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GENERALIZED FIBONACCI SEQUENCES AND LINEAR RECURRENCES

Abstract. We analyze the existing relations among particular classes of generalized Fibonacci sequences, with characteristic polynomial having two integer roots. Several examples are given. Some properties concerning in general linear recurrences are also shown.

1. Introduction

There exists a very wide literature about the generalized Fibonacci sequences: one can see about it refs. [2], where also interesting applications to number theory are shown, and [1], where such sequences are treated as a particular case of a more general class of sequences of numbers. In this paper, after a brief introduction to the generalized Fibonacci sequences, we study the properties of some sequences whose characteristic polynomial has two different integers as roots, giving several examples for clarifying the theoretical results. Denoting by $\{U_n(h, k)\}_{n\in\mathbb{N}}$ (where $h, k \in \mathbb{Z}$, $k(h^2 - 4k) \neq 0$) the linear recursive sequence with characteristic polynomial $x^2 - hx + k$, in particular we find interesting relations among the sequences $\{U_n(c - a, -ca)\}_{n\in\mathbb{N}}$, $\{U_n(c + a, ca)\}_{n\in\mathbb{N}}$ and $\{U_n(c^2 + a^2, c^2a^2)\}_{n\in\mathbb{N}}$ (where $a, c \in \mathbb{N}$ and $a < c$) and we also see a difference between the role played in this context by the terms $U_{2n}(c \pm a, \pm ca)$, $n \in \mathbb{N} \cup \{0\}$ and the role played by the terms $U_{2n+1}(c \pm a, \pm ca)$, $n \in \mathbb{N} \cup \{0\}$. Then we show a connection between the sequences $\{U_n(c + a, ca)\}_{n\in\mathbb{N}}$ and the sequences $\{U_n(c^t + a^t, c^t a^t)\}_{n\in\mathbb{N}}$ for generic $t \in \mathbb{N}$.

Finally, we review some classical properties of the class of linear recurrences, which includes the class of the generalized Fibonacci sequences.

2. Generalized Fibonacci sequences

For each pair (h, k), $h, k \in \mathbb{C}$ of complex numbers such that $k(h^2 - 4k) \neq 0$, we denote by $\{U_n(h, k)\}_{n\in\mathbb{N}}$ the generalized Fibonacci sequence defined as follows:

$$\forall n \in \mathbb{N},\ n \geq 2,\ U_n(h, k) = hU_{n-1}(h, k) - kU_{n-2}(h, k),$$

$$U_0(h, k) = 0,\ U_1(h, k) = 1.$$

An explicit expression of the nth term of $\{U_n(h, k)\}_{n\in\mathbb{N}}$ for generic $n \in \mathbb{N}\cup\{0\}$ is given by the Binet formula $U_n(h, k) = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = \dfrac{h + \sqrt{h^2 - 4k}}{2}$ and $\beta = \dfrac{h - \sqrt{h^2 - 4k}}{2}$ are the distinct roots of the polynomial $x^2 - hx + k \in \mathbb{C}[x]$, called characteristic polynomial of the sequence. Some examples of these sequences, corresponding to fixed values of h and k, are:

$$\{U_n(5,\ 2)\}_{n\in\mathbb{N}} \quad : \ 0,\ 1,\ 5,\ 23,\ 105,\ 479,\ 2185,\ 9967,\ \ldots$$

$$\{U_n(4,\ -12)\}_{n\in\mathbb{N}} \quad : \ 0,\ 1,\ 4,\ 28,\ 160,\ 976,\ 5824,\ 35008,\ \ldots$$

$$\left\{U_n\left(\frac{2}{3},\ \frac{1}{2}\right)\right\}_{n\in\mathbb{N}} \quad : \ 0,\ 1,\ \frac{2}{3},\ -\frac{1}{18},\ -\frac{10}{27},\ -\frac{71}{324},\ \frac{19}{486},\ \frac{791}{5832},\ \ldots$$

$$\{U_n(i,\ 2)\}_{n\in\mathbb{N}} \quad : \ 0,\ 1,\ i,\ -3,\ -5i,\ 11,\ 21i,\ -43,\ \ldots$$

$$\{U_n(\sqrt{2}+\sqrt{3},\ \sqrt{6})\}_{n\in\mathbb{N}} \quad : \ 0,\ 1,\ \sqrt{2}+\sqrt{3},\ 5+\sqrt{6},\ 5(\sqrt{2}+\sqrt{3}),\ 19+5\sqrt{6},\ 19(\sqrt{2}+\sqrt{3}),\ 65+19\sqrt{6},\ \ldots$$

