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On the analogue of Bernoulli polynomials

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Abstract In this paper we define the analogue of Bernoulli polynomials. We investigate some properties of the analogue of Bernoulli polynomials. Furthermore, some new relations, related to Bernoulli numbers and Euler numbers, are given. Finally, we consider the reflection symmetries of the analogue of Bernoulli polynomials.

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1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we

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normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the below notation:

$$[x] = [x : q] = \frac{1 - q^x}{1 - q}.$$

Hence, $\lim_{q \to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$\begin{aligned}
&X = \varprojlim_{N} (\mathbb{Z}/dp^N\mathbb{Z}), \\
&X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, \\
&a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \quad \text{cf. [1-4]}
\end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, set

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]}$$

and this is known to be a distribution on X due to Kim [4, 5].

Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable functions on $\mathbb{Z}_p$. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1 \text{ for some } N \geq 0\}$ is the cyclic group of order p^N. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto w^x$ (see [3, 6, 14]). Then ϕ_w has continuation to a continuous group homomorphism. For $f \in UD(\mathbb{Z}_p)$, the Kim's p-adic q-integral is defined by

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p]_q^N} \sum_{x=0}^{p^N - 1} f(x) q^x, \text{ see [2, 3, 4, 5] }.$$

Now we consider $I_1(f) = \lim_{q \to 1} I_q(f)$. From this, we can derive the below

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N - 1} f(x), \text{ see [4, 6, 14] }.$$

From the above definition, we can also derive $I_1(f_1) = I_1(f) + f'(0)$, where $f_1(x) = f(x+1)$ (see [2, 4, 6, 14]. By using $I_1(f)$-integral, many authors are studied the analogs of Bernoulli numbers and polynomials, cf.[1, 6, 7, 8, 10, 11, 14]. The remainder of the paper is organized as follows: In Section 2, we define the analogue of Bernoulli polynomials. We investigate some properties of the analogue of Bernoulli polynomials. In Section 3, we consider the reflection symmetries of the analogue of Bernoulli polynomials.

2. The analogue of Bernoulli numbers and polynomials

The purpose of this section is to introduce the analogue of Bernoulli numbers and polynomials. By using these numbers, we will give relations between Bernoulli numbers and Euler numbers. First, we start from the definition of the analogue of Bernoulli numbers as follows:

$$\frac{t}{we^t-1}=\sum_{n=0}^{\infty}B_n(w)\frac{t^n}{n!},\quad w\in T_p. \tag{1}$$

where $B_n(w)$ are called analogue of nth Bernoulli numbers. Since $I_1(f_1)=I_1(f)+f'(0)$, if we take $f(x)=e^{tx}w^x$, we easily see that

$$I_1(w^xe^{xt})=\frac{t}{we^t-1}.$$

Hence we have

$$\int_{\mathbb{Z}_p}w^xx^nd\mu_1(x)=B_n(w).$$

Now we define the analogue of Bernoulli polynomials $B_n(w,x)$ as

$$e^{xt}\frac{t}{we^t-1}=\sum_{n=0}^{\infty}B_n(w,x)\frac{t^n}{n!}. \tag{2}$$

By (1) and (2), it is not difficult to see that

$$B_n(w,x)=\sum_{l=0}^{n}\binom{n}{l}B_l(w)x^{n-l}.$$

By (2), we also have

$$\int_{\mathbb{Z}_p}w^t(x+t)^nd\mu_1(t)=B_n(w,x). \tag{3}$$

Let u be algebraic in complex number field. Then Frobenius-Euler numbers are defined by

$$e^{H(u)t} = \frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}, \tag{4}$$

This relation can be written as

$$H_0(u) = 1, \quad (H(u)+1)^k - uH_k(u) = 0 \quad (1 \le k).$$

Therefore we have

$$uH_k(u) = \sum_{i=0}^{k} \binom{k}{i} H_i(u), H_k(u) = \frac{1}{u-1}\sum_{i=0}^{k-1}\binom{k}{i}H_i(u), \text{ for } u \neq 1.$$

