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LEGENDRE TRANSFORMS AND APÉRY'S SEQUENCES

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Abstract

This article studies particular sequences satisfying polynomial recurrences, among those Apéry's sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

which is shown to be the Legendre transform of the sequence

$$c_k = \sum_{j=0}^{k} \binom{k}{j}^3.$$

This results in the construction of simultaneous approximations of $\pi^2/8$ and $\zeta(3)$.

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1. Introduction

For a sequence (c_k) we shall consider its *Legendre transform* (a_n) defined by

$$a_n = \sum_{k=0}^{n} c_k \binom{n}{k} \binom{n+k}{k}.$$

It should be noticed that each sequence (a_n) is the Legendre transform of a unique sequence (c_k) (cf. Section 2). We shall also consider the sequence of *Legendre polynomials* belonging to (c_k) defined by

$$a_n(x) = \sum_{k=0}^{n} c_k \binom{n}{k} \binom{n+k}{k} x^k.$$

The classical Legendre polynomials orthogonal on $[-1, 0]$ belongs in this way to the sequence (c_k) with $c_k = 1$.

This article is motivated by the following conjecture (see [10]): For integral r, $r \geq 2$, numerical evidence indicates that each of the sequences

$$a_n^{(r)} = \sum_{k=0}^{n} \binom{n}{k}^r \binom{n+k}{k}^r$$

is the Legendre transform of an integral sequence $(c_k^{(r)})$. Challenged by this problem it was noticed by W. Deuber, W. Thumser and B. Voigt (University of Bielefeld) that the corresponding sequence (c_k) for $r = 2$ seemed to be

$$c_k = c_k^{(2)} = \sum_{j=0}^{k} \binom{k}{j}^3. \tag{1}$$

This was then proved independently by Strehl (University of Erlangen-Nürnberg, see [13]), and myself. Strehl obtained the more general formula

$$\sum_{k=0}^{n} \binom{n}{k}^2 \frac{\binom{\alpha+\beta+n+k}{k}^2}{\binom{\alpha+k}{k}\binom{\beta+k}{k}} = \sum_{k=0}^{n} \binom{n}{k} \frac{\binom{\alpha+\beta+n+k}{k}}{\binom{\beta+k}{k}} \sum_{j=0}^{k} \binom{k}{j}^2 \frac{\binom{\beta+k}{j}}{\binom{\alpha+j}{j}},$$

where α and β are parameters. The choice $\alpha = \beta = 0$ gives the formula (1) for (c_k) for $r = 2$. In [13] Strehl also proved that $c_k^{(3)}$ is integral by establishing the formula

$$c_k^{(3)} = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}^2 \binom{2j}{k-j}.$$

It is well known (see [1, 6]) that Apéry's sequence

$$a_n = a_n^{(2)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \tag{2}$$

satisfies the recurrence relation

$$(n+1)^3 a_{n+1} - \big((n+1)^3 + n^3 + 4(2n+1)^3\big) a_n + n^3 a_{n-1} = 0 \qquad \text{for } n \geq 0. \tag{3}$$

The sequence (1) has also long been known to satisfy the recurrence relation (see [1, 2, 3, 5, 6, 12])

$$(k+1)^2 c_{k+1} - (7k^2 + 7k + 2) c_k - 8k^2 c_{k-1} = 0 \qquad \text{for } k \geq 0. \tag{4}$$

After presenting some simple properties of the Legendre transform in Section 2, we consider in Section 3 a class of three term recurrent sequences (c_k) such that the corresponding sequence (a_n) is also three term recurrent. Simple examples of this kind are described in Section 4.

In Section 5 we consider the important recurrence (4) leading through Legendre transforms to Apéry's sequences related to $\zeta(3)$. In addition to obtaining the formula

$$a_n^{(2)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3$$

we also get simultaneous approximations of $\pi^2/8$ and $\zeta(3)$.

In Section 6 we consider the simpler sequence

$$a_n = \sum_{k=0}^{n} \binom{2k}{k} \binom{n}{k} \binom{n+k}{k}, \tag{5}$$

which is the Legendre transform of the sequence

$$c_k = \sum_{j=0}^{k} \binom{k}{j}^2 = \binom{2k}{k}. \tag{6}$$

This sequence is rather peculiar, namely

$a_0 = 1^2$, $a_1 = 5 \cdot 1^2$, $a_2 = 7^2$, $a_3 = 5 \cdot 11^2$, $a_4 = 91^2$,
$a_5 = 5 \cdot 155^2$, $a_6 = 1345^2$, $a_7 = 5 \cdot 2365^2$, $a_8 = 20995^2$, $a_9 = 5 \cdot 37555^2$,
$\ldots$.

This will be explained by means of some particular sequences of orthogonal polynomials.