PROPOSITION 1. *For generic $h,\ k,\ \delta\ \in\ \mathbb{C}$ with $\delta k(h^2-4k)\neq 0$, we have $\forall\, n\in\mathbb{N}\cup\{0\}$, $U_n(\delta h,\ \delta^2 k)=\delta^{n-1}U_n(h,\ k)$.*

Proof. We first observe that, $h,\ k,\ \delta$ being fixed, we have $(\delta h)^2-4(\delta^2 k)=\delta^2(h^2-4k)\neq 0$. Then we can consider the generalized Fibonacci sequence $\{U_n(\delta h,\ \delta^2 k)\}_{n\in\mathbb{N}}$. Clearly the assertion of the proposition holds when $n=0,\ 1$. Now for generic $m\in\mathbb{N}, m\geq 2$ let us suppose that it holds when $n=m-1,\ m-2$; we assert that it is satisfied even when $n=m$. Indeed from the definition of generalized Fibonacci sequence we have:

$$\begin{aligned}
U_m(\delta h,\ \delta^2 k) &= \delta h\, U_{m-1}(\delta h,\ \delta^2 k)-\delta^2 k\, U_{m-2}(\delta h,\ \delta^2 k)\\
&= \delta h\,(\delta^{m-2}U_{m-1}(h,k))-\delta^2 k(\delta^{m-3}U_{m-2}(h,\ k))\\
&= \delta^{m-1}(hU_{m-1}(h,k)-kU_{m-2}(h,k))\\
&= \delta^{m-1}U_m(h,\ k).
\end{aligned}$$

The proposition is then proven by induction. □

COROLLARY 1. *For generic $h,\ k\ \in\ \mathbb{R}$ such that $k(h^2-4k)\neq 0$, we have $\forall\, n\in\mathbb{N}\cup\{0\}$, $U_n(ih,\ -k)=i^{n-1}U_n(h,\ k)$.*

Now let us suppose we are given $z,\ w\in\mathbb{C}$ with $zw(z^2-4w)\neq 0$ and $w/z^2\in\mathbb{R}$. Let us set $h=|z|$, $k=w|z|^2/z^2$ and $\delta=z/|z|$. Then $h,\ k\in\mathbb{R}$, $\delta k(h^2-4k)\neq 0$ and from Proposition 1 we deduce that $\forall\, n\in\mathbb{N}\cup\{0\}$, $U_n(\delta h,\ \delta^2 k)=\delta^{n-1}U_n(h,\ k)$, i.e. that $\forall\, n\in\mathbb{N}\cup\{0\}$, $U_n(z,\ w)=(z/|z|)^{n-1}U_n(h,\ k)$. This implies that $\forall\, n\in\mathbb{N}\cup\{0\}$, $|U_n(z,\ w)|=|U_n(h,\ k)|$. Moreover, we can write $\forall\, n\in\mathbb{N}\cup\{0\}$,

$$U_n(z,w)=U_n(h,k)[\cos((n-1)arg(z))+i\sin((n-1)arg(z))].$$

3. The sequences $\{U_n(c-a,\,-ca)\}_{n\in\mathbb{N}}$ and $\{U_n(c+a,\,ca)\}_{n\in\mathbb{N}}$ ($c, a \in \mathbb{N},\ a < c$)