By (3) and (4), we give a interesting formula on relationship between the $B_n(w)$ and $H_n(w)$. Since

$$\frac{t}{we^t-1} = \frac{t}{w}\frac{1}{e^t-w^{-1}} = \frac{t}{w}\frac{1}{1-w^{-1}}\frac{1-w^{-1}}{e^t-w^{-1}} = \frac{t}{w-1}\frac{1-w^{-1}}{e^t-w^{-1}},$$

we have

$$\begin{aligned}\frac{1}{w-1}\sum_{n=0}^{\infty} H_n(w^{-1})\frac{t^n}{n!} &= \frac{1}{t}\sum_{n=0}^{\infty} B_n(w)\frac{t^n}{n!} = \frac{1}{t}\sum_{n=1}^{\infty} B_n(w)\frac{t^n}{n!} \\ &= \frac{1}{t}\sum_{n=0}^{\infty} B_{n+1}(w)\frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}\frac{B_{n+1}(w)}{n+1}\frac{t^n}{n!}.\end{aligned}$$

Hence we have the following theorem.

Theorem 1. For $n \ge 1$, we have

$$(1)\ B_n(w) = \frac{n}{w-1}H_{n-1}(w^{-1}), w \neq 1,$$

$$(2)\ B_n(w) = I_1(\phi_w(x)x^n),$$

$$(3)\ B_n(w) = \frac{1}{n+1}\lim_{N\to\infty}\frac{1}{Cp^N}\sum_{x=0}^{Cp^N-1} w^x x^{n+1}.$$

In [6, 14], the I_1-integral transform of f is the function $\widehat{f}: T_p \to \mathbb{C}_p$ defined by

$$\widehat{f}(w) = I_1(f\phi_w) \text{ for all } w \in T_p, f \in UD(\mathbb{Z}_p).$$

Now, we consider the I_q-integral transform by using p-adic q-integral on $\mathbb{Z}_p$ for a variable $q \in \mathbb{C}_p$ (see [7]) . For $f \in UD(\mathbb{Z}_p)$ the p-adic q-integral was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} f(x) q^x, \text{ cf. [6] .}$$

By simple calculation, we have

$$\begin{aligned} &\lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} f(y) w^y q^y \qquad (5) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{y=0}^{p^N-1} f(y) q^y \sum_{x=0}^{p^N-1} w^{y-x} = f(x) q^x = \phi_q(x) f(x), \text{ see [8] .} \end{aligned}$$

Since $I_1(f\phi_{wq}) = \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x)$, we also have

$$\lim_{N\to\infty} \sum_{x=0}^{p^N-1} w^{-x} \int_{\mathbb{Z}_p} f(x) w^x q^x d\mu_1(x) = \lim_{N\to\infty} \sum_{w\in C_{p^N}} \phi_{w^{-1}} I_q(f\phi_{qw}) = \sum_{w\in T_p} \phi_{w^{-1}} I_1(f\phi_{qw}). \qquad (6)$$

By (5), (6), we obtain

$$\frac{\log q}{q-1} \sum_{w\in T_p} \phi_{w^{-1}} \frac{q-1}{\log q} I_1(f\phi_{qw}) = \frac{\log q}{q-1} \sum_{w\in T_p} \phi_{w^{-1}} I_q(f\phi_w) = \phi_q(x) f(x).$$

Therefore, we obtain the following I_q-integral transform.

Theorem 2. For $f \in UD(\mathbb{Z}_p), w \in T_p$, we have [7]

$$\widehat{f}(qw) = \sum_{w\in T_p} I_q(f\phi_w)\phi_{w^{-1}} = \frac{q-1}{\log q} \phi_q(x) f(x).$$

Now we introduce the convolution for any $f, g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$ due to Woodcock as follows [14] :

$$(f \otimes g)(n) = \sum_{k=0}^{n} f(k)g(n-k), n \geq 0.$$

$$(f * g)(x) = \sum_{w} \widehat{f}_w \widehat{g}_w \phi_{w^{-1}}(x),$$

where the Fourier transform $\widehat{f}_w = I_1(f\phi_q)$. From Kim and Woodcock [4, 6, 7, 14], we have

$$\Delta^{n+1}(f \otimes g)(x) = (f \otimes \Delta^{n+1}g)(x) = \sum_{j=0}^{n} \Delta^j f(x+1)\Delta^{n-j}g(0).$$

