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the q-Genocchi polynomials $G_{n,q}(x)$. Finally, we give a table for the solutions of the q-Genocchi polynomials $G_{n,q}(x)$.

1. Introduction

Many mathematicians have the studied Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Genocchi numbers and the Genocchi polynomials. The Bernoulli numbers and polynomials, the Euler numbers and polynomials, and the Genocchi polynomials posses many interesting properties and arising in many areas of mathematics and physics (see [1–12]). We introduce the new analogs of the Genocchi numbers and polynomials. In the 21st century, the computing environment would make more and more rapid progress. Using computer, a realistic study for new analogs of Genocchi numbers and polynomials is very interesting. It is the aim of this paper to observe an interesting phenomenon of "scattering" of the zeros of q-Genocchi polynomials $G_{n,q}(x)$. The outline of this paper is as follows. In Section 2, we study the q-Genocchi polynomials $G_{n,q}(x)$. In Section 3, we describe the beautiful zeros of q-Genocchi polynomials $G_{n,q}(x)$ using a numerical investigation. Also we display distribution and structure of the zeros of the q-Genocchi polynomials $G_{n,q}(x)$ by using computer. By using the results of our paper, the readers can observe the regular behaviour of the roots of q-Genocchi polynomials $G_{n,q}(x)$. Finally, we carried out computer experiments that demonstrate a remarkably regular structure of the complex roots of q-Genocchi polynomials $G_{n,q}(x)$. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in C_p$. If $q \in C$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \le 1$:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.1}$$

Compare [1, 2, 4, 10, 11, 13–16]. Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = \lim_{\overleftarrow{N}} \left(\frac{\mathbb{Z}}{dp^N\mathbb{Z}}\right), \qquad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \quad a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a\left(\operatorname{mod} dp^N\right)\right\}, \tag{1.2}$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$. For any positive integer N,

$$\mu_q\left(a + dp^N\mathbb{Z}_p\right) = \frac{q^a}{\left[dp^N\right]_q} \tag{1.3}$$

is known to be a distribution on X, cf. [1, 2, 4, 5, 9, 10, 13]. We say that g is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients $F_g(x,y) = f(x) - f(y)/(x-y)$ have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \longrightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}, \tag{1.4}$$

the q-deformed bosonic p-adic integral of the function g is defined by Kim:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{\left[p^N\right]_q} \sum_{0\le x<p^N} g(x)q^x. \tag{1.5}$$

Note that

$$qI_q(f_1) = I_q(f) + (q-1)f(0) + \frac{q-1}{\log q}f'(0), \tag{1.6}$$

where $f_1(x) = f(x+1), f'(0) = df(0)/dx$. Now, the fermionic p-adic invariant q-integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{0 \le x < p^N} g(x)(-q)^x. \tag{1.7}$$

If we take $g_1(x) = g(x+1)$ in (1.7), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.8}$$

From (1.8), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l), \tag{1.9}$$

where $g_n(x) = g(x+n)$. First, we introduce the Genocchi numbers and the Genocchi polynomials. The Genocchi numbers G_n are defined by the generating function:

$$F(t) = \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \qquad (|t| < \pi). \tag{1.10}$$

Compare [4, 9–11, 17], where we use the technique method notation by replacing G^n by $G_n (n \ge 0)$ symbolically. We consider the Genocchi polynomials $G_n(x)$ as follows:

$$F(x,t) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{1.11}$$

Note that $G_n(x) = \sum_{k=0}^{n} \binom{n}{k} G_k x^{n-k}$. In the special case $x = 0$, we define $G_n(0) = G_n$.

2. An Analogue of the Genocchi Numbers and Polynomials

The versions of q-Genocchi numbers and polynomials, which were derived from different considerations and different formulas, were defined by Kim [13, 14]. Kim [14] treated analogue of the Genocchi numbers, which is called q-analogue of the Genocchi numbers. Kim defined the q-extension of the Genocchi numbers and polynomials as follows:

$$\begin{aligned} F_q(t) &= \sum_{n=0}^{\infty} c_{n,q} \frac{t^n}{n!} = e^{t/(1-q)} \sum_{n=0}^{\infty} \frac{(2n+1)}{[2n+1]_q} [n]_q \left(\frac{1}{q-1}\right)^{n-1} \frac{t^n}{n!}, \\ F_q(x,t) &= \sum_{n=0}^{\infty} c_{n,q}(x) \frac{t^n}{n!} = e^{t/(1-q)} \sum_{n=0}^{\infty} \frac{(2n+1)}{[2n+1]_q} [n]_q \left(\frac{1}{q-1}\right)^{n-1} q^{nx} \frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

In [14], Kim introduced the q-analogue of the Genocchi polynomials as follows:

$$G_q(x,t) = [2]_q t \sum_{n=0}^{\infty} (-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.2}$$

We now consider another construction q-Genocchi numbers and polynomials. In (1.8), if we take $g(x) = e^{xt}$, then one has

$$(\log q + t) \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q (\log q + t)}{qe^t + 1}. \tag{2.3}$$

Let us define the q-Genocchi numbers and polynomials as follows:

$$(\log q + t) \int_{\mathbb{Z}_p} e^{yt} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \tag{2.4}$$

$$(\log q + t) \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(y) = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \tag{2.5}$$

Note that $G_{n,q}(0) = G_{n,q}$, $\lim_{q \to 1} G_{n,q} = G_n$, where G_n are the nth Genocchi numbers. By (2.4) and (2.5), we obtain the following Witt's formula.

Theorem 2.1. *For $q \in \mathbb{C}_p$ with $|1-q|_p \leq p^{-1/(p-1)}$, we have*

$$\begin{gathered} n \int_{\mathbb{Z}_p} x^{n-1} d\mu_{-q}(x) + \log q \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = G_{n,q}, \\ n \int_{\mathbb{Z}_p} (x+y)^{n-1} d\mu_{-q}(y) + \log q \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(y) = G_{n,q}(x). \end{gathered} \tag{2.6}$$

By the above theorem, easily see that

$$G_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} G_{k,q}. \tag{2.7}$$

Let q be a complex number with $|q| < 1$. By the meaning of (1.10) and (1.11), let us define the q-Genocchi numbers $G_{n,q}$ and polynomials $G_{n,q}(x)$ as follows:

$$\begin{gathered} F_q(t) = \frac{[2]_q (\log q + t)}{qe^t + 1} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{n!}, \\ F_q(x,t) = \frac{[2]_q (\log q + t)}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_{n,q}(x) \frac{t^n}{n!}. \end{gathered} \tag{2.8}$$

For q-Euler numbers, Kim constructed q-Euler numbers which can be uniquely determined by

$$q\left(qE_q+1\right)^n+E_{n,q}=\begin{cases}[2]_q, & \text{if } n=0,\\ 0, & \text{if } n>0,\end{cases} \tag{2.9}$$

with the usual convention of symbolically replacing E_q^n by $E_{n,q}$, where $E_{n,q}$ denotes the q-Euler numbers. For q-Genocchi numbers, we have the following theorem.

Theorem 2.2. *q-Genocchi numbers $G_{n,q}$ are defined inductively by*

$$G_{0,q}=\frac{[2]_q\log q}{1+q}, \qquad q\left(G_q+1\right)^n+G_{n,q}=\begin{cases}[2]_q, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{2.10}$$

with the usual convention about replacing $(G_q)^n$ by $G_{n,q}$ in the binomial expansion.

Proof. From (2.4), we obtain

$$\frac{[2]_q(\log q+t)}{qe^t+1}=\sum_{n=0}^{\infty}G_{n,q}\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(G_q\right)^n\frac{t^n}{n!}=e^{G_qt}, \tag{2.11}$$

which yields

$$[2]_q(\log q+t)=\left(qe^t+1\right)e^{G_qt}=qe^{(G_q+1)t}+e^{G_qt}. \tag{2.12}$$

Using the Taylor expansion of exponential function, we have

$$\begin{aligned}[2]_q\log q+[2]_qt&=\sum_{n=0}^{\infty}\left\{q\left(G_q+1\right)^n+\left(G_q\right)^n\right\}\frac{t^n}{n!}\\ &=q\left(G_q+1\right)^0+\left(G_q\right)^0+q\left(G_q+1\right)^1+\left(G_q\right)^1\\ &\quad+\sum_{n=2}^{\infty}\left\{q\left(G_q+1\right)^n+\left(G_q\right)^n\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

The result follows by comparing the coefficients.

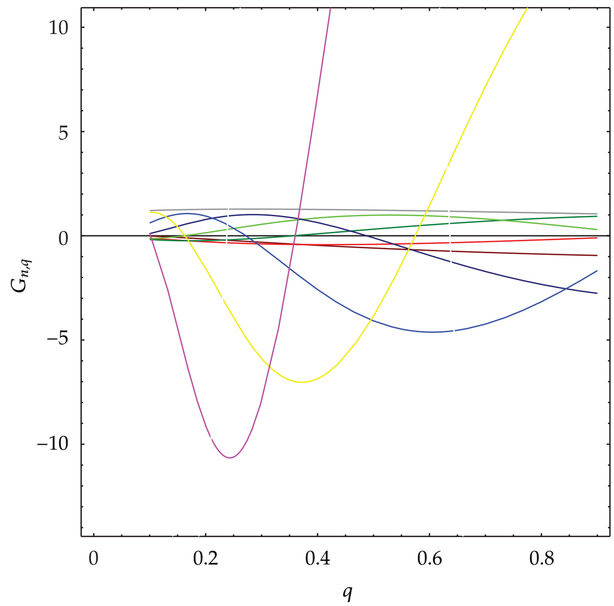

Figure 1: Curves of $G_{n,q}$.

Here is the list of the first q-Genocchi numbers $G_{n,q}$:

$$G_{0,q} = \log q,$$
$$G_{1,q} = -\frac{-1 - q + q\log q}{(1+q)},$$
$$G_{2,q} = \frac{q(-2 - 2q - \log q + q\log q)}{(1+q)^2}, \tag{2.14}$$
$$G_{3,q} = -\frac{q(3 - 3q^2 + \log q - 4q\log q + q^2\log q)}{(1+q)^3},$$
$$\vdots$$

We display the shapes of the q-Genocchi numbers $G_{n,q}$. For $n = 1, \ldots, 10$, we can draw a curve of $G_{n,q}$, $1/10 \leq q \leq 9/10$, respectively. This shows the ten curves combined into one. We display the shape of $G_{n,q}$: (Figure 1).

Because

$$\frac{\partial}{\partial x}F_q(t,x) = tF_q(t,x) = \sum_{n=0}^{\infty}\frac{d}{dx}G_{n,q}(x)\frac{t^n}{n!}, \tag{2.15}$$

it follows the important relation

$$\frac{d}{dx}G_{n,q}(x) = nG_{n-1,q}(x). \tag{2.16}$$

Here is the list of the first the q-Genocchi polynomials $G_{n,q}(x)$:

$$\begin{gathered}
G_{0,q}(x) = \log q, \\
G_{1,q}(x) = \frac{(1 + q - q\log q + x\log q + qx\log q)}{(1+q)}, \\
\vdots
\end{gathered} \tag{2.17}$$

Since

$$\begin{aligned}
\sum_{l=0}^{\infty} G_{l,q}(x+y)\frac{t^l}{l!} &= \frac{[2]_q \log q + [2]_q t}{qe^t + 1} e^{(x+y)t} \\
&= \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m \frac{t^m}{m!} \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} G_{n,q}(x)\frac{t^n}{n!} y^{l-n}\frac{t^{l-n}}{(l-n)!}\right) \\
&= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} G_{n,q}(x) y^{l-n}\right)\frac{t^l}{l!},
\end{aligned} \tag{2.18}$$

we have the following theorem. □

Theorem 2.3. *q-Genocchi polynomials $G_{n,q}(x)$ satisfy the following relation:*

$$G_{l,q}(x+y) = \sum_{n=0}^{l}\binom{l}{n} G_{n,q}(x) y^{l-n}. \tag{2.19}$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} &= \frac{[2]_q \log q + [2]_q t}{qe^t+1}e^{xt} \\ &= \frac{[2]_q}{m[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a \frac{[2]_{q^m}\log q^m + [2]_{q^m} mt}{q^m e^{mt}+1}e^{(a/m+x/m)(mt)} \\ &= \frac{[2]_q}{m[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a \sum_{n=0}^{\infty} G_{n,q^m}\left(\frac{a+x}{m}\right)\frac{(mt)^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(m^{n-1}\frac{[2]_q}{[2]_{q^m}}\sum_{a=0}^{m-1}(-1)^a q^a G_{n,q^m}\left(\frac{a+x}{m}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Hence we have the following theorem.

Theorem 2.4. *For any positive integer m(=odd), one obtains*

$$G_{n,q}(x) = m^{n-1}\frac{[2]_q}{[2]_{q^m}}\sum_{i=0}^{m-1}(-1)^i q^i G_{n,q^m}\left(\frac{i+x}{m}\right), \quad \text{for } n \geq 0. \tag{2.21}$$

3. Distribution and Structure of the Zeros

In this section, we investigate the zeros of the q-Genocchi polynomials $G_{n,q}(x)$ by using computer. We display the shapes of the q-Genocchi polynomials $G_{n,q}(x)$. For $n = 1, \ldots, 10$, we can draw a curve of $G_{n,q}(x), -2 \leq x \leq 2$, respectively. This shows the ten curves combined into one. We display the shape of $G_{n,q}(x)$ (Figures 2, 3, 4, and 5).

We plot the zeros of $G_{n,q}(x), x \in \mathbb{C}$ for $n = 10, 20, 25, 30, q = 1/3$ (Figures 6, 7, 8, and 9).

Next, we plot the zeros of $G_{n,q}(x), x \in \mathbb{C}$ for $n = 30$, $q = 1/2, 1/3, 1/4, 1/5$. (Figures 10, 11, 12, and 13).

In Figures 6, 7, 8, 9, 10, 11, 12, and 13, $G_{n,q}(x), x \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry. This translates to the following open problem: prove or disprove: $G_{n,q}(x), x \in \mathbb{C}$, has $\mathrm{Im}(x) = 0$ reflection symmetry. Our numerical results for numbers of real and complex zeros of $G_{n,q}(x), q = 1/2, 1/3$, are displayed in Table 1.

Figure 15 shows the distribution of real zeros of $G_{n,q}(x)$ for $1 \leq n \leq 20$.

In Figure 15(a), we choose $q = 1/10$. In Figure 15(b), we choose $q = 3/10$. In Figure 15(c), we choose $q = 5/10$. In Figure 15(d), we choose $q = 6/10$.

We calculated an approximate solution satisfying $G_{n,q}(x), q = 1/2, 1/3, x \in \mathbb{R}$. The results are given in Tables 2 and 3.

The plot above shows $G_{n,q}(x)$ for real $1/10 \leq q \leq 9/10$ and $-3 \leq x \leq 3$, with the zero contour indicated in black (Figure 16). In Figure 16(a), we choose $n = 2$. In Figure 16(b), we choose $n = 3$. In Figure 16(c), we choose $n = 4$. In Figure 16(d), we choose $n = 5$.

We will consider the more general open problem. In general, how many roots does $G_{n,q}(x)$ have? Prove or disprove: $G_{n,q}(x)$ has n distinct solutions. Find the numbers of complex zeros $C_{G_{n,q}(x)}$ of $G_{n,q}(x), \mathrm{Im}(x) \neq 0$. Prove or give a counterexample: *Conjecture*: since

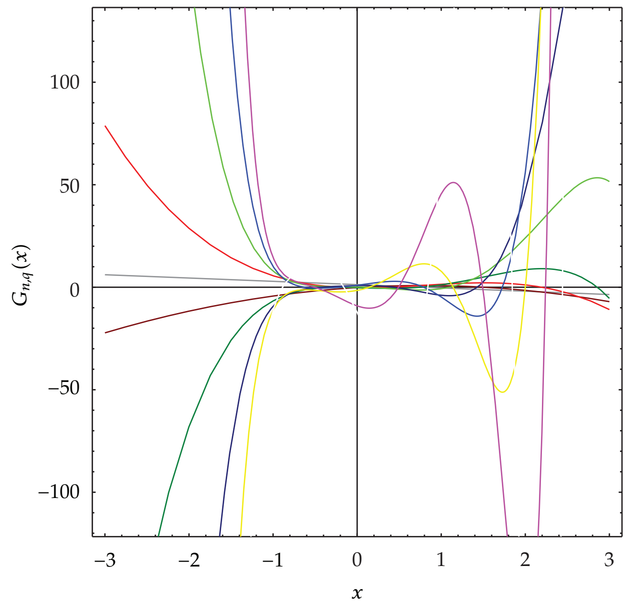

Figure 2: Curves of $G_{n,1/5}(x)$.

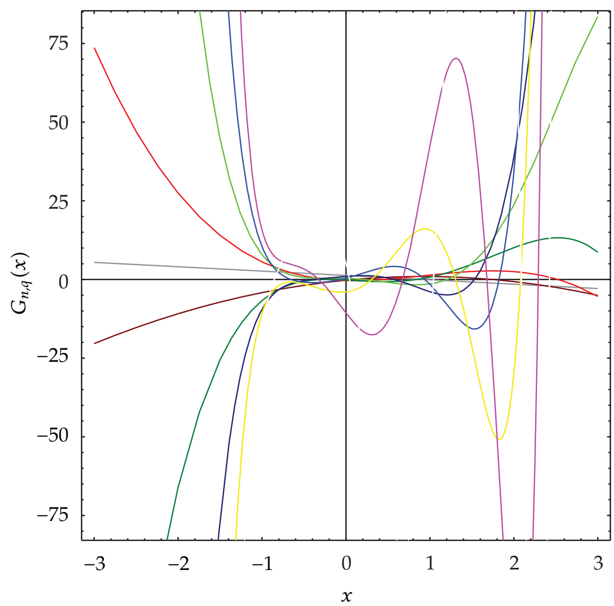

Figure 3: Curves of $G_{n,1/4}(x)$.