Let two positive integers a, c be fixed with $a < c$, and let us consider the generalized Fibonacci sequences $\{U_n(c-a,\,-ca)\}_{n\in\mathbb{N}}$, $\{U_n(c+a,\,ca)\}_{n\in\mathbb{N}}$ and $\{U_n(c^2+a^2,\,c^2a^2)\}_{n\in\mathbb{N}}$. By using the Binet formulas for the terms of the first and the third sequence, we can easily obtain:

$$\forall n \in \mathbb{N}\cup\{0\},\ U_n(c^2+a^2,\,c^2a^2) = \frac{c^{2n}-a^{2n}}{c^2-a^2} = \frac{1}{c-a}\cdot\frac{c^{2n}-(-a)^{2n}}{c+a}$$

$$= \frac{1}{c-a}U_{2n}(c-a,\,-ca). \tag{1}$$

By using the Binet formulas for the terms of the second and the third of the same Fibonacci sequences, we obtain:

$$\forall n \in \mathbb{N}\cup\{0\},\ U_n(c^2+a^2,\,c^2a^2) = \frac{c^{2n}-a^{2n}}{c^2-a^2} = \frac{1}{c+a}\cdot\frac{c^{2n}-a^{2n}}{c-a}$$

$$= \frac{1}{c+a}U_{2n}(c+a,\,ca). \tag{2}$$

From equalities (1) and (2) we can also deduce that

$$\frac{U_{2n}(c-a,\,-ca)}{c-a} = \frac{U_{2n}(c+a,\,ca)}{c+a};$$

then $(c+a)U_{2n}(c-a,\,-ca) - (c-a)U_{2n}(c+a,\,ca) = 0$.

EXAMPLE 1. Let us take $c = 5$, $a = 2$ and consider the three following sequences:

$\{U_n(3,\,-10)\}_{n\in\mathbb{N}}$: 0, 1, 3, 19, 87, 451, 2223, 11179, 55767, ...

$\{U_n(7,\,10)\}_{n\in\mathbb{N}}$: 0, 1, 7, 39, 203, 1031, 5187, 25999, 130123, ...

$\{U_n(29,\,100)\}_{n\in\mathbb{N}}$: 0, 1, 29, 741, 18589, 464981, 11625549, 290642821, ...

We verify equalities (1) and (2) for $n = 1, 2, 3, 4$: $3U_1(29,\,100) = 3\cdot 1 = 3 = U_2(3,\,-10)$; $3U_2(29,\,100) = 3\cdot 29 = 87 = U_4(3,\,-10)$; $3U_3(29,\,100) = 3\cdot 741 = 2223 = U_6(3,\,-10)$; $3U_4(29,\,100) = 3\cdot 18589 = 55767 = U_8(3,\,-10)$; $7U_1(29,\,100) = 7\cdot 1 = 7 = U_2(7,\,10)$; $7U_2(29,\,100) = 7\cdot 29 = 203 = U_4(7,\,10)$; $7U_3(29,\,100) = 7\cdot 741 = 5187 = U_6(7,\,10)$; $7U_4(29,\,100) = 7\cdot 18589 = 130123 = U_8(7,\,10)$. From such equalities we immediately deduce, for $n = 1, 2, 3, 4$, that $7U_{2n}(3,\,-10) - 3U_{2n}(7,\,10) = 0$.

For generic a fixed in $\mathbb{N}$, by posing $c = a + 1$ in equalities (1) we deduce that:

$$\forall\, n \in \mathbb{N} \cup \{0\},\ U_n(a^2 + (a+1)^2,\ [a(a+1)]^2) = U_{2n}(1,\ -a(a+1)).$$

EXAMPLE 2. Let us take $a = 2$, $c = 3$ and consider the two following sequences:

$$\{U_n(1,\ -6)\}_{n\in\mathbb{N}} \quad : 0,\ 1,\ 1,\ 7,\ 13,\ 55,\ 133,\ 463,\ 1261,\ \dots$$

$$\{U_n(13,\ 36)\}_{n\in\mathbb{N}} \quad : 0,\ 1,\ 13,\ 133,\ 1261,\ 11605,\ 105469,\ 953317,\ \dots$$

We can clearly see that $U_1(13,\ 36) = U_2(1,\ -6)$, $U_2(13,\ 36) = U_4(1,\ -6)$, $U_3(13,\ 36) = U_6(1,\ -6)$ and $U_4(13,\ 36) = U_8(1,\ -6)$.

