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COMMENTS ON A SHORT PROOF OF AN EXPLICIT FORMULA FOR BERNOULLI NUMBERS

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Abstract: The object of this paper is to motivate a proof, given by Grzegorz Rądkowski, of a formula expressing the Bernoulli numbers in terms of certain numbers. It is also shown how the original formula may be written in terms of Stirling numbers of the second kind.

The Bernoulli numbers B_n are defined by their generating function

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty} B_n\frac{t^n}{n!} \tag{1}$$

In [2] the author defines numbers $a_{n,k}$, $n=0,1,2,\ldots$, $k=1,2,3,\ldots$ by

$$a_{n,k}:=(-1)^n\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(j+1)^n\,. \tag{2}$$

He shows that

$$a_{n,1}=(-1)^n,\ a_{0,k}=0 \text{ for } n=0,1,2,\ldots \text{ and for } k=2,3,\ldots \tag{3}$$

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and that

$$a_{n+1,k+1} = ka_{n,k} - (k+1)a_{n,k+1} \tag{4}$$

for $n = 0, 1, 2, \ldots$ and for $k = 1, 2, 3, \ldots$. Using

$$\frac{1}{1+e^t} = \frac{1}{t}\frac{t}{e^t-1} - \frac{1}{t}\frac{2t}{e^{2t}-1} = \sum_{n=0}^{\infty} \frac{B_{n+1}(1-2^{n+1})}{n+1}\frac{t^n}{n!} \tag{5}$$

and showing by induction that

$$\left(\frac{1}{1+e^t}\right)^{(n)} = \sum_{k=1}^{n+1} a_{n,k} \left(\frac{1}{1+e^t}\right)^k \tag{6}$$

he gets the formula

$$B_{n+1} = -\frac{n+1}{2^{n+1}-1}\sum_{k=1}^{n+1} \frac{a_{n,k}}{2^k}. \tag{7}$$

(Note that for convergent power series $F(t) = \sum a_n \frac{t^n}{n!}$ the coefficients a_n are given by $a_n = F^{(n)}(0)$.)

Knowing (2) it is easy to see that (3) and (4) are satisfied. These equations lead to (6). So the real question is how to get (2) from (3) and (4). This topic is not touched in [2].

In the proof of the following proposition a (simple and) generic method is demonstrated that allows the determination of the $a_{n,k}$ by their boundary values $a_{0,k}$ and $a_{n,1}$ and by the recursion (4).

Proposition 1. *The double sequence* $(a_{n,k})_{\substack{n=0,1,2,\ldots \\ k=1,2,3,\ldots}}$ *satisfies* (3) *and* (4) *if, and only if,* (2) *is satisfied.*

Proof. For $k \geq 1$ put $\sigma_k(t) := \sum_{n=0}^{\infty} a_{n,k}t^n$. Then $\sigma_1(t) = \frac{1}{1+t}$ by (3). Moreover, using (4) and $a_{0,k+1} = 0$, gives

$$\sigma_{k+1}(t) = a_{0,k+1} + \sum_{n=1}^{\infty} a_{n,k+1}t^n = \sum_{n=0}^{\infty} a_{n+1,k+1}t^{n+1} =$$

$$= kt\sum_{n=0}^{\infty} a_{n,k}t^n - (k+1)t\sum_{n=0}^{\infty} a_{n,k+1}t^n = kt\sigma_k(t) - (k+1)t\sigma_{k+1}(t).$$

Thus $\sigma_{k+1}(t) = \frac{kt}{1+(k+1)t}\sigma_k(t)$ and, by induction

$$\sigma_k(t) = \frac{(k-1)!\, t^{k-1}}{(1+t)(1+2t)\ldots(1+kt)},\ k = 1, 2, 3, \ldots \tag{8}$$

σ_k may be decomposed into partial fractions

$$\sigma_k(t) = \sum_{l=1}^{k} \frac{\alpha_l}{1+lt}. \tag{9}$$

The coefficients α_l may easily be determined from

$$(10)\qquad (k-1)!\, t^{k-1} = \sum_{l=1}^{k} \alpha_l \prod_{1\le j\le k,\, j\ne l} (1+jt)$$

by substituting $t = -1/p$, $1 \le p \le k$, which leads to

$$\frac{(k-1)!\,(-1)^{k-1}}{p^{k-1}} = \alpha_p \prod_{1\le j\le k,\, j\ne p} \left(1 - \frac{j}{p}\right) = \alpha_p \frac{(p-1)!\,(-1)^{k-p}\,(k-p)!}{p^{k-1}},$$

i. e.,

$$(11)\qquad \alpha_p = (-1)^{p-1}\binom{k-1}{p-1},\ p = 1, 2, 3, \ldots, k.$$

Since $\frac{1}{1+lt} = \sum_{n=0}^{\infty}(-1)^n l^n t^n$ we get by (9) and (11) that

$$(12)\qquad \sigma_k(t) = \sum_{n=0}^{\infty}(-1)^n \left(\sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1} l^n\right) t^n.$$

Accordingly,

$$a_{n,k} = (-1)^n \sum_{l=1}^{k}(-1)^{l-1}\binom{k-1}{l-1} l^n = (-1)^n \sum_{j=0}^{k-1}(-1)^j \binom{k-1}{j}(j+1)^n. \quad \Diamond$$

Remark 1. The method used in the above proof has been applied in [1, p. 207, Th. C] to determine the Stirling numbers $S(n,k)$ of the second kind by their boundary values $S(0,0) = 1$, $S(n,0) = 0$, $n = 1, 2, \ldots$ and $S(0,k) = 0$, $k = 1, 2, \ldots$ and by the recursion relation $S(n+1,k+1) = (k+1)S(n,k+1) + S(n,k)$. The resulting generating function satisfies

$$(13)\qquad \sum_{n=0}^{\infty} S(n,k)t^n = \frac{t^k}{(1-t)(1-2t)\ldots(1-kt)}.$$

Both, the boundary values and the recursion relation come from the combinatorial interpretation that $S(n,k)$ is the number of partitions of a set with n elements into k nonempty disjoint subsets ([1, p. 206, Def. A]).

Remark 2. The explicit formula

$$(14)\qquad S(n,k) = \frac{(-1)^k}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j} j^n$$

immediately follows from decomposing $\tau_k(t) := \frac{t^k}{(1-t)(1-2t)\ldots(1-kt)}$ into partial fractions. Moreover by observing that $t\sigma_k(-t) = (-1)^{k-1}(k-1)!\,\tau_k(t)$ and using $S(0,k) = 0$ for $k \ge 1$ we also get an explicit expression of $a_{n,k}$

in terms of the Stirling numbers, namely

$$a_{n,k} = (-1)^{n-k+1}(k-1)!\,S(n+1,k). \tag{15}$$

Thus (7) expresses the Bernoulli numbers in terms of the Stirling numbers of the second kind.

It might be interesting to note that another formula with the same behavior exists.

$$B_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} k!S(n,k). \tag{16}$$

This may be proved, following [1], by writing $\frac{t}{e^t-1} = \frac{\ln\big(1+(e^t-1)\big)}{e^t-1}$ and observing $\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!}$.

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