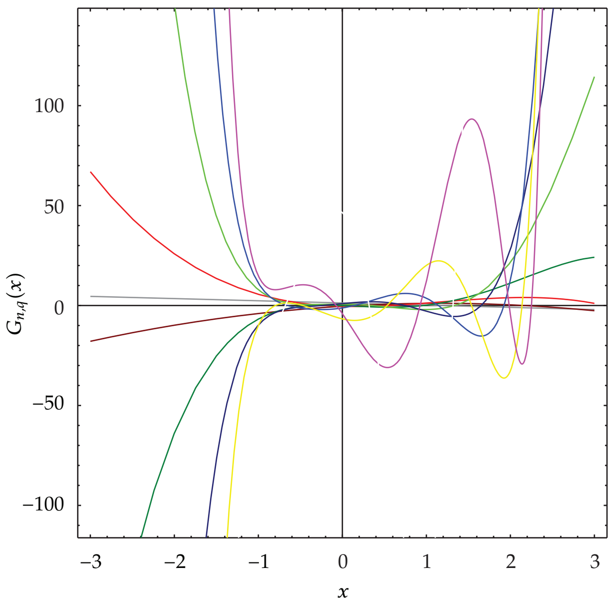

Figure 4: Curves of $G_{n,1/3}(x)$.

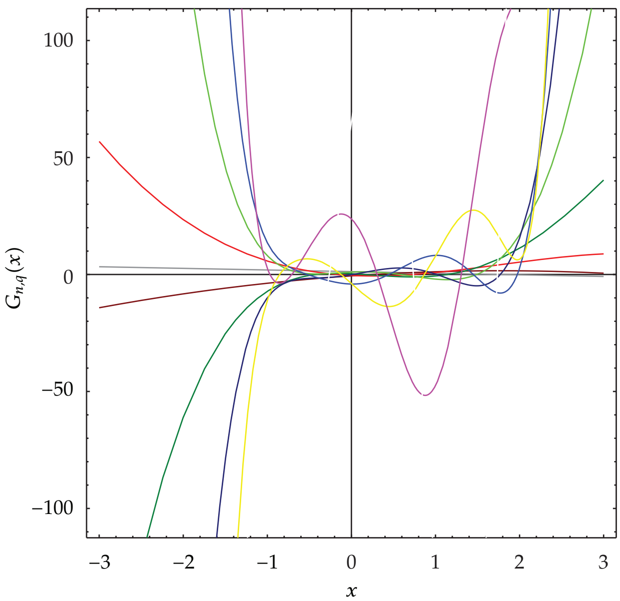

Figure 5: Curves of $G_{n,1/2}(x)$.

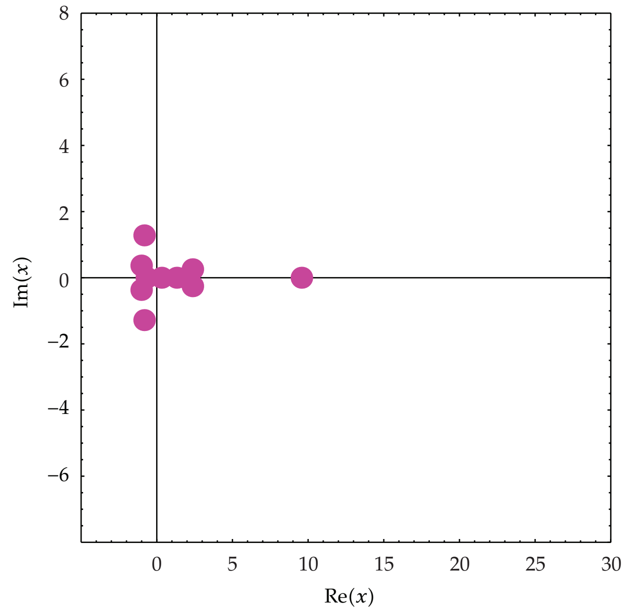

Figure 6: Zero of $G_{10,1/3}(x)$.

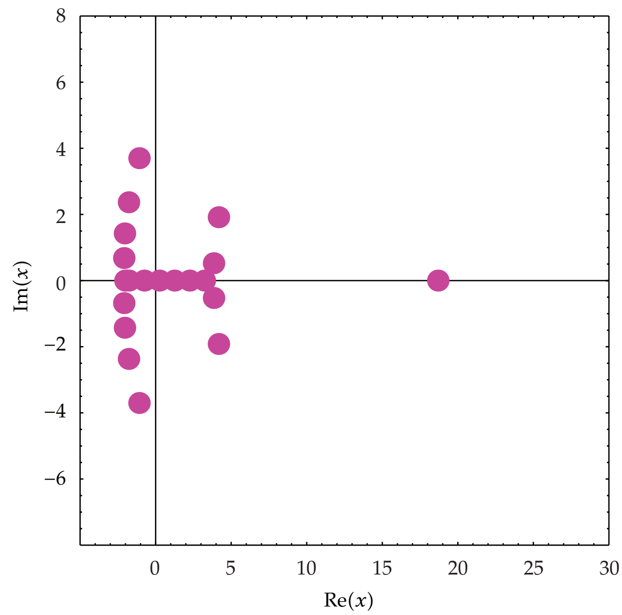

Figure 7: Zero of $G_{20,1/3}(x)$.

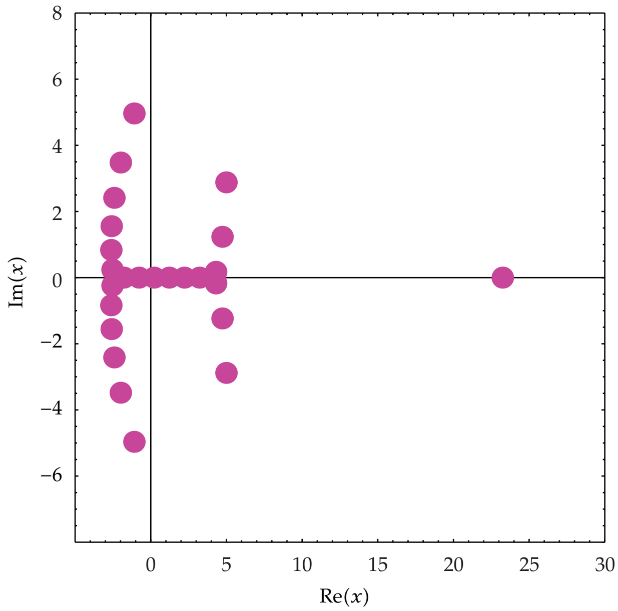

Figure 8: Zero of $G_{25,1/3}(x)$.

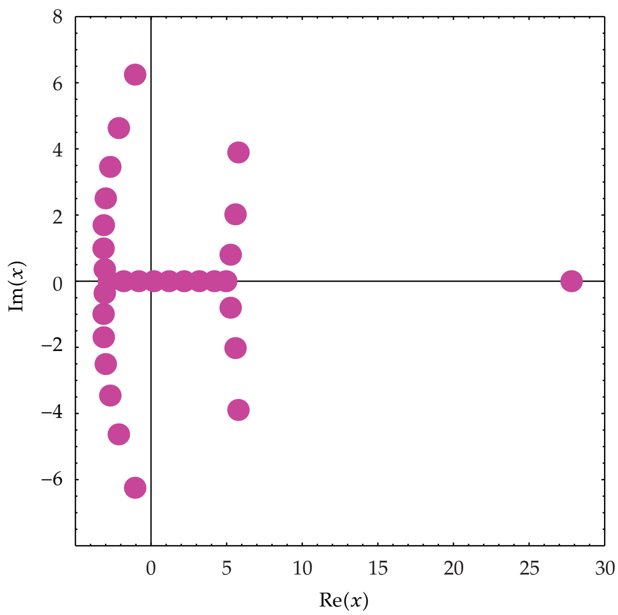

Figure 9: Zero of $G_{30,1/3}(x)$.

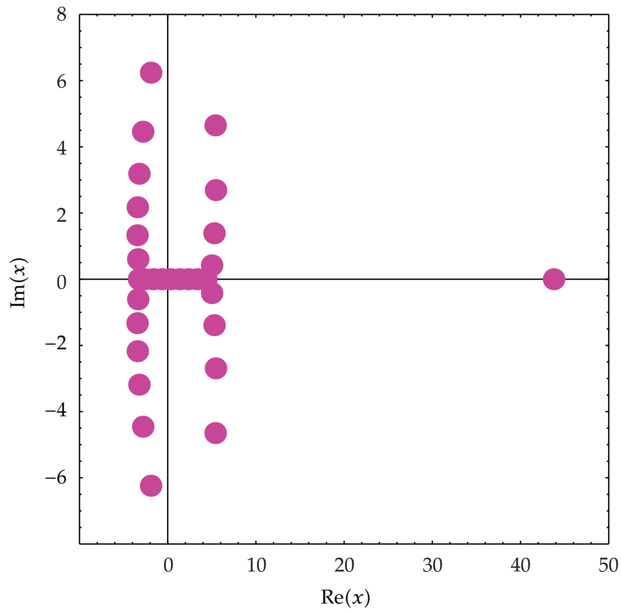

Figure 10: Zero of $G_{30,1/2}(x)$.

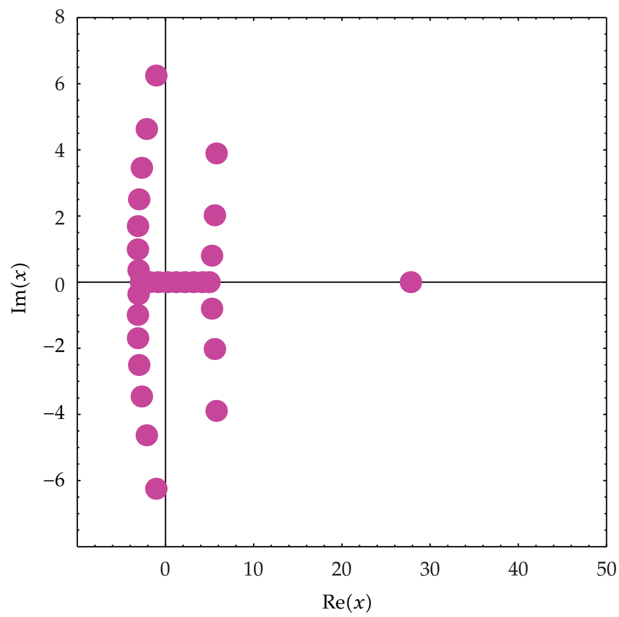

Figure 11: Zero of $G_{30,1/3}(x)$.

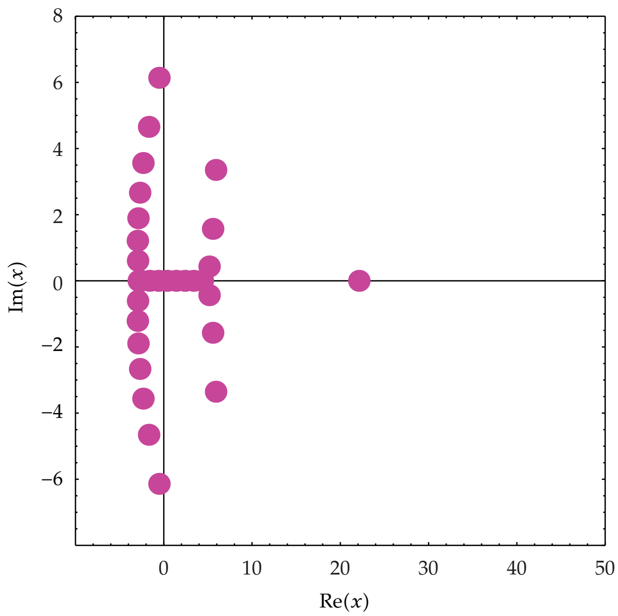

Figure 12: Zero of $G_{30,1/4}(x)$.

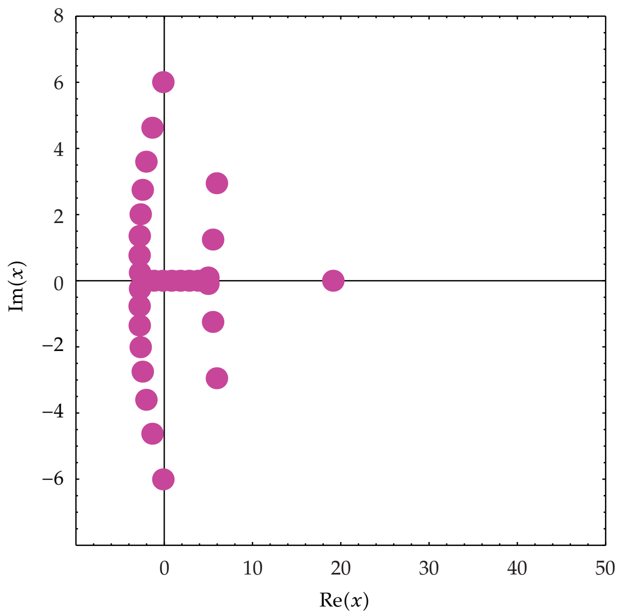

Figure 13: Zero of $G_{30,1/5}(x)$.

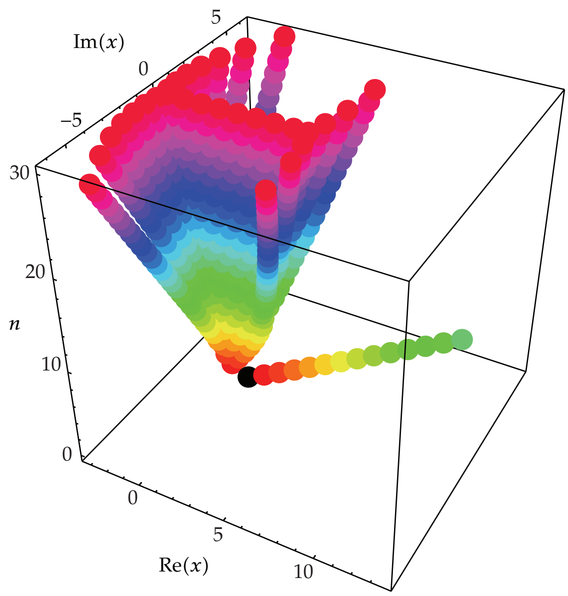

Figure 14: Stacks of zeros $G_{n,1/3}(x)$ for $1 \le n \le 30$.

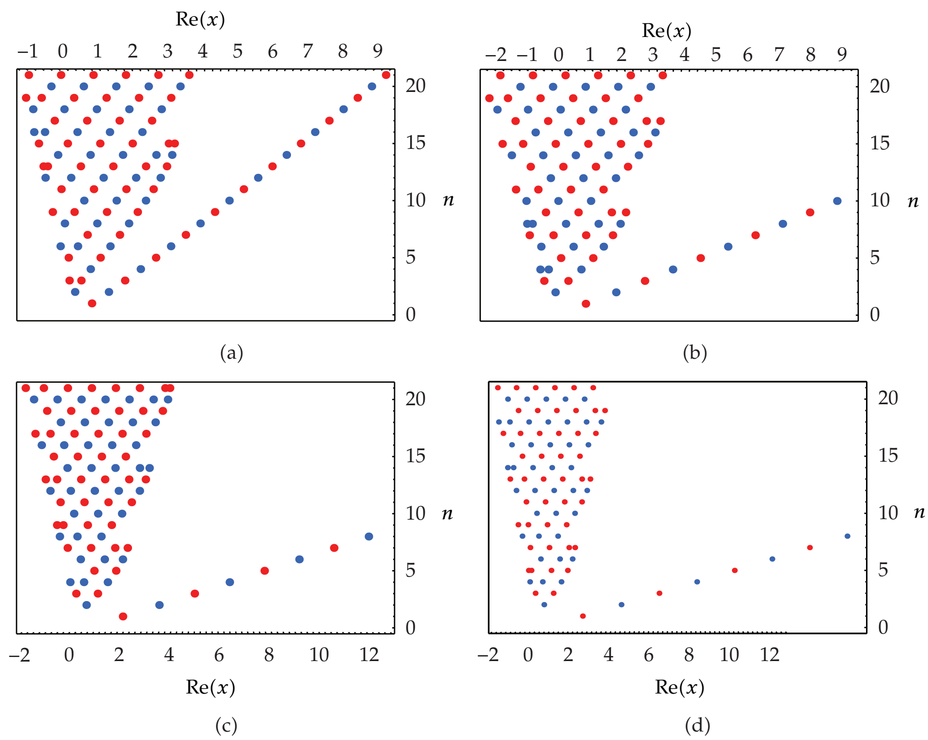

Figure 15: Plot of real zeros of $G_{n,q}(x)$ for $1 \le n \le 20$.

Table 1: Numbers of real and complex zeros of $G_{n,q}(x)$.