Supposing once more $a,\ c$ fixed in $\mathbb{N}$ with $a < c$, the following equalities are verified:

$$\begin{aligned} &\forall n \in \mathbb{N} \cup \{0\},\ (c+a)U_{2n+1}(c-a,\ -ca) - (c-a)U_{2n+1}(c+a,\ ca) \\ &= (c+a)\frac{c^{2n+1} - (-a)^{2n+1}}{c+a} - (c-a)\frac{c^{2n+1} - a^{2n+1}}{c-a} \\ &= (c^{2n+1} + a^{2n+1}) - (c^{2n+1} - a^{2n+1}) = 2a^{2n+1}. \end{aligned}$$

We summarize in the following theorem two important results obtained in this section.

THEOREM 1. *Let $a,\ c \in \mathbb{N}$ be assigned with $a < c$. Then for all $n \in \mathbb{N} \cup \{0\}$ we have:*

$$(c+a)U_{2n}(c-a,\ -ca) - (c-a)U_{2n}(c+a,\ ca) = 0$$

and

$$(c+a)U_{2n+1}(c-a,\ -ca) - (c-a)U_{2n+1}(c+a,\ ca) = 2a^{2n+1}.$$

The second part of Theorem 1 gives rise to the following corollary.

COROLLARY 2. *Let $c \in \mathbb{N}$, $c \geq 2$ be fixed. We have $(c+1)U_{2n+1}(c-1,\ -c) - (c-1)U_{2n+1}(c+1,\ c) = 2$ for all $n \in \mathbb{N} \cup \{0\}$ (in fact these equalities hold even when $c = 1$). Moreover, if $c \geq 3$ and we fix $a \in \mathbb{N}$ with $2 \leq a < c$, we have $\lim_{n\to+\infty} [(c+a)U_{2n+1}(c-a,\ -ca) - (c-a)U_{2n+1}(c+a,\ ca)] = +\infty$.*

The following fact is verified: when $2 \leq a < c$, for all even n we have $(c+a)U_n(c-a,\ -ca) - (c-a)U_n(c+a,\ ca) = 0$, while the terms of the succession of the numbers in the form $(c+a)U_n(c-a,\ -ca) - (c-a)U_n(c+a,\ ca)$ with n odd tend to infinity as $n \to +\infty$: they actually grow exponentially with respect to n.

EXAMPLE 3. By taking once more $c = 5$ and $a = 2$, we verify the second part of Theorem 1 for $n = 0,\ 1,\ 2,\ 3$. We have: $7U_1(3,\ -10) - 3U_1(7,\ 10) = 7 \cdot 1 - 3 \cdot 1 = 4 = 2 \cdot 2^1$; $7U_3(3,\ -10) - 3U_3(7,\ 10) = 7 \cdot 19 - 3 \cdot 39 = 16 = 2 \cdot 2^3$; $7U_5(3,\ -10) - 3U_5(7,\ 10) = 7 \cdot 451 - 3 \cdot 1031 = 64 = 2 \cdot 2^5$; $7U_7(3,\ -10) - 3U_7(7,\ 10) = 7 \cdot 11179 - 3 \cdot 25999 = 256 = 2 \cdot 2^7$.

EXAMPLE 4. Let us fix $c = 6$, $a = 1$ and consider the two following sequences:

$\{U_n(5, -6)\}_{n\in\mathbb{N}}$: 0, 1, 5, 31, 185, 1111, 6665, 39991, 239945, ...

$\{U_n(7, 6)\}_{n\in\mathbb{N}}$: 0, 1, 7, 43, 259, 1555, 9331, 55987, 335923, ...