The final section contains a number of computer-aided results of recurrent sequences (c_k) such that the corresponding sequence (a_n) is also recurrent. We propose to continue this investigation by extending the class of recurrent sequences (c_k) for which the corresponding sequence (a_n) of Legendre transforms is known to be recurrent (see also [9]). Such insight might also prove the conjecture about $a_n^{(r)}$ for values of $r \geq 3$.

2. Simple properties of Legendre transforms

We shall mention the following simple results:

(i) If (a_n) is the Legendre transform of (c_k) then the following *inversion formula* holds:

$$c_k = \sum_{j=0}^{k} (-1)^{k-j} \frac{2j+1}{k+j+1} \frac{\binom{k}{j}}{\binom{k+j}{j}} a_j.$$

(ii)

$$\sum_{k=0}^{n} \frac{(-1)^k}{k+1} \binom{n}{k} \binom{n+k}{k} = \begin{cases} 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

(iii) For $m \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} \frac{(-1)^k}{2k-2m+1} \binom{n}{k} \binom{n+k}{k} =$$
$$(-1)^m \frac{(2n+1-2(m-1)) \cdots (2n-1)(2n+1)(2n+3) \cdots (2n+1+2(m-1))}{(1 \cdot 3 \cdots (2m-1))^2}.$$

(iv) For $m \in \mathbb{N}$ we let

$$c_k^{(m)} = \begin{cases} 1 & k = m, \\ 0 & k \neq m. \end{cases}$$

Then obviously

$$\sum_{k=0}^{n} c_k^{(m)} \binom{n}{k} \binom{n+k}{k} = \frac{(n-m+1) \cdots (n-1)n(n+1) \cdots (n+m)}{(m!)^2}.$$

Since

$$\binom{n}{k} \binom{n+k}{k} = \binom{2k}{k} \binom{n+k}{n-k},$$

the relation (i) is an immediate consequence of the well-known relations for so-called *Legendre pairs* (cf. [8]):

$$a_n = \sum_k \binom{n+k}{n-k} b_k$$

if and only if

$$b_n = \sum_k (-1)^{n+k} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1} \right) a_k.$$

Notice that (ii) follows from (i), when applied to the sequence (a_n) with $a_0 = 1, a_n = 0$ for $n > 0$. For another derivation of (ii) see [9]. Notice also that the formulas in (iii) and (iv) together give the inverse Legendre transform of an arbitrary polynomial sequence (a_n).

3. Three term recurrences and Legendre transforms

We shall prove the following result:

THEOREM 1. *Let $A, B, C, D, E \in \mathbb{R}$, $C \neq 0$ be constants. We consider polynomials*

$$\begin{aligned} P_0(k) &= Ak^2 + Bk + C, \\ P_2(k) &= Dk^2, \\ Q_1(k) &= k(Ak + (B - A)), \\ P_1(k) &= Dk(k+1) - Q_1(k) + E, \end{aligned}$$

and polynomials

$$\begin{aligned} p_0(n) &= (n+1)P_0(n), \\ p_2(n) &= n(P_0(n) - (B - A)(2n+1)), \\ q_1(n) &= 2P_1(n) + 2Q_1(n) = 2Dn(n+1) + 2E, \\ p_1(n) &= p_0(n) + p_2(n) + (2n+1)q_1(n). \end{aligned}$$

(i) *Suppose the sequence (c_k) satisfies the recurrence*

$$P_0(k)c_{k+1} - P_1(k)c_k - P_2(k)c_{k-1} = 0 \qquad \text{for } k \geq 1 \tag{7}$$

with initial values $c_0 = 1$, $c_1 = E/C$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence

$$p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} = 0 \qquad \text{for } n \geq 1 \tag{8}$$

with initial values $a_0 = 1$, $a_1 = 1 + 2E/C$.

(ii) *Suppose the sequence (c_k) satisfies the recurrence* (7) *with initial values $c_0 = 0$, $c_1 = 1$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence*

$$p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} = C(4n+2) \qquad \text{for } n \geq 1 \tag{9}$$

with initial values $a_0 = 0$, $a_1 = 2$.