Degree n	$q = 1/2$		$q = 1/3$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	3	0
4	4	0	4	0
5	3	2	3	2
6	4	2	4	2
7	5	2	5	2
8	4	4	6	2
9	5	4	5	4
10	4	6	4	6

Table 2: Approximate solutions of $G_{n,1/2}(x) = 0, x \in \mathbb{R}$.

Degree n	x
1	1.7760
2	0.2583, 3.294
3	−0.1698, 0.7313, 4.767
4	−0.4188, 0.1527, 1.145, 6.225
5	0.5848, 1.492, 7.677
6	0.01656, 1.017, 1.772, 9.126
7	−0.5269, 0.4468, 1.452, 1.974, 10.573
8	−0.8536, −0.1221, 0.8779, 12.019
9	−0.969, −0.707, 0.3088, 1.309, 13.46
10	−0.2604, 0.7396, 1.738, 14.91

Table 3: Approximate solutions of $G_{n,1/3}(x) = 0, x \in \mathbb{R}$.

Degree n	x
1	1.1602
2	0.1523, 2.168
3	−0.2107, 0.5657, 3.126
4	−0.3621, −0.02976, 0.9703, 4.062
5	0.3561, 1.333, 4.989
6	−0.2472, 0.7504, 1.652, 5.910
7	−0.6547, 0.1435, 1.144, 1.923, 6.828
8	−0.798, −0.4682, 0.5362, 1.540, 2.139, 7.744
9	−0.07080, 0.9292, 1.98, 2.25, 8.659
10	−0.673, 0.3221, 1.322, 9.573

n is the degree of the polynomial $G_{n,q}(x)$, the number of real zeros $R_{G_{n,q}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{G_{n,q}(x)} = n - C_{G_{n,q}(x)}$, where $C_{G_{n,q}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{G_{n,q}(x)}$ and $C_{G_{n,q}(x)}$. Find the equation of envelope curves bounding the real zeros lying on the plane and the equation of a trajectory curve running through the complex zeros on any one of the arcs. For $n = 1, \ldots, 10$, we can draw a plot of the $G_{n,q}(x)$, respectively. This shows the ten curves combined into one. These figures give mathematicians

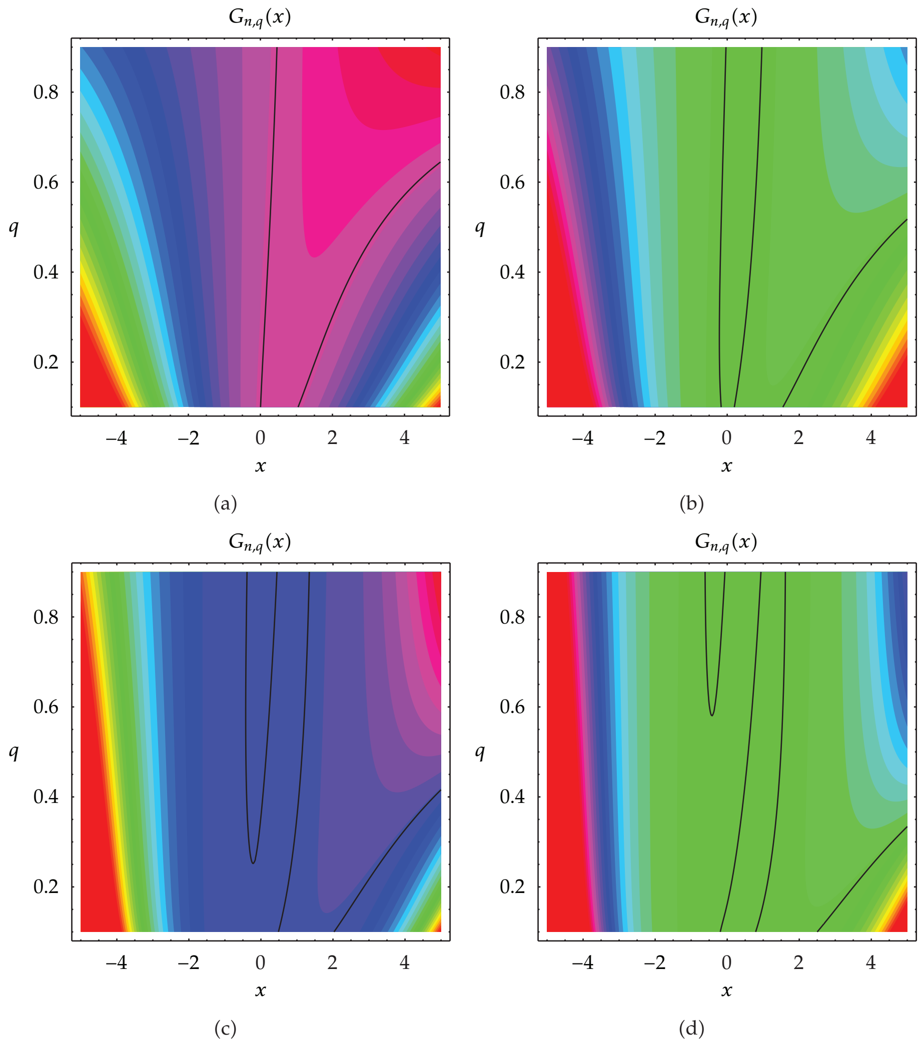

Figure 16: Zero contour of $G_{n,q}(x)$.

an unbounded capacity to create visual mathematical investigations of the behavior of the $G_{n,q}(x)$ and roots of the $G_{n,q}(x)$ (Figures 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16). Moreover, it is possible to create new mathematical ideas and analyze them in ways that generally are not possible by hand. The author has no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the q-Genocchi polynomials $G_{n,q}(x)$ to appear in mathematics and physics. For related topics the interested reader is referred to [15–19].

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Research Article

Derivation of Identities Involving Bernoulli and Euler Numbers

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We derive some new and interesting identities involving Bernoulli and Euler numbers by using some polynomial identities and p-adic integrals on $\mathbb{Z}_p$.

1. Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic absolute value $|\ |_p$ on $\mathbb{C}_p$ is normalized so that $|p|_p = 1/p$. Let $\mathbb{Z}_{>0}$ be the set of natural numbers and $\mathbb{Z}_{\geq 0} = \mathbb{Z}_{>0} \cup \{0\}$.

As is well known, the Bernoulli polynomials $B_n(x)$ are defined by the generating function as follows:

$$F(t,x) = \frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.1}$$

with the usual convention of replacing $B(x)^n$ by $B_n(x)$.

In the special case, $x = 0$, $B_n(0) = B_n$ is referred to as the nth Bernoulli number. That is, the generating function of Bernoulli numbers is given by

$$F(t) = F(t,0) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = e^{Bt}, \tag{1.2}$$

with the usual convention of replacing B^n by B_n, (cf. [1–23]).

From (1.2), we see that the recurrence formula for the Bernoulli numbers is

$$(B+1)^n - B_n = \delta_{1,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}, \tag{1.3}$$

where $\delta_{k,n}$ is the Kronecker symbol.

By (1.1) and (1.2), we easily get the following:

$$B_n(x) = (B+x)^n = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.4}$$

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.5}$$

(cf. [12]). Then it is easy to see that

$$I(f_1) = I(f) + f'(0), \tag{1.6}$$

where $f_1(x) = f(x+1)$ and $f'(0) = df(x)/dx|_{x=0}$.

By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.7}$$

(cf. [12–14]). From (1.7), we can derive the Witt's formula for the nth Bernoulli polynomial as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.8}$$

By (1.1), we have the following:

$$B_n(1-x) = (-1)^n B_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.9}$$

Thus, from (1.3), (1.4), and (1.9), we have the following:

$$B_n(1) = B_n + \delta_{1,n} = (-1)^n B_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.10}$$

By (1.4), we have the following:

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.11}$$

Especially, for $x = 1$ and $y = 1$,

$$B_n(2) = \sum_{k=0}^{n} \binom{n}{k} B_k(1) = \sum_{k=0}^{n} \binom{n}{k} (B_k + \delta_{1,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.12}$$

Therefore, from (1.9), (1.10), and (1.12), we can derive the following relation. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n} = n + (-1)^n B_n. \tag{1.13}$$

Let $f(y) = (x + y)^{n+1}$. By (1.6), we have the following:

$$\int_{\mathbb{Z}_p} (x + y + 1)^{n+1} d\mu(y) - \int_{\mathbb{Z}_p} (x + y)^{n+1} d\mu(y) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.14}$$

By (1.8) and (1.14), we have the following:

$$B_{n+1}(x + 1) - B_{n+1}(x) = (n + 1)x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.15}$$

Thus, by (1.11) and (1.15), we have the following identity.

$$x^n = \frac{1}{n + 1} \sum_{l=0}^{n} \binom{n + 1}{l} B_l(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.16}$$

As is well known, the Euler polynomials $E_n(x)$ are defined by the generating function as follows:

$$G(t, x) = \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.17}$$

with the usual convention of replacing $E(x)^n$ by $E_n(x)$.

In the special case, $x = 0$, $E_n(0) = E_n$ is referred to as the nth Euler number. That is, the generating function of Euler numbers is given by

$$G(t) = G(t, 0) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = e^{Et}, \tag{1.18}$$

with the usual convention of replacing E^n by E_n, (cf. [1–23]).

From (1.18), we see that the recurrence formula for the Euler numbers is

$$(E + 1)^n + E_n = 2\delta_{0,n}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.19}$$

By (1.17) and (1.18), we easily get the following:

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n}\binom{n}{l}E_l x^{n-l} = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}x^l, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.20}$$

Let $C(\mathbb{Z}_p)$ be the space of continuous $\mathbb{C}_p$-valued functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x, \tag{1.21}$$

(cf. [9]). Then it is easy to see that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.22}$$

where $f_1(x) = f(x+1)$.

By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\ . \tag{1.23}$$

From (1.23), we can derive the Witt's formula for the n-th Euler polynomial as follows:

$$\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y) = E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.24}$$

By (1.17), we have the following:

$$E_n(1-x) = (-1)^n E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.25}$$

Thus, from (1.19), (1.20), and (1.25), we have the following:

$$E_n(1) = -E_n + 2\delta_{0,n} = (-1)^n E_n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.26}$$

By (1.20), we have the following:

$$E_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.27}$$

Especially, for $x = 1$ and $y = 1$,

$$E_n(2) = \sum_{k=0}^{n}\binom{n}{k}E_k(1) = \sum_{k=0}^{n}\binom{n}{k}(-E_n + 2\delta_{0,k}), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.28}$$

Therefore, from (1.25), (1.26), and (1.28), we can derive the following relations. For $n \in \mathbb{Z}_{\geq 0}$,

$$(-1)^n E_n(-1) = E_n(2) = 2 - E_n(1) = 2 + E_n - 2\delta_{0,n} = 2 - (-1)^n E_n. \tag{1.29}$$

Let $f(y) = (x+y)^n$. By (1.22), we have the following:

$$\int_{\mathbb{Z}_p} (x+y+1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.30}$$

By (1.24) and (1.30), we have the following:

$$E_n(x+1) + E_n(x) = 2x^n, \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.31}$$

Thus, by (1.27) and (1.31), we get the following identity.

$$x^n = \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x) + E_n(x), \quad \text{for } n \in \mathbb{Z}_{\geq 0}. \tag{1.32}$$

The Bernstein polynomials are defined by

$$B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}, \quad \text{for } k, n \in \mathbb{Z}_{\geq 0}, \tag{1.33}$$

with $0 \leq k \leq n$ (cf. [14]).

By the definition of $B_{k,n}(x)$, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x). \tag{1.34}$$

In this paper, we derive some new and interesting identities involving Bernoulli and Euler numbers from well-known polynomial identities. Here, we note that our results are "complementary" to those in [6], in the sense that we take a fermionic p-adic integral where a bosonic p-adic integral is taken and vice versa, and we use the identity involving Euler polynomials in (1.32) where that involving Bernoulli polynomials in (1.16) is used and vice versa. Finally, we report that there have been a lot of research activities on this direction of research, namely, on derivation of identities involving Bernoulli and Euler numbers and polynomials by exploiting bosonic and fermionic p-adic integrals (cf. [6–8]).

2. Identities Involving Bernoulli Numbers

Taking the bosonic p-adic integral on both sides of (1.16), we have the following:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x)d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\int_{\mathbb{Z}_p} B_k(x)d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\int_{\mathbb{Z}_p} x^j d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j.\end{aligned} \tag{2.1}$$

Therefore, we obtain the following theorem.

Theorem 2.1. *Let $m \in \mathbb{Z}_{\geq 0}$. Then on has the following:*

$$B_m = \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}B_{k-j}B_j. \tag{2.2}$$

Let us apply (1.9) to the bosonic p-adic integral of (1.16).

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\int_{\mathbb{Z}_p} B_k(x)d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\int_{\mathbb{Z}_p} B_k(1-x)d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\int_{\mathbb{Z}_p} (1-x)^j d\mu(x)\\ &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}(-1)^j B_j(-1).\end{aligned} \tag{2.3}$$

Then, we can express (2.3) in three different ways.

By (1.13), (2.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\left(j+B_j+\delta_{1,j}\right) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\left(kB_{k-1}(1)+kB_{k-1}+\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j\right) \\
&= -\sum_{k=0}^{m-1}\binom{m}{k}\left(B_k+(-1)^kB_k\right) + \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j \\
&= -\sum_{k=0}^{m-1}\binom{m}{k}(B_k+B_k+\delta_{1,k}) + \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j \\
&= -2(B_m(1)-B_m)-(m-\delta_{1,m}) + \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j \\
&= -\delta_{1,m}-m + \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}B_j.
\end{aligned}
\tag{2.4}
$$

Thus, we have the following theorem.

Theorem 2.2. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
B_m = -\delta_{1,m} - m + \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j. \tag{2.5}
$$

Corollary 2.3. *Let m be an integer ≥ 2. Then one has the following:*

$$
B_m + m = \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j. \tag{2.6}
$$

Especially, for an odd integer m with $m \geq 3$, we obtain the following corollary.

Corollary 2.4. *Let m be an odd integer with $m \geq 3$. Then one has the following:*

$$
m(m+1) = \sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}(-1)^k B_{k-j}B_j. \tag{2.7}
$$

By (1.13), (2.3) can be written as

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu(x) &= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}B_{k-j}\left(j+(-1)^jB_j\right)\\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\left(kB_{k-1}(1)+\sum_{j=0}^{k}\binom{k}{j}(-1)^jB_{k-j}B_j\right)\\
&= -\sum_{k=0}^{m-1}\binom{m}{k}B_k+\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^jB_{k-j}B_j\\
&= -B_m(1)+B_m+\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^jB_{k-j}B_j.
\end{aligned} \tag{2.8}$$

By (1.10), (2.8) can be written as

$$\int_{\mathbb{Z}_p} x^m d\mu(x) = (-1)^{m+1}B_m+B_m+\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^jB_{k-j}B_j. \tag{2.9}$$

So, we get the following theorem.

Theorem 2.5. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}(-1)^{m+k+j}B_{k-j}B_j. \tag{2.10}$$

By (1.10), (2.8) can also be written as

$$\int_{\mathbb{Z}_p} x^m d\mu(x) = -\delta_{1,m}+\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}(-1)^k\sum_{j=0}^{k}\binom{k}{j}(-1)^jB_{k-j}B_j. \tag{2.11}$$

Thus, we have the following theorem.

Theorem 2.6. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_m = -\delta_{1,m}+\frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\binom{m+1}{k}\binom{k}{j}(-1)^{k+j}B_{k-j}B_j. \tag{2.12}$$

3. Identities Involving Euler Numbers

Taking the fermionic p-adic integral on both sides of (1.32), we have the following:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}\left(E_m(x)+\frac{1}{2}\sum_{k=0}^{m-1}\binom{m}{k}E_k(x)\right)d\mu_{-1}(x)\\ &= \sum_{l=0}^{m}\binom{m}{l}E_{m-l}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x)+\frac{1}{2}\sum_{k=0}^{m-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}E_{k-j}\int_{\mathbb{Z}_p} x^j d\mu_{-1}(x)\\ &= \sum_{l=0}^{m}\binom{m}{l}E_{m-l}E_l+\frac{1}{2}\sum_{k=0}^{m-1}\binom{m}{k}\sum_{j=0}^{k}\binom{k}{j}E_{k-j}E_j.\end{aligned} \tag{3.1}$$

So, we obtain the following theorem.

Theorem 3.1. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$E_m = \sum_{l=0}^{m}\binom{m}{l}E_{m-l}E_l+\frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}E_{k-j}E_j. \tag{3.2}$$

Let us apply (1.25) to the fermionic p-adic integral of (1.32).

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m\int_{\mathbb{Z}_p} E_m(1-x)d\mu_{-1}(x)+\frac{1}{2}\sum_{k=0}^{m-1}\binom{m}{k}(-1)^k\int_{\mathbb{Z}_p} E_k(1-x)d\mu_{-1}(x)\\ &= (-1)^m\sum_{k=0}^{m}\binom{m}{k}E_{m-k}(-1)^kE_k(-1)\\ &\quad+\frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^kE_{k-j}(-1)^jE_j(-1).\end{aligned} \tag{3.3}$$

Then, we can express (3.3) in two different ways.