If $g(0) = 0$, then we obtain

$$\Delta(f \otimes g)(x) = (f \otimes \Delta g)(x).$$

Since $I_1(f_1) = I_1(f) + f'(0)$, we have

$$I_1(\Delta f) = f'(0).$$

Hence we obtain

$$I_1(\Delta(f \otimes g))(x)) = I_1(f \otimes \Delta g)(x) = (f \otimes g)'(0).$$

On the other hand, Woodcock [8] introduced the following results.

$$(f \otimes g)' = (f \otimes g') + (f' \otimes g) + f * g,$$

$$(f * g)(z) = I_1(f(x)g(z-x)) - (f \otimes g').$$

By definition, we have $(f \otimes g)(0) = f(0)g(0)$. Hence

$$\begin{aligned}(f \otimes g)'(0) &= (f \otimes g')(0) + f'(0) \otimes g(0) + (f * g)(0)\\ &= f(0)g'(0) + f'(0) + (f * g)(0)\\ &= f(0)g'(0) + (f * g)(0).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned} I_1(f \otimes \Delta g) &= (f \otimes g)'(0) \\ &= f(0)g'(0) + (f * g)(0) \\ &= f(0)g'(0) + I_1(fg_-) - (f * g')(0) \\ &= f(0)g'(0) - f(0)g'(0) + I_1(fg_-) \\ &= I_1(fg_-), \end{aligned}$$

where $g_-(x) = g(-x)$. For $w \in T_p$, let $f = z^m \phi_w(z), g = z^n$. Then we have

$$\begin{aligned} I_1(f \otimes \Delta g)(z) &= I_1(z^m \phi_w(z)(-z)^n) \\ &= (-1)^n I_1(z^{m+n} \phi_w(z)) \\ &= (-1)^n B_{n+m}(w). \end{aligned}$$

Since $I_1(\phi_w(x)) = 0$ and

$$e^{tx} = \lim_{N \to \infty} \sum_{w \in C_{p^N}} \frac{t\phi_w(x)}{we^t - 1} = \sum_{n=0}^{\infty} \lim_{N \to \infty} \sum_{w \in C_{p^N}} I_1(x^n \phi_w(x)) \phi_w(x),$$

we obtain

$$x^n = B_n(1) + \sum_{w \in T_p, w \neq 1} \frac{1}{w-1} H_{n-1}(w^{-1}) \phi_w(x).$$

Therefore we have the following theorem.

Theorem 3. For $m, n \geq 1$, we have

$$(1)\ (-1)^n B_{m+n}(w) = \sum_{k=0}^{n-1} \binom{n}{k} I_1(z^m \phi_w(z) \otimes z^k),$$

$$\begin{aligned} (2)\ x^n &= B_n(1) + \sum_{w \in T_p, w \neq 1} \frac{1}{w-1} H_{n-1}(w^{-1}) \phi_w(x) \\ &= B_n(1) + \sum_{w \in T_p, w \neq 1} \frac{B_n(w)}{n} \phi_w(x). \end{aligned}$$

3. The reflection symmetries of the analogue of Bernoulli polynomials

In this section we consider the reflection symmetries of the analogue of Bernoulli polynomials. Let $\mathbb{R}$ be the field of real numbers and let w be the p^N-th root of unity. For $x \in \mathbb{R}$, we consider the Bernoulli polynomials $B_n(x)$ as follows:

$$F(t,x) = \frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \text{ see [9-13] }.$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty} B_n(1-x)\frac{(-t)^n}{n!} &= F(-t, 1-x) \\ &= \frac{-t}{e^{-t}-1}e^{(1-x)(-t)} \\ &= \frac{t}{e^t-1}e^{xt} \\ &= F(t,x) = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},\end{aligned}$$

we obtain that

$$B_n(1-x) = (-1)^n B_n(x). \tag{7}$$

Hence $B_n(x), x \in \mathbb{C}$, has $Re(x) = 1/2$ reflection symmetry in addition to the usual $Im(x) = 0$ reflection symmetry analytic complex functions. What happens with the reflection symmetry (7), when one considers the analogue of Bernoulli polynomials ? We are going now to reflection at $1/2$ of x on the analogue of Bernoulli polynomials. Since