(iii) *Suppose the sequence (c_k) satisfies the recurrence*

$$P_0(k)c_{k+1} - P_1(k)c_k - P_2(k)c_{k-1} = \frac{(-1)^k}{k+1} \qquad \text{for } k \geq 1 \tag{10}$$

with initial values $c_0 = 0$, $c_1 = 1/C$. Then the Legendre transform (a_n) of (c_k) satisfies the recurrence (8) *with initial values $a_0 = 0$, $a_1 = 2/C$.*

PROOF. For abbreviation we let

$$a_{n,k} = c_k \binom{n}{k}\binom{n+k}{k},$$

where (c_k) is any sequence. The Legendre transform (a_n) of (c_k) is then given by

$$a_n = \sum_{k=0}^{n} a_{n,k}.$$

We notice first that

$$(11)\ p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k} = \left(2p_0(n)\binom{n}{k-1}\binom{n+k}{k} - 2p_2(n)\binom{n}{k}\binom{n+k-1}{k-1} - (2n+1)q_1(n)\binom{n}{k}\binom{n+k}{k}\right)c_k.$$

Using the method of *creative telescoping* (cf. [5, 6]) we let

$$A_{n,k} = -\binom{n}{k}\binom{n+k}{k}(2n+1)\Big(\big(q_1(n) - 2Q_1(k)\big)c_k + 2P_2(k)c_{k-1}\Big)$$

for $0 \le k \le n$, and with the proviso that $A_{n,k} = 0$ for $k < 0$ or $k > n$. An easy rearrangement shows that identically for $0 < k < n+1$:

$$(12)\quad A_{n,k} - A_{n,k-1} = \left(2p_0(n)\binom{n}{k-1}\binom{n+k}{k} - 2p_2(n)\binom{n}{k}\binom{n+k-1}{k-1} - (2n+1)q_1(n)\binom{n}{k}\binom{n+k}{k}\right)c_k - 2\binom{n}{k-1}\binom{n+k-1}{k-1}(2n+1)\times \big(P_0(k-1)c_k - P_1(k-1)c_{k-1} - P_2(k-1)c_{k-2}\big).$$

In particular for $k = 1$, and using (11), we also obtain

$$(13)\qquad A_{n,1} - A_{n,0} = p_0(n)a_{n+1,1} - p_1(n)a_{n,1} + p_2(n)a_{n-1,1} - (4n+2)(Cc_1 - Ec_0).$$

We also notice that

$$(14)\qquad A_{n,0} = -(2n+1)q_1(n)c_0.$$

Case 1. Assume first that (c_k) satisfies (7) for $k \ge 1$. Then by (11) and (12)

$$(15.1)\qquad A_{n,k} - A_{n,k-1} = p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k}$$

for $1 < k < n+1$. By (14) the relation (15.1) also holds for $k = 0$. Using (7) we get

$$(16.1)\quad -A_{n,n} = \binom{2n}{n}(2n+1)\big(2P_1(n)c_n + 2P_2(n)c_{n-1}\big) = 2\binom{2n}{n}(2n+1)P_0(n)c_{n+1} = \binom{2n+2}{n+1}(n+1)P_0(n)c_{n+1} = p_0(n)a_{n+1,n+1},$$

so that relation (15.1) also holds for $k = n + 1$. Consequently by (13) and (15.1)

$$\begin{aligned} &p_0(n)a_{n+1} - p_1(n)a_n + p_0(n)a_{n-1} \\ &= \sum_{k=0}^{n+1} \left(A_{n,k} - A_{n,k-1}\right) + (4n+2)(Cc_1 - Ec_0) \\ &= (4n+2)(Cc_1 - Ec_0), \end{aligned}$$

which proves the two first claims of the theorem.

Case 2. Assume next that (c_k) satisfies (10) for $k \geq 1$ with $c_0 = 0$, $c_1 = 1/C$. Then (15.1) is replaced by

$$\begin{aligned} A_{n,k} - A_{n,k-1} = {} & p_0(n)a_{n+1,k} - p_1(n)a_{n,k} + p_2(n)a_{n-1,k} \\ & -(4n+2)\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{(-1)^{k-1}}{k} \end{aligned} \tag{15.2}$$

for $1 < k < n + 1$, and (16.1) is replaced by

$$-A_{n,n} = p_0(n)a_{n+1,n+1} - (4n+2)\binom{2n}{n}\frac{(-1)^n}{n+1}. \tag{16.2}$$

By (14) and (16.2) it follows that the relation (15.2) also holds for $k = n + 1$ and also for $k = 0$ when omitting the last term in (15.2). Since $Cc_1 - Ec_0 = 1$ it follows by (13), (15.2) and Section 2(ii) that

$$\begin{aligned} &p_0(n)a_{n+1} - p_1(n)a_n + p_2(n)a_{n-1} \\ &= \sum_{k=0}^{n+1} \left(A_{n,k} - A_{n,k-1}\right) + (4n+2)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k+1} \\ &= 0 \qquad \text{for } n > 0. \end{aligned}$$

This proves the last claim of the theorem.