By (1.29), (3.3) can be written as

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}(2+E_k-2\delta_{0,k}) \\
&\quad + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}(2+E_j-2\delta_{0,j}) \\
&= 2E_m + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}E_k + 2(-1)^{m+1}E_m + \sum_{k=0}^{m-1}\binom{m}{k}E_k \\
&\quad + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}E_j + \sum_{k=0}^{m-1}\binom{m}{k}(-1)^{k+1}E_k \\
&= 2E_m + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}E_k + 2(-1)^{m+1}E_m + E_m(1) - E_m \\
&\quad + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}E_j + (-1)^{m+1}(E_m(-1)-E_m) \\
&= -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}E_k + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}E_j.
\end{aligned} \tag{3.4}$$

Thus, we get the following theorem.

Theorem 3.2. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$E_m = -2 + 2\delta_{0,m} + (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}E_k + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}E_j. \tag{3.5}$$

Corollary 3.3. *Let $m \in \mathbb{Z}_{>0}$. Then one has the following:*

$$E_m + 2 = (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}E_k + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k} \binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}E_j. \tag{3.6}$$

By (1.29), (3.3) can be written as

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) &= (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{m-k}\left(2-(-1)^k E_k\right) \\
&\quad + \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^k E_{k-j}\left(2-(-1)^j E_j\right) \\
&= 2E_m + (-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_{m-k}E_k \\
&\quad + \sum_{k=0}^{m-1}\binom{m}{k}E_k - \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j}E_j \\
&= 2E_m + (-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_{m-k}E_k \\
&\quad + E_m(1) - E_m - \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j}E_j \\
&= 2\delta_{0,m} + (-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_{m-k}E_k \\
&\quad - \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^k(-1)^j E_{k-j}E_j.
\end{aligned}
\tag{3.7}
$$

So, we have the following theorem.

Theorem 3.4. *Let $m \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$
E_m = 2\delta_{0,m} + (-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_{m-k}E_k - \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^{k+j}E_{k-j}E_j. \tag{3.8}
$$

Corollary 3.5. *Let $m \in \mathbb{Z}_{>1}$. Then one has the following:*

$$
E_m = (-1)^{m+1}\sum_{k=0}^{m}\binom{m}{k}(-1)^k E_{m-k}E_k - \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=0}^{k}\binom{m}{k}\binom{k}{j}(-1)^{k+j}E_{k-j}E_j. \tag{3.9}
$$

4. Identities Involving Bernoulli and Euler Numbers

By (1.16) and (1.32), we have the following:

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^{m+n} d\mu(x) &= \int_{\mathbb{Z}_p} \left(\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x)\right)\left(E_n(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x)\right)d\mu(x) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\int_{\mathbb{Z}_p} B_k(x)E_n(x)d\mu(x) \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\binom{m+1}{k}\sum_{l=0}^{n-1}\binom{n}{l}\int_{\mathbb{Z}_p} B_k(x)E_l(x)d\mu(x) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{m+1}{k}\binom{k}{j}\binom{n}{l}B_{k-j}E_{n-l}B_{j+l} \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{j}\binom{l}{i}B_{k-j}E_{l-i}B_{j+i}.
\end{aligned} \tag{4.1}$$

Therefore, we get the following theorem.

Theorem 4.1. *Let $m, n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$\begin{aligned}
B_{m+n} &= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{j=0}^{k}\sum_{l=0}^{n}\binom{m+1}{k}\binom{k}{j}\binom{n}{l}B_{k-j}E_{n-l}B_{j+l} \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{j=0}^{k}\sum_{i=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{j}\binom{l}{i}B_{k-j}E_{l-i}B_{j+i}.
\end{aligned} \tag{4.2}$$

By (1.16) and (1.33), we have the following:

$$\begin{aligned}
\int_{\mathbb{Z}_p} x^{m}B_{k,n}(x) d\mu(x) &= \int_{\mathbb{Z}_p} \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}B_l(x)B_{k,n}(x)d\mu(x) \\
&= \frac{1}{m+1}\binom{n}{k}\sum_{l=0}^{m}\sum_{i=0}^{l}\binom{m+1}{l}\binom{l}{i}B_{l-i}\int_{\mathbb{Z}_p} x^{i+k}(1-x)^{n-k}d\mu(x) \\
&= \frac{1}{m+1}\binom{n}{k}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{j=0}^{n-k}\binom{m+1}{l}\binom{l}{i}\binom{n-k}{j}(-1)^j B_{l-i}\int_{\mathbb{Z}_p} x^{i+k+j}d\mu(x) \\
&= \frac{1}{m+1}\binom{n}{k}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{j=0}^{n-k}\binom{m+1}{l}\binom{l}{i}\binom{n-k}{j}(-1)^j B_{l-i}B_{i+k+j}.
\end{aligned} \tag{4.3}$$

By (1.33), we have the following:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m B_{k,n}(x)d\mu(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} x^{m+k}(1-x)^{n-k}d\mu(x)\\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_{\mathbb{Z}_p} x^{m+k+j}d\mu(x)\\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j B_{m+k+j}.\end{aligned}\tag{4.4}$$

By (4.3) and (4.4), we obtain the following theorem.

Theorem 4.2. *Let $m,n,k \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j B_{m+k+j} = \frac{1}{m+1}\sum_{l=0}^{m}\sum_{i=0}^{l}\sum_{j=0}^{n-k}\binom{m+1}{l}\binom{l}{i}\binom{n-k}{j}(-1)^j B_{l-i}B_{i+k+j}.\tag{4.5}$$

Especially, one has the following:

$$(m+1)B_{m+n} = \sum_{l=0}^{m}\sum_{i=0}^{l}\binom{m+1}{l}\binom{l}{i}B_{l-i}B_{i+n}.\tag{4.6}$$

By (4.2) and (4.6), we have the following theorem. Note that (4.8) in the following was obtained in [6].

Theorem 4.3. *Let $m,n \in \mathbb{Z}_{\geq 0}$. Then one has the following:*

$$B_{m+n} = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_{m+l} + \frac{1}{2}\sum_{l=0}^{n-1}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}E_{l-i}B_{m+i}.\tag{4.7}$$

In particular, we have the following:

$$B_n = \sum_{l=0}^{n}\binom{n}{l}E_{n-1}B_l + \frac{1}{2}\sum_{l=0}^{n-1}\sum_{i=0}^{l}\binom{n}{l}\binom{l}{i}E_{l-i}B_i.\tag{4.8}$$

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Research Article

Identities on the Bernoulli and Genocchi Numbers and Polynomials

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We give some interesting identities on the Bernoulli numbers and polynomials, on the Genocchi numbers and polynomials by using symmetric properties of the Bernoulli and Genocchi polynomials.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic rational integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic norm on C_p is normalized so that $|p|_p = p^{-1}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x \tag{1.1}$$

(see [1–16]). From (1.1), we have

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0) \tag{1.2}$$

(see [1–16]), where $f_1(x) = f(x+1)$.

Let us take $f(x) = e^{xt}$. Then, by (1.2), we get

$$t\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \tag{1.3}$$

where G_n are the nth ordinary Genocchi numbers (see [8, 15]).

From the same method of (1.3), we can also derive the following equation:

$$t\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2t}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \tag{1.4}$$

where $G_n(x)$ are called the nth Genocchi polynomials (see [14, 15]).

By (1.3), we easily see that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l} \tag{1.5}$$

(see [15]). By (1.3) and (1.4), we get Witt's formula for the nth Genocchi numbers and polynomials as follows:

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \quad \text{for } n \in \mathbb{Z}_+. \tag{1.6}$$

From (1.2), we have

$$\int_{\mathbb{Z}_p} (x+1)^n d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = 2\delta_{0,n}, \tag{1.7}$$

where the symbol $\delta_{0,n}$ is the Kronecker symbol (see [4, 5]).

Thus, by (1.5) and (1.7), we get

$$(G+1)^n + G_n = 2\delta_{1,n} \tag{1.8}$$

(see [15]). From (1.4), we can derive the following equation:

$$\int_{\mathbb{Z}_p} (1-x+y)^n d\mu_{-1}(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y). \tag{1.9}$$

By (1.6) and (1.9), we see that

$$\frac{G_{n+1}(1-x)}{n+1} = (-1)^n \frac{G_{n+1}(x)}{n+1}. \tag{1.10}$$

Thus, by (1.10), we get $G_{n+1}(2)/(n+1) = (-1)^n (G_{n+1}(-1)/(n+1))$.

From (1.5) and (1.8), we have

$$\frac{G_{n+1}(2)}{n+1} = 2 - \frac{G_{n+1}(1)}{n+1} = 2 + \frac{G_{n+1}}{n+1} - 2\delta_{1,n+1}. \tag{1.11}$$

The Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{t}{e^t - 1} e^{xt} = e^{B(x)t} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \tag{1.12}$$

(see [6, 9, 12]) with the usual convention about replacing $B^n(x)$ by $B_n(x)$.

In the special case, $x = 0$, $B_n(0) = B_n$ is called the n-th Bernoulli number. By (1.12), we easily see that

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l = (B + x)^n \tag{1.13}$$

(see [6]). Thus, by (1.12) and (1.13), we get reflection symmetric formula for the Bernoulli polynomials as follows:

$$B_n(1 - x) = (-1)^n B_n(x), \tag{1.14}$$

$$B_0 = 1, \quad (B + 1)^n - B_n = \delta_{1,n} \tag{1.15}$$

(see [6, 9, 12]). From (1.14) and (1.15), we can also derive the following identity:

$$(-1)^n B_n(-1) = B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n}. \tag{1.16}$$

In this paper, we investigate some properties of the fermionic p-adic integrals on $\mathbb{Z}_p$. By using these properties, we give some new identities on the Bernoulli and the Euler numbers which are useful in studying combinatorics.

2. Identities on the Bernoulli and Genocchi Numbers and Polynomials

Let us consider the following fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} B_n(x) d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1}, \quad \text{for } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \end{aligned} \tag{2.1}$$

On the other hand, by (1.14) and (1.15), we get

$$
\begin{aligned}
I_1 &= (-1)^n \int_{\mathbb{Z}_p} B_n(1-x)d\mu_{-1}(x) \\
&= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \\
&= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1} \\
&= (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \left(2 + \frac{G_{l+1}}{l+1} - 2\delta_{1,l+1}\right) \\
&= 2(-1)^n (B_n + \delta_{1,n}) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} + 2(-1)^{n+1} B_n.
\end{aligned}
\tag{2.2}
$$

Equating (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$
\left(1 + (-1)^{n+1}\right) \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1} = 2(-1)^n \delta_{1,n}. \tag{2.3}
$$

By using the reflection symmetric property for the Euler polynomials, we can also obtain some interesting identities on the Euler numbers.

Now, we consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the polynomials as follows:

$$
\begin{aligned}
I_2 &= \int_{\mathbb{Z}_p} G_n(x)d\mu_{-1}(x) \\
&= \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\
&= \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \frac{G_{l+1}}{l+1}, \quad \text{for } n \in \mathbb{Z}_+.
\end{aligned}
\tag{2.4}
$$

On the other hand, by (1.8), (1.10), and (1.11), we get

$$
\begin{aligned}
I_2 &= (-1)^{n-1} \int_{\mathbb{Z}_p} G_n(1-x)d\mu_{-1}(x) \\
&= (-1)^{n-1} \sum_{l=0}^{n} \binom{n}{l} G_{n-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu_{-1}(x) \\
&= (-1)^{n-1} \sum_{l=0}^{n} \binom{n}{l} G_{n-l} (-1)^l \frac{G_{l+1}(-1)}{l+1}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{n-1}\sum_{l=0}^{n}\binom{n}{l}G_{n-l}\left(2+\frac{G_{l+1}}{l+1}-2\delta_{1,l+1}\right)\\
&= 2(-1)^{n-1}(2\delta_{1,n}-G_n)+2(-1)^nG_n\\
&\quad+(-1)^{n-1}\sum_{l=0}^{n}\binom{n}{l}G_{n-l}\frac{G_{l+1}}{l+1}.
\end{aligned}
\tag{2.5}
$$

Equating (2.4) and (2.5), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$
(1+(-1)^n)\sum_{l=0}^{n}\binom{n}{l}G_{n-l}\frac{G_{l+1}}{l+1} = 4(-1)^nG_n + 4(-1)^{n+1}\delta_{1,n}. \tag{2.6}
$$

Let us consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of $B_n(x)$ and $G_n(x)$ as follows:

$$
\begin{aligned}
I_3 &= \int_{\mathbb{Z}_p} B_m(x)G_n(x)\,d\mu_{-1}(x)\\
&= \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}G_{n-l}\int_{\mathbb{Z}_p} x^{k+l}\,d\mu_{-1}(x)\\
&= \sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}G_{n-l}\frac{G_{k+l+1}}{k+l+1}.
\end{aligned}
\tag{2.7}
$$

On the other hand, by (1.10) and (1.14), we get

$$
\begin{aligned}
I_3 &= \int_{\mathbb{Z}_p} B_m(x)G_n(x)\,d\mu_{-1}(x)\\
&= (-1)^{n+m-1}\int_{\mathbb{Z}_p} B_m(1-x)G_n(1-x)\,d\mu_{-1}(x)\\
&= (-1)^{n+m-1}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}G_{n-l}\int_{\mathbb{Z}_p}(1-x)^{k+l}\,d\mu_{-1}(x)\\
&= 2(-1)^{n+m-1}B_m(1)G_n(1)+2(-1)^{m+n}B_mG_n\\
&\quad+(-1)^{n+m-1}\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}G_{n-l}\frac{G_{k+l+1}}{k+l+1}.
\end{aligned}
\tag{2.8}
$$

By (2.7) and (2.8), we easily see that

$$\begin{aligned}
&\left(1+(-1)^{n+m+1}\right)\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}G_{n-l}\frac{G_{k+l+1}}{k+l+1}\\
&\quad=2(-1)^{m+n-1}(\delta_{1,m}+B_m)(2\delta_{1,n}-G_n)+2(-1)^{m+n}B_mG_n\\
&\quad=4(-1)^{m+n-1}B_m\delta_{1,n}+2(-1)^{m+n}B_mG_n+4(-1)^{m+n-1}\delta_{1,m}\delta_{1,n}\\
&\qquad+2(-1)^{m+n}\delta_{1,m}G_n+2(-1)^{m+n}B_mG_n.
\end{aligned}\tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.3. *For $n,m\in\mathbb{Z}_+$, one has*

$$\begin{aligned}
&\left(1+(-1)^{n+m+1}\right)\sum_{k=0}^{m}\sum_{l=0}^{n}\binom{m}{k}\binom{n}{l}B_{m-k}\frac{G_{n-l+1}}{n-l+1}\frac{G_{k+l+1}}{k+l+1}\\
&\quad=4(-1)^{m+n}B_mG_n+4(-1)^{m+n-1}B_m\delta_{1,n}+4(-1)^{m+n-1}\delta_{1,m}\delta_{1,n}\\
&\qquad+2(-1)^{m+n}\delta_{1,m}G_n.
\end{aligned}\tag{2.10}$$

Corollary 2.4. *For $n,m\in\mathbb{N}$, one has*

$$\sum_{k=0}^{2m}\sum_{l=0}^{2n}\binom{2m}{k}\binom{2n}{l}B_{2m-k}G_{2n-l}\frac{G_{k+l+1}}{k+l+1}=2B_{2m}G_{2n}.\tag{2.11}$$

Let us consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of the Bernoulli polynomials and the Bernstein polynomials. For $n,k\in\mathbb{Z}_+$, with $0\le k\le n$, $B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}$ are called the Bernstein polynomials of degree n, see [11]. It is easy to show that $B_{k,n}(x)=B_{n-k,n}(1-x)$,

$$\begin{aligned}
I_4&=\int_{\mathbb{Z}_p}B_m(x)B_{k,n}(x)\,d\mu_{-1}(x)\\
&=\binom{n}{k}\sum_{l=0}^{m}\binom{m}{l}B_{m-l}\int_{\mathbb{Z}_p}x^{k+l}(1-x)^{n-k}\,d\mu_{-1}(x)\\
&=\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^jB_{m-l}\int_{\mathbb{Z}_p}x^{k+l+j}\,d\mu_{-1}(x)\\
&=\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^jB_{m-l}\frac{G_{k+l+j+1}}{k+l+j+1}.
\end{aligned}\tag{2.12}$$

On the other hand, by (1.14) and (2.12), we get

$$\begin{aligned}
I_4 &= (-1)^m \int_{\mathbb{Z}_p} B_m(1-x) B_{n-k,n}(1-x)\, d\mu_{-1}(x) \\
&= (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l}\binom{k}{j} (-1)^j B_{m-l} \int_{\mathbb{Z}_p} (1-x)^{n-k+l+j}\, d\mu_{-1}(x) \\
&= (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l}\binom{k}{j} (-1)^j B_{m-l} \\
&\quad \times \left(2 - 2\delta_{1,n-k+l+j+1} + \frac{G_{n-k+l+j+1}}{n-k+l+j+1}\right) \\
&= 2(-1)^m \binom{n}{k} B_m(1)\delta_{0,k} + 2(-1)^{m+1}\binom{n}{k} B_m \delta_{k,n} \\
&\quad + (-1)^m \binom{n}{k} \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l}\binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned} \tag{2.13}$$

Equating (2.12) and (2.13), we see that

$$\begin{aligned}
&\sum_{l=0}^{m} \sum_{j=0}^{n-k} \binom{m}{l}\binom{n-k}{j} (-1)^j B_{m-l} \frac{G_{k+l+j+1}}{k+l+j+1} \\
&\quad = 2(-1)^m B_m(1)\delta_{0,k} + 2(-1)^{m+1} B_m \delta_{k,n} \\
&\qquad + (-1)^m \sum_{l=0}^{m} \sum_{j=0}^{k} \binom{m}{l}\binom{k}{j} (-1)^j B_{m-l} \frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned} \tag{2.14}$$

Thus, from (2.14), we obtain the following theorem.