We have: $7U_1(5, -6) - 5U_1(7, 6) = 7 \cdot 1 - 5 \cdot 1 = 2$; $7U_2(5, -6) - 5U_2(7, 6) = 7 \cdot 5 - 5 \cdot 7 = 0$; $7U_3(5, -6) - 5U_3(7, 6) = 7 \cdot 31 - 5 \cdot 43 = 2$; $7U_4(5, -6) - 5U_4(7, 6) = 7 \cdot 185 - 5 \cdot 259 = 0$; $7U_5(5, -6) - 5U_5(7, 6) = 7 \cdot 1111 - 5 \cdot 1555 = 2$; $7U_6(5, -6) - 5U_6(7, 6) = 7 \cdot 6665 - 5 \cdot 9331 = 0$; $7U_7(5, -6) - 5U_7(7, 6) = 7 \cdot 39991 - 5 \cdot 55987 = 2$; $7U_8(5, -6) - 5U_8(7, 6) = 7 \cdot 239945 - 5 \cdot 335923 = 0$.

EXAMPLE 5. Let us fix $c = 5$, $a = 3$ and consider the two following sequences:

$\{U_n(2, -15)\}_{n\in\mathbb{N}}$: 0, 1, 2, 19, 68, 421, 1862, 10039, 48008, ...

$\{U_n(8, 15)\}_{n\in\mathbb{N}}$: 0, 1, 8, 49, 272, 1441, 7448, 37969, 192032, ...

Now for $n = 1, 2, \ldots, 8$ we calculate the value $\frac{c+a}{2}U_n(c-a, -ca) - \frac{c-a}{2}U_n(c+a, ca)$. We obtain: $4U_1(2, -15) - U_1(8, 15) = 4 \cdot 1 - 1 = 3 = 3^1$; $4U_2(2, -15) - U_2(8, 15) = 4 \cdot 2 - 8 = 0$; $4U_3(2, -15) - U_3(8, 15) = 4 \cdot 19 - 49 = 27 = 3^3$; $4U_4(2, -15) - U_4(8, 15) = 4 \cdot 68 - 272 = 0$; $4U_5(2, -15) - U_5(8, 15) = 4 \cdot 421 - 1441 = 243 = 3^5$; $4U_6(2, -15) - U_6(8, 15) = 4 \cdot 1862 - 7448 = 0$; $4U_7(2, -15) - U_7(8, 15) = 4 \cdot 10039 - 37969 = 2187 = 3^7$; $4U_8(2, -15) - U_8(8, 15) = 4 \cdot 48008 - 192032 = 0$.

4. Relation between $\{U_n(c+a, ca)\}_{n\in\mathbb{N}}$ and $\{U_n(c^t + a^t, c^t a^t)\}_{n\in\mathbb{N}}$ ($c, a, t \in \mathbb{N}, a < c$)

Let us fix $a, c, t \in \mathbb{N}$ with $a < c$. Then by applying Binet formulas to the terms of the sequences $\{U_n(c+a, ca)\}_{n\in\mathbb{N}}$ and $\{U_n(c^t + a^t, c^t a^t\}_{n\in\mathbb{N}}$, we can write:

$$\forall n \in \mathbb{N} \cup \{0\},\ U_n(c^t + a^t, c^t a^t) = \frac{c^{tn} - a^{tn}}{c^t - a^t} = \frac{(c^{tn} - a^{tn})/(c-a)}{(c^t - a^t)/(c-a)} = \frac{U_{tn}(c+a, ca)}{U_t(c+a, ca)}.$$

EXAMPLE 6. Let us take $a = 1$, $c = 2$, $t = 3$ and consider the two following sequences:

$\{U_n(3, 2)\}_{n\in\mathbb{N}}$: 0, 1, 3, 7, 15, 31, 63, 127, 255, 511, ...

$\{U_n(9, 8)\}_{n\in\mathbb{N}}$: 0, 1, 9, 73, 585, 4681, 37449, 299593, 2396745, ...

We have $U_3(3, 2) = 7 = 7U_1(9, 8)$, $U_6(3, 2) = 63 = 7U_2(9, 8)$, $U_9(3, 2) = 511 = 7U_3(9, 8)$, and so on.