REMARK 1. Assume that $B = 2A$, which implies that $p_2(n+1) = p_0(n)$. Assume further that $p_0(n) \neq 0$ for $n \geq 0$, and that $a_n \neq 0$ for $n \geq 0$. To distinguish the three sequences (a_n) in $(i) - (iii)$ they will here be denoted (a_n), (a'_n), (a''_n), respectively. The following formulas are easily deduced:

$$d''_n := \begin{vmatrix} a_n & a''_n \\ a_{n+1} & a''_{n+1} \end{vmatrix} = \frac{2C}{p_0(n)},$$

$$D_n := \begin{vmatrix} a_{n-1} & a'_{n-1} & a''_{n-1} \\ a_n & a'_n & a''_n \\ a_{n+1} & a'_{n+1} & a''_{n+1} \end{vmatrix} = \frac{-4C^2(2n+1)}{p_0(n-1)p_0(n)},$$

$$d_n' := \begin{vmatrix} a_n & a_n' \\ a_{n+1} & a_{n+1}' \end{vmatrix} = \frac{2C}{p_0(n)} \sum_{\nu=0}^{n} (2\nu+1)a_\nu ,$$

$$\alpha'' := \lim \frac{a_n''}{a_n} = 2C \sum_{n=0}^{\infty} \frac{1}{p_0(n)a_n a_{n+1}} ,$$

$$\alpha' := \lim \frac{a_n'}{a_n} = 2C \sum_{n=0}^{\infty} \frac{1}{p_0(n)a_n a_{n+1}} \sum_{\nu=0}^{n} (2\nu+1)a_\nu ,$$

$$\alpha'' - \frac{a_n''}{a_n} = 2C \sum_{\nu=n}^{\infty} \frac{1}{p_0(\nu)a_\nu a_{\nu+1}} ,$$

$$\alpha' = \sum_{n=0}^{\infty} (2n+1)(a_n\alpha'' - a_n'') .$$

The formulas concerning infinite series are purely formal, and convergence must therefore be ascertained when applied.

4. Examples

EXAMPLE 1. (Classical and generalized Legendre polynomials.) For $A = B = D = 0$, $C = 1$, $E = x$ the recurrence

$$c_{k+1} - xc_k - 0 \cdot c_{k-1} = 0$$

has the solution $c_k = x^k$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k$$

then satisfies the recurrence

$$(n+1)a_{n+1} - (2n+1)(1+2x)a_n + na_{n-1} = 0.$$

The polynomials $a_n = a_n(x)$ (Legendre polynomials) are orthogonal with respect to Lebesgue measure on $[-1, 0]$.

For $A = B = 0, C = 1, D = x_1, E = x_0$ we get the recurrence

$$c_{k+1} - (x_0 + k(k+1)x_1)c_k - k^2 x_1 c_{k-1} = 0.$$

The corresponding sequence of generalized Legendre polynomials a_n then satisfies the recurrence

$$(n+1)a_{n+1} - (2n+1)(1 + 2(x_0 + n(n+1)x_1)a_n + na_{n-1} = 0.$$

Compare [9] for a wider class of generalized Legendre polynomials.

EXAMPLE 2. (Orthogonal polynomials related to Bernoulli numbers.) For $A = D = 0$, $B = C = 1$, $E = x$ the recurrence

$$(k+1)c_{k+1} - (x-k)c_k - 0 \cdot c_{k-1} = 0$$

has the solution $c_k = \binom{x}{k}$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\binom{x}{k}$$

therefore satisfies the recurrence

$$(n+1)^2 a_{n+1} - (2n+1)(1+2x)a_n - n^2 a_{n-1} = 0.$$

When defining a linear functional s on $\mathbb{R}[x]$ by

$$s\left(\binom{x}{k}\right) = \frac{(-1)^k}{k+1},$$

it follows easily by Section 2 (ii) and the recurrence relation that the polynomials $a_n = a_n(x)$ are orthogonal with respect to the functional s, and that

$$s(a_n(x)^2) = \frac{(-1)^n}{2n+1}.$$

Since (compare [7])

$$x^n = \sum_{k=0}^{n} A_{nk}\binom{x}{k},$$

where

$$A_{nk} = \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

it follows that

$$s_n = s(x^n) = \sum_{k=0}^{n} A_{nk}\frac{(-1)^k}{k+1} = \sum_{k=0}^{n}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^n = B_n.$$

EXAMPLE 3. (Recurrent sequences related to Pell's equation.) Let p be a prime number and $m \in \mathbb{N}$. Suppose that $(x_1, x_2) \in \mathbb{Z}^2$ is an arbitrary solution to Pell's equation

$$x_1^2 - px_2^2 = \varepsilon, \quad \varepsilon = \pm 1.$$

Let $A = 1$, $B = 0$, $C = -p^{2m-1}$ be fixed. We consider two cases of values for D and E corresponding to a specific solution to Pell's equation:
Case (i). For $D = \varepsilon p x_2^2$, $E = -p^m x_1 x_2 - \varepsilon p^{2m} x_2^2$, the recurrence (7) has the integral solution $c_k = D^k(x_1 k + \varepsilon p^m x_2)$, $k \geq 0$.
Case (ii). For $D = -\varepsilon x_1^2$, $E = -p^m x_1 x_2 + \varepsilon p^{2m-1} x_1^2$, the recurrence (7) has the integral solution $c_k = D^k(x_2 k - \varepsilon p^{m-1} x_1)$, $k \geq 0$.