Theorem 2.5. *For $n, m \in \mathbb{N}$, one has*

$$\sum_{l=0}^{2m} \sum_{j=0}^{n} \binom{2m}{l}\binom{n}{j} (-1)^j B_{2m-l} \frac{G_{l+j+1}}{l+j+1} = 2B_{2m}(1) + \sum_{l=0}^{2m} \binom{2m}{l} B_{2m-l} \frac{G_{n+l+1}}{n+l+1}. \tag{2.15}$$

Finally, we consider the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of the Euler polynomials and the Bernstein polynomials as follows:

$$\begin{aligned}
I_5 &= \int_{\mathbb{Z}_p} G_m(x) B_{k,n}(x)\, d\mu_{-1}(x) \\
&= \binom{n}{k} \sum_{l=0}^{m} \binom{m}{l} G_{m-l} \int_{\mathbb{Z}_p} x^{k+l}(1-x)^{n-k}\, d\mu_{-1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^{j}G_{m-l}\int_{\mathbb{Z}_p} x^{k+l+j}\,d\mu_{-1}(x) \\
&= \binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^{j}G_{m-l}\frac{G_{k+l+j+1}}{k+l+j+1}.
\end{aligned}
\tag{2.16}
$$

On the other hand, by (1.10) and (2.12), we get

$$
\begin{aligned}
I_5 &= (-1)^{m-1}\int_{\mathbb{Z}_p} G_m(1-x)B_{n-k,n}(1-x)\,d\mu_{-1}(x) \\
&= (-1)^{m-1}\binom{n}{k}\sum_{l=0}^{m}\binom{m}{l}G_{m-l}\sum_{j=0}^{k}\binom{k}{j}(-1)^{j}\int_{\mathbb{Z}_p}(1-x)^{n-k+l+j}\,d\mu_{-1}(x) \\
&= (-1)^{m-1}\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^{j}G_{m-l} \\
&\quad\times\left(2+\frac{G_{n-k+l+j+1}}{n-k+l+j+1}-2\delta_{1,n-k+l+j+1}\right) \\
&= 2(-1)^{m-1}\binom{n}{k}G_m(1)\delta_{0,k}+2(-1)^{m}\binom{n}{k}G_m\delta_{k,n} \\
&\quad+(-1)^{m-1}\binom{n}{k}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^{j}G_{m-l}\frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned}
\tag{2.17}
$$

Equating (2.16) and (2.17), we obtain

$$
\begin{aligned}
&\sum_{l=0}^{m}\sum_{j=0}^{n-k}\binom{m}{l}\binom{n-k}{j}(-1)^{j}G_{m-l}\frac{G_{k+l+j+1}}{k+l+j+1} \\
&\qquad= 2(-1)^{m-1}G_m(1)\delta_{0,k}+2(-1)^{m}G_m\delta_{k,n} \\
&\qquad\quad+(-1)^{m-1}\sum_{l=0}^{m}\sum_{j=0}^{k}\binom{m}{l}\binom{k}{j}(-1)^{j}G_{m-l}\frac{G_{n-k+l+j+1}}{n-k+l+j+1}.
\end{aligned}
\tag{2.18}
$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.6. *For $n, m \in \mathbb{N}$, one has*

$$
\sum_{l=0}^{2m}\sum_{j=0}^{n}\binom{2m}{l}\binom{n}{j}(-1)^{j}G_{2m-l}\frac{G_{l+j+1}}{l+j+1} = -2G_{2m}(1)-\sum_{l=0}^{2m}\binom{2m}{l}G_{2m-l}\frac{G_{n+l+1}}{n+l+1}.
\tag{2.19}
$$

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Research Article

Bernoulli Basis and the Product of Several Bernoulli Polynomials

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We develop methods for computing the product of several Bernoulli and Euler polynomials by using Bernoulli basis for the vector space of polynomials of degree less than or equal to n.

1. Introduction

It is well known that, the nth Bernoulli and Euler numbers are defined by

$$\sum_{l=0}^{n}\binom{n}{l}B_l - B_n = \delta_{1,n}, \qquad \sum_{l=0}^{n}\binom{n}{l}E_l + E_n = 2\delta_{0,n}, \tag{1.1}$$

where $B_0 = E_0 = 1$ and $\delta_{k,n}$ is the Kronecker symbol (see [1–20]).

The Bernoulli and Euler polynomials are also defined by

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \qquad E_n(x) = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}x^l. \tag{1.2}$$

Note that $\{B_0(x), B_1(1), \ldots, B_n(x)\}$ forms a basis for the space $\mathbb{P}_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$.

So, for a given $p(x) \in \mathbb{P}_n$, we can write

$$p(x) = \sum_{k=0}^{n} a_k B_k(x), \tag{1.3}$$

(see [8–12]) for uniquely determined $a_k \in \mathbb{Q}$.

Further,

$$\begin{aligned} a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\}, \quad \text{where } p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \\ a_0 &= \int_0^1 p(t)dt, \quad \text{where } k = 1, 2, \ldots, n. \end{aligned} \tag{1.4}$$

Probably, $\{1, x, \ldots, x^n\}$ is the most natural basis for the space $\mathbb{P}_n$. But $\{B_0(x), B_1(x), \ldots, B_n(x)\}$ is also a good basis for the space $\mathbb{P}_n$, for our purpose of arithmetical and combinatorial applications.

What are common to $B_n(x)$, $E_n(x)$, x^n? A few proportion common to them are as follows:

(i) they are all monic polynomials of degree n with rational coefficients;

(ii) $(B_n(x))' = nB_{n-1}(x)$, $(E_n(x))' = nE_{n-1}(x)$, $(x^n)' = nx^{n-1}$;

(iii) $\int B_n(x)dx = B_{n+1}(x)/(n+1) + c$, $\int E_n(x)dx = E_{n+1}(x)/(n+1) + c$, $\int x^n dx = x^{n+1}/(n+1) + c$.

In [5, 6], Carlitz introduced the identities of the product of several Bernoulli polynomials:

$$\begin{aligned} B_m(x)B_n(x) &= \sum_{r=0}^{\infty}\left\{\binom{m}{2r}n + \binom{n}{2r}m\right\}\frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1} \\ &\quad \times \frac{m!n!}{(m+n)!}B_{m+n} \quad (m+n \geq 2), \\ \int_0^1 B_m(x)B_n(x)B_p(x)B_q(x)dx &= (-1)^{m+n+p+q}\sum_{r,s=0}^{\infty}\left\{\binom{m}{2r}n + \binom{n}{2r}m\right\}\left\{\binom{p}{2s}q + \binom{q}{2s}p\right\} \\ &\quad \times \frac{(m+n-2r-1)!(p+q-2s-1)!}{(m+n+p+q-2r-2s)!}B_r B_s B_{m+n+p+q-2r-2} \\ &\quad + (-1)^{m+p}\frac{m!n!}{(m+n)!}\frac{p!q!}{(p+q)!}B_{m+n}B_{p+q}. \end{aligned} \tag{1.5}$$

In this paper, we will use (1.4) to derive the identities of the product of several Bernoulli and Euler polynomials.

2. The Product of Several Bernoulli and Euler Polynomials

Let us consider the following polynomials of degree n:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x), \tag{2.1}$$

where the sum runs over all nonnegative integers $i_1,\ldots,i_r,\ j_1,\ldots j_s$ satisfying $i_1+\cdots+i_r+j_1+\cdots+j_s=n,\ r+s=1,\ r,s\geq 0$.

Thus, from (2.1), we have

$$\begin{aligned} p^{(k)}(x) &= (n+r+s-1)(n+r+s-2)\cdots(n+r+s-k) \\ &\quad\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}^{\infty} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \end{aligned} \tag{2.2}$$

For $k=1,2,\ldots,n$, by (1.4), we get

$$\begin{aligned} a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1)-p^{(k-1)}(0)\right\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{\binom{n+r+s}{k}}{n+r+s}\left\{\sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ k+r-n-1\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right. \\ &\qquad\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n+a+1-k-r}^{\infty} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\ &\qquad\left.-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}. \end{aligned} \tag{2.3}$$

From (2.3), we note that

$$\begin{aligned} a_n &= \frac{\binom{n+r+s}{n}}{n+r+s}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s}\left\{\left(-\frac{1}{2}+1\right)r-\left(-\frac{1}{2}\right)s-\left(-\frac{1}{2}\right)(r+s)\right\} \\ &= \frac{\binom{n+r+s}{n}}{n+r+s}(r+s)=\binom{n+r+s-1}{n}, \end{aligned}$$

$$
\begin{aligned}
a_{n-1} &= \frac{1}{n+r+s}\binom{n+r+s}{n-1} \\
&\quad \times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=2} \left\{ B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \right\} \\
&= \frac{1}{n+r+s}\binom{n+r+s}{n-1}\left\{ \frac{1}{6}r + \frac{1}{2}\frac{1}{2}\binom{r+s}{2} - \frac{1}{6}r - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2} \right\} = 0, \\
a_0 &= \int_0^1 p(t)dt \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}^{\infty} \sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \\
&\quad \times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots+p_s+1}.
\end{aligned}
\tag{2.4}
$$

Therefore, by (1.3), (2.1), (2.3), and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\
&\quad = \frac{1}{n+r+s}\sum_{k=1}^{n-2}\binom{n+r+s}{k} \\
&\qquad \times \left\{ \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ k+r-n-1\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \right. \\
&\qquad\quad \left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}^{\infty} B_{i_1}\cdots B_{i_r}E_{k_1}\cdots E_{j_s} \right\} B_k(x) + \binom{n+r+s-1}{n}B_n(x) \\
&\qquad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} \sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \\
&\qquad \times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots p_s+1}.
\end{aligned}
\tag{2.5}
$$

Let us take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t, \tag{2.6}$$

Then, from (2.6), we have

$$\begin{aligned} p^{(k)}(x) &= (n+r+s)(n+r+s-1)\cdots(n+r+s-k+1) \\ &\quad\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t, \end{aligned} \tag{2.7}$$

By (1.4) and (2.7), we get, for $k = 1, 2, \ldots, n$,

$$\begin{aligned} a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\} \\ &= \frac{1}{n+r+s+1}\binom{n+r+s+1}{k} \\ &\quad\times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1} \left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\ &= \frac{\binom{n+r+s+1}{k}}{n+r+s+1}\left\{ \sum_{\substack{0\le a\le r \\ 0\le c\le s \\ k+r-n-1\le a\le r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \right. \\ &\qquad\qquad \times \sum_{t=0}^{n+a+1-k-r} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\ &\qquad\qquad \left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \right\}, \end{aligned} \tag{2.8}$$

Now, we look at a_n and a_{n-1}.

$$\begin{aligned} a_n &= \frac{\binom{n+r+s+1}{n}}{n+r+s+1} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=1} \left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\ &= \frac{\binom{n+r+s+1}{n}}{n+r+s+1}\left\{\frac{1}{2}(r+s) + 1 - \left(-\frac{1}{2}\right)(r+s)\right\} \\ &= \frac{r+s+1}{n+r+s+1}\binom{n+r+s+1}{n} = \binom{n+r+s}{n}, \end{aligned}$$

$$
\begin{aligned}
a_{n-1} &= \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=2} \left\{ B_{i_1}(1)\cdots B_{i_r}(1) E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s} 0^t \right\} \\
&= \frac{\binom{n+r+s+1}{n-1}}{n+r+s+1} \left\{ \frac{1}{6}r + 1 + \frac{1}{2}\frac{1}{2}\binom{r+s}{2} + \frac{1}{2}(r+s) - \frac{1}{6}r - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2} \right\} \\
&= \frac{1}{n+r+s+1} \binom{n+r+s+1}{n-1} \frac{r+s+2}{2} = \frac{1}{2}\binom{n+r+s}{n-1},
\end{aligned}
\tag{2.9}
$$

From (2.6), we note that

$$
\begin{aligned}
a_0 &= \int_0^1 p(t)\,dt \\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \\
&\quad \times B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} E_{j_s-p_s} \frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}.
\end{aligned}
\tag{2.10}
$$

Therefore, by (1.3), (2.6), (2.8), (2.9), and (2.10), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} B_{i_1}(x)\cdots B_{i_r}(x) E_{j_1}(x)\cdots E_{j_s}(x) x^t \\
&\quad = \frac{1}{n+r+s+1} \sum_{k=1}^{n-2} \binom{n+r+s+1}{k} \\
&\qquad \times \left\{ \sum_{\substack{0\leq a\leq r \\ 0\leq c\leq s \\ k+r-n-1\leq a\leq r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{t=0}^{n+a+1-k-r} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} B_{i_1}\cdots B_{i_a} E_{j_1}\cdots E_{j_c} \right. \\
&\qquad\qquad \left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} B_{i_1}\cdots B_{i_r} E_{j_1}\cdots E_{j_s} \right\} B_k(x) \\
&\qquad + \frac{1}{2}\binom{n+r+s}{n-1} B_{n-1}(x) + \binom{n+r+s}{n} B_n(x) \\
&\qquad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} \sum_{l_1=0}^{i_1} \cdots \sum_{l_r=0}^{i_r} \sum_{p_1=0}^{j_1} \cdots \sum_{p_s=0}^{j_s} \left\{ \binom{i_1}{l_1} \cdots \binom{i_r}{l_r} \binom{j_1}{p_1} \cdots \binom{j_s}{p_s} \right. \\
&\qquad\qquad \times B_{i_1-l_1} \cdots B_{i_r-l_r} E_{j_1-p_1} \cdots E_{j_s-p_s} \\
&\qquad\qquad \left. \times \frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1} \right\}.
\end{aligned}
\tag{2.11}
$$

Consider the following polynomial of degree n:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}^{\infty} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.12}$$

Then, from (2.12), one has

$$p^{(k)}(x) = (r+s)^k \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} \frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!}. \tag{2.13}$$

By (1.4) and (2.13), one gets, for $k = 1, 2, \ldots, n$,

$$\begin{aligned}
a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\} \\
&= \frac{(r+s)^{k-1}}{k!} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1} \left\{ \frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} \right\} \\
&= \frac{(r+s)^{k-1}}{k!} \left\{ \sum_{\substack{0\le a\le r \\ 0\le c\le s \\ k+r-n-1\le a\le r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!i_2!\cdots i_a!j_1!\cdots j_c!} \right. \\
&\qquad \left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \right\}.
\end{aligned} \tag{2.14}$$

Now look at a_n and a_{n-1}:

$$\begin{aligned}
a_n &= \frac{(r+s)^{n-1}}{n!} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1} \left\{ \frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} \right\} \\
&= \frac{(r+s)^{n-1}}{n!} \left\{ \frac{1}{2}(r+s) - \left(-\frac{1}{2}\right)(r+s) \right\} = \frac{(r+s)^n}{n!}, \\
a_{n-1} &= \frac{(r+s)^{n-2}}{(n-1)!} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=2} \left\{ \frac{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} \right\} \\
&= \frac{(r+s)^{n-2}}{(n-1)!} \left\{ \frac{1}{2}\frac{1}{6}r + \frac{1}{2}\frac{1}{2}\binom{r+s}{2} - \frac{1}{2}\frac{1}{6}r - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2} \right\} = 0.
\end{aligned} \tag{2.15}$$

It is easy to show that

$$a_0 = \int_0^1 p(t)dt = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} \frac{1}{i_1!\cdots i_r! j_1!\cdots j_s!} \times \sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\left\{\frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{l_1+\cdots+l_r+p_1+\cdots p_s+1}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}\right\}. \tag{2.16}$$

Therefore, by (1.3), (2.14), and (2.15), one obtains the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x) \\
&= \sum_{k=1}^{n-2}\frac{(r+s)^{k-1}}{k!}\left\{\sum_{\substack{0\leq a\leq r\\ 0\leq c\leq s\\ k+r-n-1\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right. \\
&\qquad \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a+1-k-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!i_2!\cdots i_a!j_1!\cdots j_c!} \\
&\qquad - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!} \\
&\qquad \left.\times B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\vphantom{\sum_{\substack{0\\0\\0}}}\right\} B_k(x) + \frac{(r+s)^n}{n!}B_n(x) \\
&+ \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s} \\
&\times \frac{B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!(l_1+\cdots+l_r+p_1+\cdots p_s+1)}.
\end{aligned} \tag{2.17}$$

Take the polynomial $p(x)$ of degree n as follows:

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t. \tag{2.18}$$

Then, from (2.18), one gets

$$p^{(k)}(x) = (r+s+1)^k \times \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t. \tag{2.19}$$