5. Linear recurrences

Let $k \in \mathbb{N}$ be fixed; for generic $c_1, c_2, \ldots, c_k \in \mathbb{C}$ with $c_k \neq 0$, let us set $p(x) = x^k - c_1x^{k-1} - c_2x^{k-2} - \ldots - c_k \in \mathbb{C}[x]$. Let $\beta \in \mathbb{C}$, $t \in \mathbb{N}$ be given such that $(x-\beta)^t | p(x)$. Then the following proposition holds.

PROPOSITION 2. *If we pose, for every* $n \in \mathbb{N}$, $A_n = \left[\prod_{i=0}^{t-2}(n-i)\right]\beta^n$ *(where the empty product is posed equal to* 1*), then for all* $n \in \mathbb{N}$, $n \geq k$ *the equality* $A_n = c_1A_{n-1} + c_2A_{n-2} + \ldots + c_kA_{n-k}$ *is satisfied.*

Proof. For each fixed $n \geq k$, let us consider the polynomial $f(x) = x^{n-k}p(x) = x^n - c_1x^{n-1} - c_2x^{n-2} - \ldots - c_kx^{n-k} \in \mathbb{C}[x]$. Since $(x-\beta)^t | f(x)$, β is a root of $f^{(t-1)}(x)$, and then

$$\left[\prod_{i=0}^{t-2}(n-i)\right]\beta^{n-(t-1)} = \sum_{j=1}^{k} c_j \left[\prod_{i=0}^{t-2}(n-j-i)\right]\beta^{n-j-(t-1)}.$$

By multiplying the two members of the latter equality by β^{t-1}, we obtain

$$\left[\prod_{i=0}^{t-2}(n-i)\right]\beta^{n} = \sum_{j=1}^{k} c_j \left[\prod_{i=0}^{t-2}(n-j-i)\right]\beta^{n-j},$$

i.e. $A_n = \sum_{j=1}^{k} c_jA_{n-j}$. □

THEOREM 2. *If* k, $c_1, c_2, \ldots, c_k$, $p(x)$, β, t *are given as above and we pose, for every* $n \in \mathbb{N}$, $B_n = n^{t-1}\beta^n$, *then for all* $n \geq k$ *we have* $B_n = c_1B_{n-1} + c_2B_{n-2} + \ldots + c_kB_{n-k}$.

Proof. When $t = 1$ *or* 2 the theorem is clearly equivalent to Proposition 2. Now after fixing $t \in \mathbb{N}$, $t \geq 3$ let us suppose the theorem to hold for any positive integer lower than t, and let us deduce its validity for the integer t. We can set $q_t(x) = \prod_{i=0}^{t-2}(x-i) = x^{t-1} + g_t(x) \in \mathbb{C}[x]$, with $\deg(g_t(x)) = t-2$. For each fixed $n \in \mathbb{N}$, $n \geq k$, from Proposition 2 we obtain the equality $q_t(n)\beta^n = \sum_{j=1}^{k} c_jq_t(n-j)\beta^{n-j}$, i.e.

$$[n^{t-1} + g_t(n)]\beta^n = \sum_{j=1}^{k} c_j[(n-j)^{t-1} + g_t(n-j)]\beta^{n-j}. \tag{3}$$

Since $\deg(g_t(x)) = t - 2$ and we have supposed that the theorem holds for all integers lower than t, the equality

$$g_t(n)\beta^n = \sum_{j=1}^{k} c_j g_t(n-j)\beta^{n-j} \tag{4}$$

is easily verified to be satisfied. Finally, from equalities (3) and (4) we obtain

$$n^{t-1}\beta^n = \sum_{j=1}^{k} c_j (n-j)^{t-1}\beta^{n-j},$$

which means $B_n = \sum_{j=1}^{k} c_j B_{n-j}$. The theorem is then proven by induction over t. □

From Theorem 2 one immediately derives the following classical result [3] on linear recurrences.