In particular for $C = -2$ (that is $p = 2$, $m = 1$) the fundamental solution $(x_1, x_2) = (1, 1)$ with $\varepsilon = -1$ gives in case (i) $(D, E) = (-2, 2)$, and thus the recurrence $(k^2 - 2)c_{k+1} + (3k^2 + k - 2)c_k + 2k^2 c_{k-1} = 0$ has integral solution $c_k = (-2)^k(k - 2)$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} (-2)^k (k-2) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n+1)(n^2-2)a_{n+1} + (2n+1)(3n^2+3n-2)a_n + n(n^2+2n-1)a_{n-1} = 0.$$

In case (ii), $(D, E) = (1, -4)$, and thus the recurrence

$$(k^2-2)c_{k+1} - (2k-4)c_k - k^2 c_{k-1} = 0$$

has integral solution $c_k = k + 1$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} (k+1) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n+1)(n^2-2)a_{n+1} - (2n+1)(3n^2+3n-10)a_n + n(n^2+2n-1)a_{n-1} = 0.$$

Analogously for $C = -2$ the solution $(x_1, x_2) = (3, 2)$ with $\varepsilon = 1$ gives in case (i) $(D, E) = (8, -28)$, and thus the recurrence

$$(k^2-2)c_{k+1} - (7k^2+9k-28)c_k - 8k^2 c_{k-1} = 0$$

has integral solution $c_k = 8^k(3k + 4)$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} 8^k (3k+4) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n+1)(n^2-2)a_{n+1} - (2n+1)(17n^2+17n-58)a_n + n(n^2+2n-1)a_{n-1} = 0.$$

In case (ii), $(D, E) = (-9, 6)$, and thus the recurrence

$$(k^2 - 2)c_{k+1} + (10k^2 + 8k - 6)c_k + 9k^2 c_{k-1} = 0$$

has integral solution $c_k = (-9)^k(2k - 3)$. The corresponding sequence

$$a_n = \sum_{k=0}^{n} (-9)^k (2k - 3) \binom{n}{k} \binom{n+k}{k}$$

therefore satisfies the recurrence

$$(n + 1)(n^2 - 2)a_{n+1} + (2n + 1)(17n^2 + 17n - 10)a_n + n(n^2 + 2n - 1)a_{n-1} = 0.$$

EXAMPLE 4. For $A = D = 2$, $B = 3$, $C = -2$, $E = -\frac{3}{2}x$ the recurrence is

$$(k + 2)(2k - 1)c_{k+1} - (k - 3x/2)c_k - 2k^2 c_{k-1} = 0.$$

The corresponding sequence (a_n) then satisfies the recurrence

$$(n + 1)(n + 2)(2n - 1)a_{n+1} - (2n + 1)\big(2(n^2 + n - 1) + (2n - 1)(2n + 3)x\big)a_n + n(n - 1)(2n + 3)a_{n-1} = 0.$$

The polynomials $a_n = a_n(x)$ are orthogonal with respect to a measure μ concentrated in the single point $x = -2/3$.

5. Apéry's sequences

By applying Theorem 1 for $A = C = 1$, $B = E = 2$, $D = 8$, that is for

$$P_0 = (k + 1)^2, \qquad P_1 = 7k^2 + 7k + 2, \qquad P_2 = 8k^2,$$

we get

THEOREM 2. (i) *The Legendre transform* (a_n) *of the sequence* (c_k) *in* (1) *(which has* $c_0 = 1$, $c_1 = 2$*) satisfies the recurrence* (3) *with initial values* $a_0 = 1$, $a_1 = 5$.

(ii) *The Legendre transform* (a_n) *of the sequence* (c_k) *satisfying the recurrence* (4) *and having initial values* $c_0 = 0$, $c_1 = 3$ *satisfies the following recurrence*

$$(n + 1)^3 a_{n+1} - \big((n + 1)^3 + n^3 + 4(2n + 1)^3\big)a_n + n^3 a_{n-1} = 3(4n + 2) \qquad \textit{for } n \geq 0 \tag{17}$$

with initial values $a_0 = 0$, $a_1 = 6$.