By (1.4) and (2.19), one gets, for $k = 1,\ldots,n$,

$$\begin{aligned}
a_k &= \frac{1}{k!}\left\{p^{(k-1)}(1) - p^{(k-1)}(0)\right\} \\
&= \frac{(r+s+1)^{k-1}}{k!} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k+1} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!} \\
&\quad \times \left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\
&= \frac{(r+s+1)^{k-1}}{k!} \left\{ \sum_{\substack{0\le a\le r \\ 0\le c\le s \\ k+r-n-1\le a\le r}} \binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c} \sum_{t=0}^{n+a+1-k-r} \frac{1}{(n+a+1-k-r-t)!} \right. \\
&\quad \times \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} \frac{1}{i_1!i_2!\cdots i_a!j_1!\cdots j_c!} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\
&\quad \left. - \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s} \right\}.
\end{aligned} \tag{2.20}$$

Now look at a_n and a_{n-1}:

$$\begin{aligned}
a_n &= \frac{(r+s+1)^{n-1}}{n!} \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=1} \frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!} \\
&\quad \times \left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1) - B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\
&= \frac{(r+s+1)^{n-1}}{n!}\left\{\frac{1}{2}(r+s) + 1 - \left(-\frac{1}{2}\right)(r+s)\right\} \\
&= \frac{(r+s+1)^{n-1}}{n!}(r+s+1) = \frac{(r+s+1)^n}{n!},
\end{aligned}$$

$$\begin{aligned}
a_{n-1} &= \frac{(r+s+1)^{n-2}}{(n-1)!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=2}\frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!}\\
&\quad\times\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)-B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\}\\
&= \frac{(r+s+1)^{n-2}}{(n-1)!}\left\{\frac{1}{2}\frac{1}{6}r+\frac{1}{2}+\frac{1}{2}\frac{1}{2}\binom{r+s}{2}+\frac{1}{2}(r+s)-\frac{1}{2}\frac{1}{6}r-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)\binom{r+s}{2}\right\}\\
&= \frac{(r+s+1)^{n-2}}{(n-1)!}\frac{r+s+1}{2}\\
&= \frac{(r+s+1)^{n-1}}{2(n-1)!}.
\end{aligned}\tag{2.21}$$

From (2.18), one can derive the following identity:

$$\begin{aligned}
a_0 &= \int_0^1 p(t)dt\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!t!}\int_0^1 B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t dt\\
&= \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!t!}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\\
&\quad\times\sum_{p_s=0}^{j_s}\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}\frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}.
\end{aligned}\tag{2.22}$$

Therefore, by (1.3), (2.20), (2.21), and (2.22), one obtains the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{1}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!t!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t\\
&\quad=\sum_{k=1}^{n-2}\frac{(r+s+1)^{k-1}}{k!}\left\{\sum_{\substack{0\leq a\leq r\\0\leq c\leq s\\k+r-n-1\leq a\leq r}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\sum_{t=0}^{n+a+1-k-r}\frac{1}{(n+a+1-k-r-t)!}\right.\\
&\qquad\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}\frac{1}{i_1!i_2!\cdots i_a!j_1!\cdots j_c!}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\qquad\left.-\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k+1}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!i_2!\cdots i_r!j_1!\cdots j_s!}\right\}B_k(x)
\end{aligned}$$

$$+\frac{(r+s+1)^{n-1}}{2(n-1)!}B_{n-1}(x)+\frac{(r+s+1)^{n}}{n!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\sum_{l_1=0}^{i_1}\cdots\sum_{l_r=0}^{i_r}\sum_{p_1=0}^{j_1}\cdots\sum_{p_s=0}^{j_s}$$

$$\times\frac{\binom{i_1}{l_1}\cdots\binom{i_r}{l_r}\binom{j_1}{p_1}\cdots\binom{j_s}{p_s}}{i_1!\cdots i_r!j_1!\cdots j_s!t!}B_{i_1-l_1}\cdots B_{i_r-l_r}E_{j_1-p_1}E_{j_s-p_s}\frac{1}{l_1+\cdots+l_r+p_1+\cdots p_s+t+1}. \tag{2.23}$$

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Research Article

Symmetry Fermionic p-Adic q-Integral on $\mathbb{Z}_p$ for Eulerian Polynomials

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Kim et al. (2012) introduced an interesting p-adic analogue of the Eulerian polynomials. They studied some identities on the Eulerian polynomials in connection with the Genocchi, Euler, and tangent numbers. In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, defined by Kim (2008), we show a symmetric relation between the q-extension of the alternating sum of integer powers and the Eulerian polynomials.

1. Introduction

The Eulerian polynomials $A_n(t), n = 0, 1, \ldots$, which can be defined by the generating function

$$\frac{1-t}{e^{(t-1)x}-t} = \sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!}, \tag{1.1}$$

have numerous important applications in number theory, combinatorics, and numerical analysis, among other areas. From (1.1), we note that

$$(A(t)+(t-1))^n - tA_n(t) = (1-t)\delta_{0,n}, \tag{1.2}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [1]). Thus far, few recurrences for the Eulerian polynomials other than (1.2) have been reported in the literature. Other recurrences are of importance as they might reveal new aspects and properties of the Eulerian polynomials, and they can help simplify the proofs of known properties. For more important properties, see, for instance, [1] or [2].

Let p be a fixed odd prime number. Let $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ be the ring of p-adic integers, the field of p-adic numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $|\cdot|_p$ be the p-adic valuation on $\mathbb{Q}$, where $|p|_p = p^{-1}$. The extended valuation on $\mathbb{C}_p$ is denoted by the same symbol $|\cdot|_p$. Let q be an indeterminate, where $|1-q|_p < 1$. Then, the q-number is defined by

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}. \tag{1.3}$$

For a uniformly (or strictly) differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$ (see [1, 3–6]), the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x. \tag{1.4}$$

Then, it is easy to see that

$$\frac{1}{q} I_{-1/q}(f_1) + I_{-1/q}(f) = [2]_{1/q} f(0), \tag{1.5}$$

where $f_1(x) = f(x+1)$.

By using the same method as that described in [1], and applying (1.5) to f, where

$$f(x) = q^{(1-\omega)x} e^{-x(1+q)\omega t} \tag{1.6}$$

for $\omega \in \mathbb{Z}_{>0}$, we consider the generalized Eulerian polynomials on $\mathbb{Z}_p$ by using the fermionic p-adic q-integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{(1-\omega)x} e^{-x(1+q)\omega t} d\mu_{-1/q}(x) &= \frac{1+q}{q^{1-\omega} e^{-(1+q)\omega t} + q} \\ &= \sum_{n=0}^{\infty} A_n(-q,\omega) \frac{t^n}{n!}. \end{aligned} \tag{1.7}$$

By expanding the Taylor series on the left-hand side of (1.7) and comparing the coefficients of the terms $t^n/n!$, we get

$$\int_{\mathbb{Z}_p} q^{(1-\omega)x} x^n d\mu_{-1/q}(x) = \frac{(-1)^n}{\omega^n (1+q)^n} A_n(-q,\omega). \tag{1.8}$$

We note that, by substituting $\omega = 1$ into (1.8),

$$A_n(-q,1) = A_n(-q) = (-1)^n(1+q)^n \int_{\mathbb{Z}_p} x^n d\mu_{-1/q}(x) \tag{1.9}$$

is the Witt's formula for the Eulerian polynomials in [1, Theorem 1]. Recently, Kim et al. [1] investigated new properties of the Eulerian polynomials $A_n(-q)$ at $q = 1$ associated with the Genocchi, Euler, and tangent numbers.

Let $T_{k,1/q}(n)$ denote the q-extension of the alternating sum of integer powers, namely,

$$T_{k,1/q}(n) = \sum_{i=0}^{n} (-1)^i i^k q^{-i} = 0^k q^0 - 1^k q^{-1} + \cdots + (-1)^n n^k q^{-n}, \tag{1.10}$$

where $0^0 = 1$. If $q \to 1$, $T_{k,q}(n) \to T_k(n) = \sum_{i=0}^{n}(-1)^i i^k$ is the alternating sum of integer powers (see [4]). In particular, we have

$$T_{k,1/q}(0) = \begin{cases} 1, & \text{for } k = 0, \\ 0, & \text{for } k > 0. \end{cases} \tag{1.11}$$

Let ω_1, ω_2 be any positive odd integers. Our main result of symmetry between the q-extension of the alternating sum of integer powers and the Eulerian polynomials is given in the following theorem, which is symmetric in ω_1 and ω_2.

Theorem 1.1. *Let ω_1, ω_2 be any positive odd integers and $n \geq 0$. Then, one has*

$$\begin{aligned} &\sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_1) T_{n-i,q^{-\omega_2}}(\omega_1 - 1)\omega_2^{n-i}(-1-q)^{n-i} \\ &\quad = \sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_2) T_{n-i,q^{-\omega_1}}(\omega_2 - 1)\omega_1^{n-i}(-1-q)^{n-i}. \end{aligned} \tag{1.12}$$

Observe that Theorem 1.1 can be obtained by the same method as that described in [4]. If $q = 1$, Theorem 1.1 reduces to the form stated in the remark in [4, page 1275].

Using (1.11), if we take $\omega_2 = 1$ in Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let ω_1 be any positive odd integer and $n \geq 0$. Then, one has*

$$A_n(-q) = \sum_{i=0}^{n} \binom{n}{i} A_i(-q,\omega_1) T_{n-i,q^{-1}}(\omega_1 - 1)(-1-q)^{n-i}. \tag{1.13}$$

2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need the following two identities (see (2.4) and (2.5)) related to the Eulerian polynomials and the q-extension of the alternating sum of integer powers.

Let ω_1, ω_2 be any positive odd integers. From (1.7), we obtain

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x} e^{-x(1+q)\omega_1 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \frac{1 + \left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1 + q^{-\omega_1} e^{-(1+q)\omega_1 t}}. \tag{2.1}$$

This has an interesting p-adic analytic interpretation, which we shall discuss below (see Remark 2.1). It is easy to see that the right-hand side of (2.1) can be written as

$$\begin{aligned} \frac{1 + \left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1 + q^{-\omega_1} e^{-(1+q)\omega_1 t}} &= \sum_{i=0}^{\omega_2 - 1} (-1)^i q^{-\omega_1 i} e^{-(1+q)\omega_1 t i} \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{\omega_2 - 1} (-1)^i i^k \left(q^{\omega_1}\right)^{-i} \omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

In (1.10), let $q = q^{\omega_1}$. The left-hand, right-hand side, by definition, becomes

$$\frac{1 + \left(q^{-\omega_1} e^{-(1+q)\omega_1 t}\right)^{\omega_2}}{1 + q^{-\omega_1} e^{-(1+q)\omega_1 t}} = \sum_{k=0}^{\infty} \left(T_{k, q^{-\omega_1}}(\omega_2 - 1) \omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.3}$$

A comparison of (2.1) and (2.3) yields the identity

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x} e^{-x(1+q)\omega_1 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \sum_{k=0}^{\infty} \left(T_{k, q^{-\omega_1}}(\omega_2 - 1) \omega_1^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.4}$$

By slightly modifying the derivation of (2.4), we can obtain the following identity:

$$\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_2)x} e^{-x(1+q)\omega_2 t} d\mu_{-1/q}(x)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} = \sum_{k=0}^{\infty} \left(T_{k, q^{-\omega_2}}(\omega_1 - 1) \omega_2^k (-1)^k (1+q)^k \right) \frac{t^k}{k!}. \tag{2.5}$$

Remark 2.1. The derivations of identities are based on the fermionic p-adic q-integral expression of the generating function for the Eulerian polynomials in (1.7) and the quotient of integrals in (2.4), (2.5) that can be expressed as the exponential generating function for the q-extension of the alternating sum of integer powers.

Observe that similar identities related to the Eulerian polynomials and the q-extension of the alternating sum of integer powers in (2.4) and (2.5) can be found, for instance, in [3, (1.8)], [4, (21)], and [6, Theorem 4].

Proof of Theorem 1.1. Let ω_1, ω_2 be any positive odd integers. Using the iterated fermionic p-adic q-integral on $\mathbb{Z}_p$ and (1.7), we have

$$\begin{aligned}
&\frac{\int\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1+(1-\omega_2)x_2} e^{-(1+q)(\omega_1 x_1+\omega_2 x_2)t} d\mu_{-1/q}(x_1) d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)} \\
&\quad = [2]_{1/q} \frac{q^{-\omega_1\omega_2} e^{-(1+q)\omega_1\omega_2 t}+1}{\left(q^{-\omega_1} e^{-(1+q)\omega_1 t}+1\right)\left(q^{-\omega_2} e^{-(1+q)\omega_2 t}+1\right)}.
\end{aligned} \tag{2.6}$$

Now, we put

$$I^* = \frac{\int\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1+(1-\omega_2)x_2} e^{-(1+q)(\omega_1 x_1+\omega_2 x_2)t} d\mu_{-1/q}(x_1) d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}. \tag{2.7}$$

From (1.7) and (2.5), we see that

$$\begin{aligned}
I^* &= \left(\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1} e^{-(1+q)(\omega_1 x_1)t} d\mu_{-1/q}(x_1)\right) \times \left(\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_2)x_2} e^{-(1+q)(\omega_2 x_2)t} d\mu_{-1/q}(x_2)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}\right) \\
&= \left(\sum_{k=0}^{\infty} A_k(-q,\omega_1)\frac{t^k}{k!}\right) \times \left(\sum_{l=0}^{\infty}\left(T_{l,q^{-\omega_2}}(\omega_1-1)\omega_2^l(-1)^l(1+q)^l\right)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} A_i(-q,\omega_1) T_{n-i,q^{-\omega_2}}(\omega_1-1)\omega_2^{n-i}(1+q)^{n-i}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.8}$$

On the other hand, from (1.7) and (2.4), we have

$$\begin{aligned}
I^* &= \left(\int_{\mathbb{Z}_p} q^{(1-\omega_2)x_2} e^{-(1+q)(\omega_2 x_2)t} d\mu_{-1/q}(x_2)\right) \times \left(\frac{\int_{\mathbb{Z}_p} q^{(1-\omega_1)x_1} e^{-(1+q)(\omega_1 x_1)t} d\mu_{-1/q}(x_1)}{\int_{\mathbb{Z}_p} q^{(1-\omega_1\omega_2)x} e^{-x(1+q)\omega_1\omega_2 t} d\mu_{-1/q}(x)}\right) \\
&= \left(\sum_{k=0}^{\infty} A_k(-q,\omega_2)\frac{t^k}{k!}\right) \times \left(\sum_{l=0}^{\infty}\left(T_{l,q^{-\omega_1}}(\omega_2-1)\omega_1^l(-1)^l(1+q)^l\right)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} A_i(-q,\omega_2) T_{n-i,q^{-\omega_1}}(\omega_2-1)\omega_1^{n-i}(1+q)^{n-i}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.9}$$

By comparing the coefficients on both sides of (2.8) and (2.9), we obtain the result in Theorem 1.1. □

3. Concluding Remarks

Note that many other interesting symmetric properties for the Euler, Genocchi, and tangent numbers are derivable as corollaries of the results presented herein. For instance, considering [1, (5)],

$$A_n(-1,\omega) = (-2\omega)^n E_n \quad (n \geq 0), \tag{3.1}$$

where E_n denotes the nth Euler number defined by $E_n := E_n(0)$, and the Euler polynomials are defined by the generating function

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{3.2}$$

and on putting $q = 1$ in Theorem 1.1 and Corollary 1.2, we obtain

$$\sum_{i=0}^{n}\binom{n}{i}\omega_1^i E_i T_{n-i}(\omega_1 - 1)\omega_2^{n-i} = \sum_{i=0}^{n}\binom{n}{i}\omega_2^i E_i T_{n-i}(\omega_2 - 1)\omega_1^{n-i}, \tag{3.3}$$

$$E_n = \sum_{i=0}^{n}\binom{n}{i}\omega_1^i E_i T_{n-i}(\omega_1 - 1). \tag{3.4}$$

These formulae are valid for any positive odd integers ω_1, ω_2. The Genocchi numbers G_n may be defined by the generating function

$$\frac{2t}{e^t+1} = \sum_{n=0}^{\infty} G_n\frac{t^n}{n!}, \tag{3.5}$$

which have several combinatorial interpretations in terms of certain surjective maps on finite sets. The well-known identity

$$G_n = 2(1-2^n)B_n \tag{3.6}$$

shows the relation between the Genocchi and the Bernoulli numbers. It follows from (3.6) and the Staudt-Clausen theorem that the Genocchi numbers are integers. It is easy to see that

$$G_n = 2nE_{2n-1} \quad (n \geq 1), \tag{3.7}$$

and from (3.2), (3.5) we deduce that

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{G_{k+1}}{k+1}x^{n-k}. \tag{3.8}$$

It is well known that the tangent coefficients (or numbers) T_n, defined by

$$\tan t = \sum_{n=1}^{\infty} (-1)^{n-1} T_{2n} \frac{t^{2n-1}}{(2n-1)!}, \tag{3.9}$$

are closely related to the Bernoulli numbers, that is, (see [1])

$$T_n = 2^n (2^n - 1) \frac{B_n}{n}. \tag{3.10}$$

Ramanujan ([7, page 5]) observed that $2^n(2^n-1)B_n/n$ and, therefore, the tangent coefficients, are integers for $n \geq 1$. From (3.3), (3.6), (3.7), and (3.10), the obtained symmetric formulae involve the Bernoulli, Genocchi, and tangent numbers (see [1]).

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Research Article

Euler Basis, Identities, and Their Applications

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Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. We show that $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ is a good basis for the space V_n, for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n, then $p(x) = \sum_{l=0}^{n} b_l E_l(x)$ for some uniquely determined $b_l \in \mathbb{Q}$. In this paper we develop method for computing b_l from the information of $p(x)$.