COROLLARY 3. *Let k, c_1, c_2, ..., c_k, $p(x)$ be assigned as above. Let us pose $p(x) = (x-\beta_1)^{t_1}(x-\beta_2)^{t_2}...(x-\beta_s)^{t_s}$ with β_1, β_2, ..., β_s distinct complex numbers, t_1, t_2, ..., $t_s \in \mathbb{N}$ and $t_1 + t_2 + ... + t_s = k$. Then the set of the sequences $\{B_n\}_{n\in\mathbb{N}}$ which satisfy for every $n \geq k$ the equality $B_n = c_1 B_{n-1} + c_2 B_{n-2} + ... + c_k B_{n-k}$ is exactly the set of the sequences*

$$n \mapsto \left[\left(\sum_{j=0}^{t_1-1} a_{1,j} n^j\right)\beta_1^n\right] + \left[\left(\sum_{j=0}^{t_2-1} a_{2,j} n^j\right)\beta_2^n\right] + ... + \left[\left(\sum_{j=0}^{t_s-1} a_{s,j} n^j\right)\beta_s^n\right],$$

where all numbers $a_{i,j}$ with $1 \leq i \leq s$ and $0 \leq j \leq t_i - 1$ are complex constants independent of n.

The following example shows another well known result [3], which is a particular case of Corollary 3.

EXAMPLE 7. Let us take a generic $\beta \in \mathbb{C}\backslash\{0\}$ and consider the sequence $\{B_n(\beta)\}_{n\in\mathbb{N}}$ defined as follows:

$$\forall n \in \mathbb{N},\ n \geq 2,\ B_n(\beta) = 2\beta B_{n-1}(\beta) - \beta^2 B_{n-2}(\beta),$$

$$B_0(\beta) = 0,\ B_1(\beta) = 1.$$

This is not a generalized Fibonacci sequence, because for $h = 2\beta$ and $k = \beta^2$ we have $h^2 - 4k = 0$. By posing $k = 2$, $c_1 = 2\beta$, $c_2 = -\beta^2$, $p(x) = x^2 - c_1 x - c_2 = (x-\beta)^2$ and using the notations of Corollary 3 we obtain $s = 1$, $t_1 = 2$, $\beta_1 = \beta$. It can easily be verified by induction that $\forall\, n \in \mathbb{N} \cup \{0\}$, $B_n(\beta) = n\beta^{n-1}$. This result is consistent with Corollary 3 and can also be obtained in the following way: for all $\alpha, \beta \in \mathbb{C}$

with $\alpha \neq \beta$, let us consider the generalized Fibonacci sequence $\{U_n(\alpha+\beta,\, \alpha\beta)\}_{n\in\mathbb{N}}$: we have $U_0(\alpha+\beta,\, \alpha\beta) = 0$, $U_1(\alpha+\beta,\, \alpha\beta) = 1$ and $\forall n \geq 2$, $U_n(\alpha+\beta,\, \alpha\beta) = (\alpha+\beta)U_{n-1}(\alpha+\beta,\, \alpha\beta) - \alpha\beta U_{n-2}(\alpha+\beta,\, \alpha\beta)$. From the Binet formulas we clearly have $\forall n \in \mathbb{N}\cup\{0\}$, $U_n(\alpha+\beta,\, \alpha\beta) = \dfrac{\alpha^n - \beta^n}{\alpha-\beta}$. From the definition of $\{B_n(\beta)\}_{n\in\mathbb{N}}$, for each fixed $n \in \mathbb{N}\cup\{0\}$ we can say that the term $B_n(\beta)$ is the limit for $\alpha \to \beta$ of the number (considered as a function of α) $U_n(\alpha+\beta,\, \alpha\beta)$, and then the following equalities hold:

$$B_n(\beta) = \lim_{\alpha\to\beta} U_n(\alpha+\beta,\, \alpha\beta) = \lim_{\alpha\to\beta} \frac{\alpha^n - \beta^n}{\alpha-\beta} = n\beta^{n-1}.$$

References

[1] HORADAM A. F., *Basic properties of a certain generalized sequence of numbers*, The Fibonacci Quarterly **3.3** (1965), 161–176.

[2] LUCAS E., *Théorie des fonctions numeriques simplement periodiques*, Amer. J. Math **1** (1878), 184–240, 289–321.

[3] ROSEN K. H., *Discrete mathematics and its applications*, McGraw-Hill 1999.

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