(iii) *The Legendre transform* (a_n) *of the sequence* (c_k) *satisfying the recurrence*

$$(18)\qquad (k+1)^2c_{k+1} - \big(7k^2+7k+2\big)c_k - 8k^2c_{k-1} = \frac{(-1)^k 3}{k+1} \qquad \textit{for } k \geq 1$$

and having initial values $c_0 = 0$, $c_1 = 3$ *satisfies the recurrence* (3) *for* $n \geq 1$ *with initial values* $a_0 = 0$, $a_1 = 6$.

REMARK 2. . As in Remark 1 we denote the three sequences (a_n) in (i)–(iii) by (a_n), (a'_n), (a''_n), respectively, and similarly for the sequences (c_k).

Since $B = 2A$ the formulas in Remark 1 applies with $p_0(n) = (n+1)^3$. In this case it is well known (cf. [6]) that

$$\lim \frac{c'_k}{c_k} = \pi^2/8, \qquad \alpha'' := \lim \frac{a''_n}{a_n} = \zeta(3).$$

Therefore also

$$\alpha' := \lim \frac{a'_n}{a_n} = \pi^2/8, \qquad \lim \frac{c''_k}{c_k} = \zeta(3).$$

The simultaneous approximation of $\pi^2/8$ and $\zeta(3)$ is illustrated in the following two tables:

k	c_k	c'_k	c''_k	$\pi^2/8 - c'_k/c_k$	$\zeta(3) - c''_k/c_k$
0	1	0	0	1.233700550	1.202056903
1	2	3	3	−0.266299450	−0.297943097
2	10	12	93/8	0.033700550	0.039556903
3	56	208/3	1217/18	−0.004394688	−0.005284367
4	346	1280/3	239429/576	0.000559895	0.000683067

n	a_n	a'_n	a''_n	$\pi^2/8 - a'_n/a_n$	$\zeta(3) - a''_n/a_n$
0	1	0	0	1.233700550	1.202056903
1	5	6	6	0.033700550	0.002056903
2	73	90	351/4	0.000823838	0.000002109
3	1445	5348/3	62531/36	0.000021196	0.000000002
4	33001	122140/3	11424695/288	0.000000561	0.000000000

6. A peculiar sequence

We consider now the sequence $a_n = \sum_{k=0}^n \binom{2k}{k}\binom{n}{k}\binom{n+k}{k}$. To explain the properties mentioned in the introduction we consider the following sequences of polynomials

$$a_n(x) = \sum_{k=0}^{n} \binom{2k}{k}\binom{n}{k}\binom{n+k}{k} x^k, \tag{19}$$

$$P_n^-(x) = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j-\frac{1}{2}}{j} x^j, \tag{20}$$

$$P_n^+(x) = \sum_{j=0}^{n} \binom{n}{j}\binom{n+j+\frac{1}{2}}{j} x^j. \tag{21}$$

Then we claim that

$$a_{2n}(x) = P_n^-(4x)^2, \tag{22}$$

$$a_{2n+1}(x) = (1+4x)P_n^+(4x)^2. \tag{23}$$

Since $a_n = a_n(1)$ we get in particular

$$a_{2n} = P_n^-(4)^2, \qquad a_{2n+1} = 5P_n^+(4)^2, \tag{24}$$

which explains the peculiarities of the sequence a_n.

The polynomials $P_n^-(x)$ and $P_n^+(x)$ are expressible in terms of the Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} F[-n, n+\alpha+\beta+1; \alpha+1; (1-x)/2].$$

In fact

$$P_n^-(x) = P_n^{(0,-\frac{1}{2})}(2x+1) = F[-n, n+\frac{1}{2};\ 1;\ -x],$$

$$P_n^+(x) = P_n^{(0,+\frac{1}{2})}(2x+1) = F[-n, n+\frac{3}{2};\ 1;\ -x].$$

Therefore formula (22) follows immediately from Clausen's formula (cf. [11])

$$F^2[a, b; a+b+\frac{1}{2}; z] = {}_3F_2[2a, 2b, a+b; 2a+2b, a+b+\frac{1}{2}; z]$$

with $a = -n,\ b = n + \frac{1}{2}, z = -4x$.

The polynomial sequence $u_n = P_n^-(x)$ satisfies the recurrence relation

$$\begin{aligned}(n+1)(2n+1)(4n-1)u_{n+1} - (4n+1)\Big(\frac{1}{2}(4n-1)(4n+3)x + 4n^2 + 2n - 1\Big)u_n \\ +n(2n-1)(4n+3)u_{n-1} = 0 \quad \text{for } n \geq 0,\end{aligned} \tag{25}$$

and is thus (cf. [14]) a sequence of orthogonal polynomials with respect to a normalized measure m^- on $\mathbb{R}$. The measure is given by

$$dm^-(t) = \begin{cases} dt/2\sqrt{1+t}, & t \in]-1,0], \\ 0 & \text{otherwise}, \end{cases} \tag{26}$$

the corresponding moments being

$$\mu_n^- = \frac{(-1)^n 4^n}{(2n+1)\binom{2n}{n}}, \qquad n \geq 0. \tag{27}$$

Also

$$\|P_n^-(x)\|^2 = \frac{1}{4n+1}, \qquad n \geq 0. \tag{28}$$

Analogously formula (22) follows from a more general formula of Orr (see [11, Theorem III]), and contiguous relations.