$$F_w(t,x) = \frac{t}{we^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(w,x)\frac{t^n}{n!},$$

by simple calculation, we have

$$\begin{aligned}F_{w^{-1}}(-t, 1-x) &= \frac{-t}{w^{-1}e^{-t}-1}e^{(1-x)(-t)} \\ &= \frac{-t}{w^{-1}e^{-t}-1}e^{(-t)}e^{xt} \\ &= w\frac{t}{we^t-1}e^{xt} \\ &= wF_w(t,x).\end{aligned}$$

Hence we obtain the following theorem.

Theorem 4. For $n \geq 0$, we have

$$B_n(w, x) = (-1)^n w^{-1} B_n(w^{-1}, 1 - x). \tag{8}$$

We have the following corollary.

Corollary 5. If $B_n(w, x) = 0$, then $B_n(w^{-1}, 1 - x) = 0$.

Finally, we shall consider the more general problems. Prove or disprove: Since n is the degree of the polynomial $B_n(w, x)$, the number of real zeros $re_{B_n(w,x)}$ lying on the real plane $Im(x) = 0$ is then $re_{B_n(w,x)} = n - c_{B_n(w,x)}$, where $c_{B_n(w,x)}$ denotes complex zeros. In general, how many roots does $B_n(w, x)$ have ? Find the numbers of complex zeros $c_{B_n(w,x)}$ of the $B_n(w, x), Im(x) \neq 0$. Using numerical experiments, we hope to investigate the structure of the complex roots of the analogue of Bernoulli polynomials $B_n(w, x)$. For related topics the interested reader is referred to [9]. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the analogue of Bernoulli polynomials $B_n(w, x)$ to appear in mathematics and physics.

References

[1] M. Cenkci, M. Can, V. Kurt, 'p-adic interpolation functions and Kummer type congruences for q-twisted and q-generalized twisted Euler numbers', *Advan. Stud. Contemp. Math.*, **9**, 203-216 (2004).

[2] T. Kim, 'Non-Arichimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials', *Russ. J. Math. Phys.*, **10**, 91-98 (2003).

[3] T. Kim et al, 'Introduction to non-archimedean analysis ', *Kyowoo Publ. Company*, 2004.

[4] T. Kim, 'On a q-analogue of the p-adic log gamma functions and related integrals', *J. Number Theory*,**76**, 320-329 (1999).

[5] T. Kim, 'q-Volkenborn integration', *Russ. J. Math. Phys.*, **9**, 288-299 (2002).

[6] T. Kim, 'An analogue of Bernoulli numbers and their congruences', *Rep. Fac. Sci. Engrg. Saga Univ. Math.*,**22**, 7-13 (1994).

[7] T. Kim, 'A note on the Fourier transform of p-adic q-integrals', *arXiv: math.NT/0511573(23 Nov. 2005)*, 1-4 (2005).

[8] N. Koblitz, 'A new proof of certain formulas for p-adic L-function ', *Duke Math. J.*,**46**, 455-468 (1979).

[9] C. S. Ryoo, H. song, R. P. Argawal , 'On the real roots of the Changhee-Barnes' q-Bernoulli polynomials', *Advan. Stud. Contemp. Math.*, **9**, 153-163, (2004).

[10] K. Shiratani, S. Yamamoto , 'On a p-adic interpolation of Euler numbers and its derivatives', *Mem. Fac. Sci. Kyushu Univ.*, **39**, 113-125, (1985).

[11] Y. Simsek , 'Theorems on Twisted L-function and Twisted Bernoulli Numbers', *Advan. Stud. Contemp. Math.*, **11**, 205-218, (2005).

[12] P. G. Todorov , 'On the theory of the Bernoulli polynomials and numbers', *J. Math. Anal. Appl.*, **104**, 175-180, (1984).

[13] E. T. Whittaker and G. N. Waston , 'A Course of Morden Analysis', *Cambridge Univ. Press*, 1963.

[14] C. F. Woodcock, 'Convolutions on the ring of p-adic integers', *J. London Math. Soc.* **20**(1979), 101-108.