1. Introduction

The Euler polynomials, $E_n(x)$, are given by

$$\frac{2}{e^t+1}e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.1}$$

(see [1–20]) with the usual convention about replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0, E_n(0) = E_n$ are called the nth Euler numbers. The Bernoulli numbers are also defined by

$$\frac{t}{e^t-1} = e^{Bt} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \tag{1.2}$$

(see [1–20]) with the usual convention about replacing B^n by B_n. As is well known, the Bernoulli polynomials are given by

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}B_l x^{n-l} = \sum_{l=0}^{n}\binom{n}{l}B_{n-l}x^l, \tag{1.3}$$

(see [9–15]) From (1.1), (1.2), and (1.3), we note that

$$B_n(1) - B_n = \delta_{1,n}, \qquad E_n(1) + E_n = 2\delta_{0,n}, \tag{1.4}$$

where $\delta_{k,n}$ is the kronecker symbol.

Let $m, n \in \mathbb{Z}_+$ with $m + n \geq 2$. The formula

$$B_m(x)B_n(x) = \sum_r \left\{ \binom{m}{2r} n + \binom{n}{2r} m \right\} \frac{B_{2r}B_{m+n-2r}(x)}{m+n-2r} + (-1)^{m+1}\frac{m!n!B_{m+n}}{(m+n)!}, \tag{1.5}$$

is proved in [4–6]. Let $V_n = \{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \leq n\}$ be the $(n+1)$-dimensional vector space over $\mathbb{Q}$. Probably, $\{1, x, \ldots, x^n\}$ is the most natural basis for this space. But $\{E_0(x), E_1(x), \ldots, E_n(x)\}$ is also a good basis for the space V_n, for our purpose of arithmetical and combinatorial applications. Thus, if $p(x) \in \mathbb{Q}[x]$ is of degree n, then

$$p(x) = \sum_{l=0}^{n} b_l E_l(x), \tag{1.6}$$

for some uniquely determined $b_l \in \mathbb{Q}$. Further,

$$b_k = \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} \qquad (k = 0, 1, 2, \ldots, n), \tag{1.7}$$

where $p^{(k)}(x) = d^k p(x)/dx^k$. In this paper we develop methods for computing b_l from the information of $p(x)$. Apply these results to arithmetically and combinatorially interesting identities involving $E_0(x), E_1(x), \ldots, E_n(x), B_0(x), \ldots, B_n(x)$. Finally, we give some applications of those obtained identities.

2. Euler Basis, Identities, and Their Applications

Let us take $p(x)$ the polynomial of degree n as follows:

$$p(x) = \sum_{k=0}^{n} B_k(x)B_{n-k}(x). \tag{2.1}$$

From (2.1), we have

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!}\sum_{l=k}^{n} B_{l-k}(x)B_{n-l}(x). \tag{2.2}$$

By (1.7) and (2.2), we get

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} \\ &= \frac{1}{2}\binom{n+1}{k}\sum_{l=k}^{n}\{(B_{l-k} + \delta_{1,l-k})(B_{n-l} + \delta_{1,n-l}) + B_{l-k}B_{n-l}\}, \end{aligned} \tag{2.3}$$

Thus, we have

$$b_k = \binom{n+1}{k}\left(\sum_{l=k}^{n}B_{l-k}B_{n-l} + B_{n-k-1}\right), \quad (0 \le k \le n-3), \tag{2.4}$$

$$b_{n-2} = \frac{7}{72}n\left(n^2-1\right), \qquad b_n = n+1, \qquad b_{n-1} = 0. \tag{2.5}$$

By (1.6), (2.1), (2.3), and (2.4), we get

$$\begin{aligned} &\sum_{k=0}^{n}B_k(x)B_{n-k}(x) \\ &\quad = \sum_{k=0}^{n-3}\binom{n+1}{k}\left(\sum_{l=k}^{n}B_{l-k}B_{n-l} + B_{n-k-1}\right)E_k(x) + \frac{7}{72}n\left(n^2-1\right)E_{n-2}(x) + (n+1)E_n(x). \end{aligned} \tag{2.6}$$

Let us consider the following triple identities:

$$p(x) = \sum_{r+s+t=n}B_r(x)B_s(x)B_t(x) = \sum_{k=0}^{n}b_kE_k(x), \tag{2.7}$$

where the sum runs over all $r, s, t \in \mathbb{Z}_+$ with $r+s+t=n$. Thus, by (2.7), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k}B_r(x)B_s(x)B_t(x). \tag{2.8}$$

From (1.7) and (2.8), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} \\ &= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{B_r(1)B_s(1)B_t(1) + B_rB_sB_t\} \\ &= \frac{\binom{n+2}{k}}{2}\left\{2\sum_{r+s+t=n-k}B_rB_sB_t + \sum_{r+s+t=n-k}\delta_{1,r}B_sB_t + \sum_{r+s+t=n-k}B_r\delta_{1,s}B_t\right. \end{aligned}$$

$$+\sum_{r+s+t=n-k} B_r B_s \delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s} B_t + \sum_{r+s+t=n-k} \delta_{1,r} B_s \delta_{1,t}$$
$$\left. + \sum_{r+s+t=n-k} B_r \delta_{1,s}\delta_{1,t} + \sum_{r+s+t=n-k} \delta_{1,r}\delta_{1,s}\delta_{1,t} \right\}. \tag{2.9}$$

Therefore, by (2.7) and (2.9), we obtain the following theorem.

Theorem 2.1. *For $r, s, t \in \mathbb{Z}_+$, and $n \in \mathbb{N}$ with $n \geq 3$, one has*

$$\sum_{r+s+t=n} B_r(x)B_s(x)B_t(x)$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s+t=n-k} B_r B_s B_t + 3\sum_{r+s=n-k-1} B_r B_s + 3B_{n-k-2} + \delta_{k,n-3}\right\}E_k(x) \tag{2.10}$$
$$+\binom{n+2}{2}E_n(x).$$

Let us take the polynomial $p(x)$ as follows:

$$p(x) = \sum_{r+s+t=n} B_r(x)B_s(x)E_t(x). \tag{2.11}$$

Then, by (2.11), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k} B_r(x)B_s(x)E_t(x). \tag{2.12}$$

From (1.6), (1.7), and (2.12), we have

$$b_k = \frac{1}{2k!}\left\{p^{(k)}(1) + p^{(k)}(0)\right\} = \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{B_r(1)B_s(1)E_t(1) + B_r B_s E_t\}$$
$$= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\{(B_r+\delta_{1,r})(B_s+\delta_{1,s})(-E_t + 2\delta_{0,t}) + B_r B_s E_t\}$$
$$= \frac{\binom{n+2}{k}}{2}\left\{-\sum_{r+s+t=n-k}\delta_{1,r}B_s E_t - \sum_{r+s+t=n-k} B_r\delta_{1,s}E_t + 2\sum_{r+s+t=n-k} B_r B_s \delta_{0,t}\right. \tag{2.13}$$
$$-\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}E_t + 2\sum_{r+s+t=n-k}\delta_{1,r}B_s\delta_{0,t} + 2\sum_{r+s+t=n-k} B_r\delta_{1,s}\delta_{0,t}$$
$$\left. +2\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}\delta_{0,t}\right\}.$$

Note that

$$\begin{aligned} b_{n-1} &= \binom{n+2}{n-1}\left\{-\sum_{s+t=0} B_s E_t - \sum_{r+t=0} B_r E_t + 2\sum_{r+s=1} B_r B_s - 0 + 2B_0 + 2B_0 + 2\cdot 0\right\} \\ &= \frac{1}{2}\binom{n+2}{n-1}\{-1-1+2(B_1+B_1)+2+2\} = 0. \end{aligned} \tag{2.14}$$

Therefore, we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned} &\sum_{r+s+t=n} B_r(x)B_s(x)E_t(x) \\ &\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s=n-k} B_r B_s - 2\sum_{r+t=n-k-1} B_r E_t - E_{n-k-2} + 4B_{n-k-1} + 2\delta_{k,n-2}\right\} E_k(x) \\ &\qquad + \binom{n+2}{2} E_n(x). \end{aligned} \tag{2.15}$$

Remark 2.3. By the same method, we obtain the following identities.

(I)

$$\begin{aligned} &\sum_{r+s+t=n} B_r(x)E_s(x)E_t(x) \\ &\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{2\sum_{r+s+t=n-k} B_r E_s E_t + \sum_{s+t=n-k-1} E_s E_t - 4\sum_{r+s=n-k} B_r E_s + 4B_{n-k} - 4E_{n-k-1}\right\} E_k(x) \\ &\qquad + \binom{n+2}{2} E_n(x). \end{aligned} \tag{2.16}$$

(II)

$$\begin{aligned} &\sum_{r+s+t=n} E_r(x)E_s(x)E_t(x) \\ &\quad = 3\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{r+s=n-k} E_r E_s - 2E_{n-k}\right\} E_k(x) + \binom{n+2}{2} E_n(x). \end{aligned} \tag{2.17}$$

Let us consider the polynomial $p(x)$ as follows:

$$p(x) = \sum_{r+s+t=n} B_r(x)B_s(x)x^t. \tag{2.18}$$

Thus, by (2.18), we get

$$p^{(k)}(x) = (n+2)(n+1)n(n-1)\cdots(n-k+3)\sum_{r+s+t=n-k} B_r(x)B_s(x)x^t. \tag{2.19}$$

From (1.6), (1.7), (2.18), and (2.19), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} = \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\left\{B_r(1)B_s(1)+B_rB_s0^t\right\} \\ &= \frac{\binom{n+2}{k}}{2}\sum_{r+s+t=n-k}\left\{(B_r+\delta_{1,r})(B_s+\delta_{1,s})+B_rB_s0^t\right\} \\ &= \frac{\binom{n+2}{k}}{2}\left\{\sum_{r+s+t=n-k}B_rB_s+\sum_{r+s+t=n-k}B_r\delta_{1,s}+\sum_{r+s+t=n-k}\delta_{1,r}B_s+\sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s}+\sum_{r+s+t=n-k}B_rB_s0^t\right\}. \end{aligned} \tag{2.20}$$

Here we note that

$$\begin{gathered} \sum_{r+s+t=n-k}B_rB_s = \sum_{t=0}^{n-k}\sum_{r+s=n-k-t}B_rB_s = \sum_{t=0}^{n-k}\sum_{r+s=t}B_rB_s \\ \sum_{r+s+t=n-k}B_r\delta_{1,s} = \begin{cases}\sum_{r=0}^{n-k-1}B_r, & \text{if } k\le n-1,\\ 0, & \text{if } k=n,\end{cases} \\ \sum_{r+s+t=n-k}B_s\delta_{1,r} = \begin{cases}\sum_{r=0}^{n-k-1}B_r, & \text{if } k\le n-1,\\ 0, & \text{if } k=n,\end{cases} \\ \sum_{r+s+t=n-k}\delta_{1,r}\delta_{1,s} = \begin{cases}1, & \text{if } k\le n-2,\\ 0, & \text{if } k=n-1 \text{ or } n,\end{cases} \\ \sum_{r+s+t=n-k}B_rB_s0^t = \sum_{r+s=n-k}B_rB_s, \quad \forall k. \end{gathered} \tag{2.21}$$

It is easy to show that

$$\begin{aligned} b_{n-1} &= \frac{1}{2}\binom{n+2}{n-1}\left\{\sum_{r+s=0}B_rB_s+2\sum_{r+s=1}B_rB_s+2B_0\right\} \\ &= \frac{1}{2}\binom{n+2}{n-1}\{1+2(B_1+B_2)+2\} = \frac{1}{2}\binom{n+2}{n-1}. \end{aligned} \tag{2.22}$$

Therefore, by (1.6), (2.18), (2.20), and (2.22), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\sum_{r+s+t=n} B_r(x)B_s(x)x^t$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{t=0}^{n-k-1}\sum_{r+s=t} B_rB_s + 2\sum_{r+s=n-k} B_rB_s + 2\sum_{r=0}^{n-k-1} B_r + 1\right\}E_k(x) \tag{2.23}$$
$$+ \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{n}E_n(x).$$

Remark 2.5. By the same method, we can derive the following identities.

(I)

$$\sum_{r+s+t=n} B_r(x)E_s(x)x^t$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{-\sum_{t=0}^{n-k-1}\sum_{r+s=t} B_rE_s - \sum_{s=0}^{n-k-1} E_s + 2\sum_{r=0}^{n-k} B_r + 2\right\}E_k(x) \tag{2.24}$$
$$+ \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{n}E_n(x).$$

(II)

$$\sum_{r+s+t=n} E_r(x)E_s(x)x^t$$
$$= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+2}{k}\left\{\sum_{t=0}^{n-k-1}\sum_{r+s=t} E_rE_s + 2\sum_{r+s=n-k} E_rE_s - 4\sum_{r=0}^{n-k} E_r + 4\right\}E_k(x) \tag{2.25}$$
$$+ \frac{1}{2}\binom{n+2}{n-1}E_{n-1}(x) + \binom{n+2}{2}E_n(x).$$

Now we generalize the above consideration to the completely arbitrary case. Let

$$p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x), \tag{2.26}$$

where the sum runs over all nonnegative integers $i_1, i_2, \ldots, i_r, j_1, \ldots, j_s$ satisfying $i_1 + i_2 + \cdots + i_r + j_1 + \cdots + j_s = n$. From (2.26), we note that

$$p^{(k)}(x) = (n+r+s-1)\cdots(n+r+s-k)\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} B_{i_1}(x)\cdots B_{i_r}(x)\times E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.27}$$

By (1.6), (1.7), (2.18), and (2.27), we get

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\}\\
&= \frac{1}{2}\binom{n+r+s-1}{k}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}\\
&= \frac{1}{2}\binom{n+r+s-1}{k}\\
&\quad\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\left\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\right.\\
&\qquad\qquad\left.\times\left(-E_{j_1}+2\delta_{0,j_1}\right)\cdots\left(-E_{j_s}+2\delta_{0,j_s}\right)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}\\
&= \frac{1}{2}\binom{n+r+s-1}{k}\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\right.\\
&\qquad\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.
\end{aligned}\tag{2.28}$$

Note that

$$\begin{aligned}
b_n &= \frac{1}{2}\binom{n+r+s-1}{n}\left\{\sum_{0\le c\le s}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c}\right.\\
&\qquad\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}\\
&= \frac{1}{2}\binom{n+r+s-1}{n}\left((2-1)^s+1\right)=\binom{n+r+s-1}{n},\\
b_{n-1} &= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{\sum_{\substack{r-1\le a\le r\\0\le c\le s}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\right.\\
&\qquad\qquad\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=1+a-r}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\qquad\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}
\end{aligned}$$

$$= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{r(2-1)^s + \sum_{0\le c\le s}\binom{s}{c}(-1)^c 2^{s-c}\left[-\frac{1}{2}(r+c)\right] - \frac{1}{2}(r+s)\right\}$$
$$= \frac{1}{2}\binom{n+r+s-1}{n-1}\left\{r - \frac{1}{2}r + \frac{1}{2}s - \frac{1}{2}r - \frac{1}{2}s\right\} = 0. \tag{2.29}$$

Therefore, by (1.6), (2.28), and (2.29), we obtain the following theorem.

Theorem 2.6. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)\\
&= \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+r+s-1}{k}\left\{\sum_{\substack{0\le a\le r\\0\le c\le s\\a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right.\\
&\qquad\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}E_k(x)\\
&+\binom{n+r+s-1}{n}E_n(x).
\end{aligned} \tag{2.30}$$

Let us consider the polynomial $p(x)$ of degree n as

$$p(x) = \sum_{t+i_1+\cdots+i_r+j_1+\cdots+j_s=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t. \tag{2.31}$$

Then, from (2.31), we have

$$\begin{aligned}
p^{(k)}(x) &= (n+r+s)(n+r+s-1)\cdots(n+r+s-k+1)\\
&\quad\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t.
\end{aligned} \tag{2.32}$$

By (1.7) and (2.32), we get

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\} \\
&= \frac{1}{2}\binom{n+r+s}{k}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k}\left\{B_{i_1}(1)\cdots B_{i_r}(1)E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\right\} \\
&= \frac{1}{2}\binom{n+r+s}{k} \\
&\quad\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n-k}\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r}) \\
&\qquad\times\left(-E_{j_0}+2\delta_{0,j_1}\right)\cdots\left(-E_{j_s}+2\delta_{1,j_s}\right)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}0^t\}
\end{aligned} \tag{2.33}$$

From (2.33), we can derive the following equation:

$$\begin{aligned}
b_k = \frac{1}{2}\binom{n+r+s}{k}\Bigg\{&\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{t=0}^{n+a-k-r}\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t}B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c} \\
&+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\Bigg\}.
\end{aligned} \tag{2.34}$$

Observe now that

$$\begin{aligned}
b_n &= \frac{1}{2}\binom{n+r+s}{n}\Bigg\{\sum_{c=0}^{s}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c} \\
&\qquad+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\Bigg\} \\
&= \frac{1}{2}\binom{n+r+s}{n}\left[(2-1)^s+1\right]=\binom{n+r+s}{n},
\end{aligned} \tag{2.35}$$

$$
\begin{aligned}
b_{n-1} &= \frac{1}{2}\binom{n+r+s}{n-1}\Bigg\{\sum_{\substack{r-1\le a\le r\\ 0\le c\le s}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{t=0}^{1+a-r}\ \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\qquad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\Bigg\}\\
&= \frac{1}{2}\binom{n+r+s}{n-1}\left\{r+1-\frac{1}{2}r+\frac{1}{2}s-\frac{1}{2}r-\frac{1}{2}s\right\} = \frac{1}{2}\binom{n+r+s}{n-1}.
\end{aligned}
\tag{2.36}
$$

Therefore, by (1.6), (2.31), (2.34), (2.35), and (2.36), we obtain the following theorem.