Similarly the polynomial sequence $u_n = P_n^+(x)$ satisfies the recurrence relation

$$\begin{aligned}(n+1)(2n+3)(4n+1)u_{n+1} - (4n+3)\big(\tfrac{1}{2}(4n+1)(4n+5)x + 4n^2 + 6n + 1\big)u_n \\ + n(2n+1)(4n+5)u_{n-1} = 0 \qquad \text{for } n \geq 0,\end{aligned} \tag{29}$$

and is a sequence of orthogonal polynomials with respect to a normalized measure m^+ on $\mathbb{R}$. The measure is given by

$$dm^+(t) = \begin{cases} \frac{3}{2}\sqrt{1+t}\,dt, & t \in]-1,0] \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

the corresponding moments being

$$\mu_n^+ = \frac{(-1)^n 3 \cdot 4^n}{(2n+1)(2n+3)\binom{2n}{n}}, \qquad n \geq 0. \tag{31}$$

Also

$$\|P_n^+(x)\|^2 = \frac{3}{4n+3}, \qquad n \geq 0. \tag{32}$$

7. Some computer-aided results

We shall mention some further examples of recurrences for Legendre transforms.

EXAMPLE 5. For the sequence (cf. Section 6) $c_k = \binom{2k}{k}x^k$ and satisfying the recurrence

$$(k+1)c_{k+1} - (4k+2)xc_k = 0,$$

the corresponding Legendre transform (a_n) satisfies the four term recurrence

$$\begin{aligned}&(2n+1)(n+2)^2a_{n+2}-(2n+3)(3n^2+6n+2+4(2n+1)(2n+3)x)a_{n+1}\\&\quad+(2n+1)(3n^2+6n+2+4(2n+1)(2n+3)x)a_n-(2n+3)n^2a_{n-1}=0.\end{aligned}$$

This follows by a telescopic argument using

$$A_{n,k}=-4(2n+1)(2n+3)\binom{n+1}{k}\binom{n+k}{k}(4k+2)xc_k.$$

EXAMPLE 6. For the sequence (c_k) of Fibonacci numbers with $c_0=c_1=1$ and satisfying the recurrence

$$c_{k+1}-c_k-c_{k-1}=0,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$\begin{aligned}&(2n-1)(n+1)(n+2)a_{n+2}-4(2n-1)(2n+3)(n+1)a_{n+1}\\&\quad-2(2n+1)(n^2+n-1)a_n-4(2n-1)(2n+3)na_{n-1}\\&\quad+(2n+3)(n-1)na_{n-2}=0.\end{aligned}$$

This follows by a telescopic argument using

$$A_{n,k}=A_{n,0}\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{1}{k^2}\left((n^2+n-k(k-1))c_k+k^2c_{k-1}\right)$$

with

$$A_{n,0}=-4(2n-1)(2n+1)(2n+3).$$

EXAMPLE 7. For the sequence (c_k) of Legendre polynomials satisfying the recurrence

$$(k+1)c_{k+1}-(2k+1)(2x+1)c_k+kc_{k-1}=0,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$\begin{aligned}&(2n-1)(n+2)^2a_{n+2}-(3+4x)(2n-1)(2n+3)^2a_{n+1}\\&\quad+(2n+1)(38n^2+38n-29+8(2n-1)(2n+3)x)a_n\\&\quad-(3+4x)(2n-1)^2(2n+3)a_{n-1}+(2n+3)(n-1)^2a_{n-2}=0.\end{aligned}$$

This follows by a telescopic argument using

$$A_{n,k}=A_{n,0}\binom{n}{k-1}\binom{n+k-1}{k-1}\frac{1}{k^2}\left((n^2+n-k(k-1)-k(2k+1)(2x+1))c_k+k^2c_{k-1}\right)$$

with

$$A_{n,0}=4(2n-1)(2n+1)(2n+3).$$

EXAMPLE 8. For a sequence (c_k) satisfying a recurrence

$$P_0(k)c_{k+1} - P_1(k)xc_k - P_2(k)x^2c_{k-1} = 0,$$

for $k \geq 0$, where

$$P_0(k) = (k+1)^2, \quad P_1(k) = \alpha k^2 + \alpha k + \beta, \quad P_2(k) = \gamma k^2,$$

the corresponding Legendre transform (a_n) satisfies the five term recurrence

$$\begin{aligned}
&(2n-1)n(n+2)^3 a_{n+2} \\
&\quad -(2n-1)(2n+3)\big(2n^3+6n^2+4n-1+2nP_1(n+1)x\big)a_{n+1} \\
&\quad -(2n+1)\big(4\gamma(2n-1)(2n+3)n(n+1)x^2 \\
&\quad -2(2n-1)(2n+3)(P_1(n)-2P_1(0))x-(6n^4+12n^3-2n^2-8n+3)\big)a_n \\
&\quad -(2n-1)(2n+3)\big(2n^3-2n+1+2(n+1)P_1(n-1)x\big)a_{n-1} \\
&\quad +(2n+3)(n+1)(n-1)^3 a_{n-2} = 0.
\end{aligned}$$

This follows by a telescopic argument using

$$\begin{aligned}
A_{n,k} = {}&-4(2n-1)(2n+1)(2n+3)\binom{n+1}{k}\binom{n+k-1}{k} \\
&\times \big((\gamma(n^2+n-k(k-1))x^2 + P_1(k)x)c_k + P_2(k)x^2c_{k-1}\big).
\end{aligned}$$

Important examples are

$$c_k = \sum_{j=0}^{k}\binom{k}{j}^3 x^k$$

satisfying the above recurrence with $(\alpha, \beta, \gamma) = (7, 2, 8)$, and

$$c_k = \sum_{j=0}^{k}\binom{k}{j}^2\binom{k+j}{j}x^k$$

satisfying the above recurrence with $(\alpha, \beta, \gamma) = (11, 3, 1)$ (cf. [6]).

EXAMPLE 9. For the sequence

$$c_k = \sum_{j=0}^{k}\binom{k}{j}^4 x^k,$$

satisfying the three term recurrence (see [1, 2, 4, 5, 6, 12])

$$P_0(k)c_{k+1} - P_1(k)xc_k - P_2(k)x^2c_{k-1} = 0,$$

where

$$P_0(k) = (k+1)^3, \quad P_1(k) = 2(2k+1)(3k^2+3k+1), \quad P_2(k) = (4k-1)4k(4k+1),$$

the corresponding Legendre transform (a_n) satisfies the seven term recurrence

$$\begin{aligned}
&(2n-3)(2n-1)n(n+3)^4 a_{n+3}\\
&\quad -(2n-1)(2n-3)(2n+5)\big(3n^4+22n^3+52n^2+33n-16\\
&\quad +2nP_1(n+2)x\big)a_{n+2}\\
&\quad -(2n-3)(2n+3)\big((1024n^5+6144n^4+10944n^3+2944n^2-5040n)x^2\\
&\quad -(192n^5+1104n^4+1728n^3-180n^2-1332n+420)x\\
&\quad -(15n^5+85n^4+120n^3-60n^2-126n+80)\big)a_{n+1}\\
&\quad +2(2n-3)(2n+1)(2n+5)\big(8(2n-1)(2n+3)(16n^2+16n-15)x^2\\
&\quad -(72n^4+144n^3-98n^2-170n+126)x-(5n^4+10n^3-10n^2-15n+16)\big)a_n\\
&\quad -(2n-1)(2n+5)\big((1024n^5-1024n^4-3392n^3+3264n^2+2448n-2160)x^2\\
&\quad -(192n^5-144n^4-768n^3+660n^2+756n-756)x\\
&\quad -(15n^5-10n^4-70n^3+60n^2+89n-96)\big)a_{n-1}\\
&\quad -(2n-3)(2n+3)(2n+5)\big(3n^4-10n^3+4n^2+17n-16\\
&\quad +2(n+1)P_1(n-2)x\big)a_{n-2}\\
&\quad +(2n+3)(2n+5)(n+1)(n-2)^4 a_{n-3} = 0.
\end{aligned}$$

This follows by a telescopic argument using

$$\begin{aligned}
A_{n,k} = -8(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)\binom{n+1}{k-1}\binom{n+k-2}{k-1}\frac{1}{k^2}\\
\times\big(4(4k+3)(4k+5)(n^2+n-(k-1)(k-2))x^2c_k\\
+k(P_1(k)xc_k+P_2(k)x^2c_{k-1})\big)
\end{aligned}$$

with

$$A_{n,0} = -480(2n-3)(2n-1)(2n+1)(2n+3)(2n+5).$$

The computations were performed by means of the GP-PARI system using the methods in [5].

Added in proof. It has been pointed out to me by Michael Stoll (University of Bonn) that arguments taken from R. P. Stanley, 'Differentiably finite power series', *European J. Combin.* **7** (1980), 175–188, lead to the result (illustrated by the examples above) that the set of polynomially recursive sequences is invariant under the Legendre transform and the inverse Legendre transform.

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