Theorem 2.7. *For $n \in \mathbb{N}$ with $n \ge 2$, one has*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n} B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t\\
&\quad = \frac{1}{2}\sum_{k=0}^{n-2}\binom{n+r+s}{k}\Bigg\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\\
&\qquad\qquad \times\sum_{t=0}^{n+a-k-r}\ \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}\\
&\qquad\qquad + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\Bigg\}E_k(x)\\
&\qquad + \frac{1}{2}\binom{n+r+s}{n-1}E_{n-1}(x) + \binom{n+r+s}{n}E_n(x).
\end{aligned}
\tag{2.37}
$$

Let us consider the following polynomial of degree n.

$$
p(x) = \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.38}
$$

Thus, by (2.38), we get

$$
p^{(k)}(x) = (r+s)^k\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\times B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x). \tag{2.39}
$$

From (1.7), we have

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1)+p^{(k)}(0)\right\}\\
&= \frac{(r+s)^k}{2k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
&\quad\times\left\{B_{i_1}(1)\cdots B_{i_r}(1)\times E_{j_1}(1)\cdots E_{j_s}(1)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}\\
&= \frac{(r+s)^k}{2k!}\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{1}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
&\quad\times\left\{(B_{i_1}+\delta_{1,i_1})\cdots(B_{i_r}+\delta_{1,i_r})\times\left(-E_{j_1}+2\delta_{0,j_1}\right)\cdots\left(-E_{j_s}+2\delta_{0,j_s}\right)+B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}\right\}.
\end{aligned}\tag{2.40}$$

Thus, by (2.40), we get

$$\begin{aligned}
b_k = \frac{(r+s)^k}{2k!}&\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c2^{s-c}\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}\right.\\
&\left.\quad+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\}.
\end{aligned}\tag{2.41}$$

Now, we note that

$$\begin{aligned}
b_n &= \frac{(r+s)^n}{2n!}\left\{\sum_{c=0}^{s}\binom{s}{c}(-1)^c2^{s-c}\right.\\
&\qquad\left.\times\sum_{i_1+\cdots+i_r+j_1+\cdots+j_c=0}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_r!j_1!\cdots j_c!}+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=0}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\}\\
&= \frac{(r+s)^n}{2n!}\left[(2-1)^s+1\right]=\frac{(r+s)^n}{n!},
\end{aligned}$$

$$
\begin{aligned}
b_{n-1} &= \frac{(r+s)^{n-1}}{2(n-1)!}\left\{\sum_{\substack{r-1\le a\le r\\ 0\le c\le s}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right.\\
&\qquad \left.\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=1+a-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=1}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\}\\
&= \frac{(r+s)^{n-1}}{2(n-1)!}\left\{r(2-1)^s+\sum_{c=0}^{s}\binom{s}{c}(-1)^c 2^{s-c}\left[-\frac{1}{2}(r+c)\right]-\frac{1}{2}(r+s)\right\}=0.
\end{aligned}
\tag{2.42}
$$

Therefore, by (1.6), (2.38), (2.41), and (2.42), we obtain the following theorem.

Theorem 2.8. *For $n\in\mathbb{N}$ with $n\ge 2$, one has*

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n}\frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)}{i_1!\cdots i_r!j_1!\cdots j_s!}\\
&\quad=\frac{1}{2}\sum_{k=0}^{n-2}\frac{(r+s)^k}{k!}\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\times\sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=n+a-k-r}\frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a!j_1!\cdots j_c!}\right.\\
&\qquad\qquad\left.+\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k}\frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r!j_1!\cdots j_s!}\right\}E_k(x)\\
&\quad+\frac{(r+s)^n}{n!}E_n(x).
\end{aligned}
\tag{2.43}
$$

By the same method, we can obtain the following identity:

$$
\begin{aligned}
&\sum_{i_1+\cdots+i_r+j_1+\cdots+j_s+t=n}\frac{B_{i_1}(x)\cdots B_{i_r}(x)E_{j_1}(x)\cdots E_{j_s}(x)x^t}{i_1!\cdots i_r!j_1!\cdots j_s!t!}\\
&\quad=\frac{1}{2}\sum_{k=0}^{n-2}\frac{(r+s+1)^k}{k!}\left\{\sum_{\substack{0\le a\le r\\ 0\le c\le s\\ a\ge k+r-n}}\binom{r}{a}\binom{s}{c}(-1)^c 2^{s-c}\right.
\end{aligned}
$$

$$\times \sum_{t=0}^{n+a-k-r} \frac{1}{(n+a-k-r-t)!} \sum_{i_1+\cdots+i_a+j_1+\cdots+j_c=t} \frac{B_{i_1}\cdots B_{i_a}E_{j_1}\cdots E_{j_c}}{i_1!\cdots i_a! j_1!\cdots j_c!}$$

$$\left. + \sum_{i_1+\cdots+i_r+j_1+\cdots+j_s=n-k} \frac{B_{i_1}\cdots B_{i_r}E_{j_1}\cdots E_{j_s}}{i_1!\cdots i_r! j_1!\cdots j_s!} \right\} E_k(x)$$

$$+ \frac{(r+s+1)^{n-1}}{2(n-1)!} E_{n-1}(x) + \frac{(r+s+1)^n}{n!} E_n(x). \tag{2.44}$$

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Research Article

Integral Formulae of Bernoulli and Genocchi Polynomials

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Recently, some interesting and new identities are introduced in the work of Kim et al. (2012). From these identities, we derive some new and interesting integral formulae for Bernoulli and Genocchi polynomials.

1. Introduction

As it is well known, the Bernoulli polynomials are defined by generating functions as follows:

$$\frac{t}{e^t-1}e^{xt}=e^{B(x)t}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{1.1}$$

(see [1–5]) with the usual convention about replacing $B^n(x)$ by $B_n(x)$. In the special case, $x=0$, $B_n(0)=B_n$ are called the nth Bernoulli numbers.

The Genocchi polynomials are also defined by

$$\frac{2t}{e^t+1}e^{xt}=e^{G(x)t}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!} \tag{1.2}$$

(see [1, 6–10]) with the usual convention about replacing $G^n(x)$ by $G_n(x)$. In the special case, $x=0$, $G_n(0)=G_n$ are called the nth Genocchi numbers.

From (1.1), we note that

$$B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l} \tag{1.3}$$

(see [1–5]). Thus, by (1.3), we get

$$\frac{d}{dx} B_n(x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} B_l x^{n-1-l} = n B_{n-1}(x) \tag{1.4}$$

(see [2]). From (1.2), we note that

$$G_n(x) = \sum_{l=0}^{n} \binom{n}{l} G_l x^{n-l}. \tag{1.5}$$

From (1.5), we can derive the following equation:

$$\frac{d}{dx} G_n(x) = n \sum_{l=0}^{n-1} \binom{n-1}{l} G_l x^{n-1-l} = n G_{n-1}(x). \tag{1.6}$$

By the definition of Bernoulli and Genocchi numbers, we get the following recurrence formulae:

$$B_0 = 1, \quad B_n(1) - B_n = \delta_{1,n}, \quad G_0 = 0, \quad G_n(1) + G_n = 2\delta_{1,n}, \tag{1.7}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [2]). From (1.4), (1.6), and (1.7), we note that

$$\int_0^1 B_n(x)dx = \frac{\delta_{0,n}}{n+1} \quad (n \geq 0), \qquad \int_0^1 G_n(x)dx = -\frac{2G_{n+1}}{n+1} \quad (n \geq 1). \tag{1.8}$$

From the identities of Bernoulli and Genocchi polynomials, we derive some new and interesting integral formulae of an arithmetical nature on the Bernoulli and Genocchi polynomials.

2. Integral Formula of Bernoulli and Genocchi Polynomials

From (1.1) and (1.2), we note that

$$\begin{aligned}
\frac{t}{e^t-1}e^{xt} &= \frac{1}{2}\left(\frac{2te^{xt}}{e^t+1}\right)+\frac{1}{t}\left(\frac{t}{e^t-1}\right)\left(\frac{2te^{xt}}{e^t+1}\right)\\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\right)+\frac{1}{t}\left(\sum_{l=0}^{\infty}B_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}G_m(x)\frac{t^m}{m!}\right)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}+\frac{1}{t}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\binom{n}{l}G_l(x)B_{n-l}\frac{t^n}{n!}\\
&= \frac{1}{2}\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\left(-\frac{1}{2}G_n(x)+\sum_{\substack{l=0\\ l\neq n}}^{n+1}\frac{\binom{n+1}{l}G_l(x)B_{n+1-l}}{n+1}\right)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{\substack{l=0\\ l\neq n}}^{n+1}\binom{n+1}{l}\frac{G_l(x)B_{n+1-l}}{n+1}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

By comparing the coefficients on the both sides of (2.1), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$B_n(x) = \sum_{l=0\ l\neq n}^{n+1}\binom{n+1}{l}\frac{G_l(x)B_{n+1-l}}{n+1}. \tag{2.2}$$

From (1.1) and (1.2), also notes that

$$\begin{aligned}
\frac{2t}{e^t+1}e^{xt} &= \frac{1}{t}\left(\frac{2t(e^t-1)}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right) = \frac{1}{t}\left(2t-2\frac{2t}{e^t+1}\right)\left(\frac{te^{xt}}{e^t-1}\right)\\
&= \frac{1}{t}\left(2t-2\sum_{l=0}^{\infty}G_l\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right)\\
&= \frac{1}{t}\left(-2\sum_{l=1}^{\infty}\frac{G_{l+1}}{l+1}\frac{t^{l+1}}{l!}\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right)\\
&= \sum_{n=1}^{\infty}\left(-2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}B_{n-l}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

By comparing the coefficients on the both sides of (2.3), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{N}$, one has*

$$G_n(x) = -2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}B_{n-l}(x). \tag{2.4}$$

Let one take the definite integral from 0 to 1 on both sides of Theorem 2.1. For $n \geq 2$,

$$0 = -2\sum_{\substack{l=1\\ l\neq n}}^{n+1}\binom{n+1}{l}\frac{G_{l+1}}{l+1}\frac{B_{n+1-l}}{n+1} = -B_nG_2 - 2\sum_{\substack{l=1\\ l\neq n-1}}^{n}\binom{n}{l}\frac{B_{n-l}G_{l+2}}{(l+1)(l+2)}. \tag{2.5}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$B_n = 2\sum_{\substack{l=1\\ l\neq n-1}}^{n}\binom{n}{l}\frac{B_{n-l}G_{l+2}}{(l+1)(l+2)}. \tag{2.6}$$

3. p-Adic Integral on $\mathbb{Z}_p$ Associated with Bernoulli and Genocchi Numbers

Let p be a fixed odd prime number. Throughout this section, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x) \tag{3.1}$$

(see [2, 5, 11]). From (3.1), we can derive the following integral equation:

$$I(f_n) = I(f) + \sum_{i=0}^{n-1}f'(i) \qquad (n \in \mathbb{N}), \tag{3.2}$$

where $f_n(x) = f(x+n)$ and $f'(i) = ((df(x))/dx)|_{x=i}$ (see [2]). Let us take $f(y) = e^{t(x+y)}$. Then we have

$$\int_{\mathbb{Z}_p} e^{t(x+y)}d\mu(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \tag{3.3}$$

(see [2, 5]). From (3.3), we have

$$\int_{\mathbb{Z}_p} (x+n)^n d\mu(y) = B_n(x), \qquad \int_{\mathbb{Z}_p} y^n d\mu(y) = B_n \tag{3.4}$$

(see [2, 5]). Thus, by (3.2) and (3.4), we get

$$\int_{\mathbb{Z}_p} (x+n)^m d\mu(x) = \int_{\mathbb{Z}_p} x^m d\mu(x) + m\sum_{i=0}^{n-1} i^{m-1}, \tag{3.5}$$

(see [2]). From (3.5), we have

$$B_m(n) - B_m = m\sum_{i=0}^{n-1} i^{m-1} \quad (n \in \mathbb{Z}_+) \tag{3.6}$$

(see [2]). The fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows [2, 8, 9]:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{3.7}$$

From (3.7), we obtain the following integral equation:

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} f(l) \tag{3.8}$$

(see [2]), where $f_n(x) = f(x+n)$. Thus, by (3.8), we have

$$\int_{\mathbb{Z}_p} (x+n)^m d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} x^m d\mu_{-1}(x) + 2\sum_{l=0}^{n-1} (-1)^{n-l-1} l^m \tag{3.9}$$

(see [2]). Let us take $f(y) = e^{t(x+y)}$. Then we have

$$t\int_{\mathbb{Z}_p} e^{t(x+y)} d\mu_{-1}(y) = \frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}. \tag{3.10}$$

From (3.10), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}, \int_{\mathbb{Z}_p} y^n d\mu_{-1}(y) = \frac{G_{n+1}}{n+1}. \tag{3.11}$$

Thus, by (3.9) and (3.11), we get

$$\frac{G_{m+1}(n)}{m+1} = (-1)^n \left(\frac{G_{n+1}}{n+1} + 2\sum_{l=0}^{n-1} (-1)^{l-1} l^m \right). \tag{3.12}$$

Let us consider the following p-adic integral on $\mathbb{Z}_p$:

$$K_1 = \int_{\mathbb{Z}_p} B_n(x)d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \int_{\mathbb{Z}_p} x^l d\mu(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l. \tag{3.13}$$

From Theorem 2.1 and (3.13), one has

$$\begin{aligned} K_1 &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \binom{k}{l} G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu(x) \\ &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} B_l G_{k-l}}{n+1}. \end{aligned} \tag{3.14}$$

Therefore, by (3.13) and (3.14), we obtain the following theorem.

Theorem 3.1. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n} \binom{n}{l} B_{n-l} B_l = \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} B_l G_{k-l}}{n+1}. \tag{3.15}$$

Now, one sets

$$K_2 = \int_{\mathbb{Z}_p} B_n(x)d\mu_{-1}(x) = \sum_{l=0}^{n} \binom{n}{l} B_{n-l} \frac{G_{l+1}}{l+1}. \tag{3.16}$$

By Theorem 2.1, one gets

$$\begin{aligned} K_2 &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \binom{n+1}{k} \frac{B_{n+1-k}}{n+1} \sum_{l=0}^{k} \binom{k}{l} G_{k-l} \int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) \\ &= \sum_{\substack{k=0 \\ k\neq n}}^{n+1} \sum_{l=0}^{k} \binom{n+1}{k} \binom{k}{l} \frac{B_{n+1-k} G_{k-l} G_{l+1}}{(n+1)(l+1)}. \end{aligned} \tag{3.17}$$

Therefore, by (3.16) and (3.17), we obtain the following theorem.

Theorem 3.2. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}B_{n-l}\frac{G_{l+1}}{l+1} = \sum_{\substack{k=0\\ k\neq n}}^{n+1}\sum_{l=0}^{k}\binom{n+1}{k}\binom{k}{l}\frac{B_{n+1-k}G_{k-l}G_{l+1}}{(n+1)(l+1)}. \tag{3.18}$$

Let us consider the following p-adic integral on $\mathbb{Z}_p$:

$$K_3 = \int_{\mathbb{Z}_p} G_n(x)d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}G_{n-l}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x) = \sum_{l=0}^{n}\binom{n}{l}G_{n-l}\frac{G_{l+1}}{l+1}. \tag{3.19}$$

From Theorem 2.2, one has

$$\begin{aligned} K_3 &= -2\sum_{l=1}^{n}\binom{n}{l}\frac{G_{l+1}}{l+1}\sum_{k=0}^{n-l}\binom{n-l}{k}B_{n-l-k}\int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) \\ &= -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}B_{n-l-k}\frac{G_{l+1}G_{k+1}}{(l+1)(k+1)}. \end{aligned} \tag{3.20}$$

Therefore, by (3.19) and (3.20), we obtain the following theorem.

Theorem 3.3. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}\frac{G_{n-l}G_{l+1}}{l+1} = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{B_{n-l-k}G_{l+1}G_{k+1}}{(l+1)(k+1)}. \tag{3.21}$$

Now, one sets

$$K_4 = \int_{\mathbb{Z}_p} G_n(x)d\mu(x) = \sum_{l=0}^{n}\binom{n}{l}G_{n-l}B_l. \tag{3.22}$$

By Theorem 2.2, one gets

$$K_4 = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{G_{l+1}}{l+1}B_{n-l-k}B_k. \tag{3.23}$$

Therefore, by (3.22) and (3.23), we obtain the following corollary.

Corollary 3.4. *For $n \in \mathbb{Z}_+$, one has*

$$\sum_{l=0}^{n}\binom{n}{l}G_{n-l}B_l = -2\sum_{l=1}^{n}\sum_{k=0}^{n-l}\binom{n}{l}\binom{n-l}{k}\frac{G_{l+1}B_{n-l-k}B_k}{l+1}. \tag{3.24}$$

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