# UNBOUNDED DISCREPANCY IN FROBENIUS NUMBERS

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**Abstract**

Let $g_j$ denote the largest integer that is represented exactly $j$ times as a non-negative integer linear combination of $\{x_1, \ldots, x_n\}$. We show that for any $k > 0$, and $n = 5$, the quantity $g_0 - g_k$ is unbounded. Furthermore, we provide examples with $g_0 > g_k$ for $n \geq 6$ and $g_0 > g_1$ for $n \geq 4$.

## 1. Introduction

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers with $\gcd(x_1, x_2, \ldots, x_n) = 1$. The *Frobenius number* $g(x_1, x_2, \ldots, x_n)$ is defined to be the largest integer that cannot be expressed as a non-negative integer linear combination of the elements of $X$. For example, $g(6, 9, 20) = 43$.

The Frobenius number — the name comes from the fact that Frobenius mentioned it in his lectures, although he apparently never wrote about it — is the subject of a huge literature, which is admirably summarized in the book of Ramírez Alfonsín [5].

Recently, Brown et al. [2] considered a generalization of the Frobenius number, defined as follows: $g_j(x_1, x_2, \ldots, x_n)$ is largest integer having exactly $j$ representations as a non-negative integer linear combination of $x_1, x_2, \ldots, x_n$. (If no such integer exists, Brown et al. defined $g_j$ to be 0, but for our purposes, it seems more reasonable to leave it undefined.) Thus $g_0$ is just $g$, the ordinary Frobenius number. They observed that, for a fixed $n$-tuple $(x_1, x_2, \ldots, x_n)$, the function $g_j(x_1, x_2, \ldots, x_n)$ need not be increasing (considered as a function of $j$). For example, they gave the example $g_{35}(4, 7, 19) = 181$ while $g_{36}(4, 7, 19) = 180$. They asked

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if there are examples for which $g_1 < g_0$. Although they did not say so, it makes sense to impose the condition that

no $x_i$ can be written as a non-negative integer linear combination of the others, $(*)$

for otherwise we have trivial examples such as $g_0(4,5,8,10) = 11$ and $g_1(4,5,8,10) = 9$. We call a tuple satisfying $(*)$ a *reasonable* tuple.

In this note we show that the answer to the question of Brown et al. is "yes," even for reasonable tuples. For example, it is easy to verify that $g_0(8,9,11,14,15) = 21$, while $g_1(8,9,11,14,15) = 20$. But we prove much more: we show that

$$g_0(2n-2, 2n-1, 2n, 3n-3, 3n) = n^2 - O(n),$$

while for any fixed $k \geq 1$ we have $g_k(2n-2, 2n-1, 2n, 3n-3, 3n) = O(n)$. It follows that for this parameterized 5-tuple and all $k \geq 1$, we have $g_0 - g_k \to \infty$ as $n \to \infty$.

For other recent work on the generalized Frobenius number, see [1, 3, 4].

## 2. The Main Result

We define $X_n = \{2n-2, 2n-1, 2n, 3n-3, 3n\}$. It is easy to see that this is a reasonable 5-tuple for $n \geq 5$. If we can write $t$ as a non-negative linear combination of the elements of $X_n$, we say $t$ has a representation or is representable.

We define $R(j)$ to be the number of distinct representations of $j$ as a non-negative integer linear combination of the elements of $X_n$.

**Theorem 1** *(a)* $g_k(X_n) = (6k+3)n - 1$ *for* $n > 6k+3$, $k \geq 1$.

*(b)* $g_0(X_n) = n^2 - 3n + 1$ *for* $n \geq 6$*;*

Before we prove Theorem 1, we need some lemmas.

**Lemma 2** *(a)* $R((6k+3)n-1) \geq k$ *for* $n \geq 4$ *and* $k \geq 1$.

*(b)* $R((6k+3)n-1) = k$ *for* $n > 6k+3$ *and* $k \geq 1$.

*Proof.* First, we note that

$$(6k+3)n - 1 = 1 \cdot (2n-1) + (3t-1) \cdot (2n) + (2(k-t)+1) \cdot (3n) \tag{1}$$

for any integer $t$ with $1 \leq t \leq k$. This provides at least $k$ distinct representations for $(6k+3)n-1$ and proves (a). We call these $k$ representations *special*.

To prove (b), we need to see that the $k$ special representations given by (1) are, in fact, all representations that can occur.

Suppose that $(a, b, c, d, e)$ is a 5-tuple of non-negative integers such that

$$a(2n-2) + b(2n-1) + c(2n) + d(3n-3) + e(3n) = (6k+3)n - 1. \qquad (2)$$

Reducing this equation modulo $n$, we get $-2a - b - 3d \equiv -1 \pmod{n}$. Hence there exists an integer $m$ such that $2a + b + 3d = mn + 1$. Clearly $m$ is non-negative. There are two cases to consider: $m = 0$ and $m \geq 1$.

If $m = 0$, then $2a + b + 3d = 1$, which, by the non-negativity of the coefficients $a, b, d$ implies that $a = d = 0$ and $b = 1$. Thus by (2) we get $2n - 1 + 2cn + 3en = (6k+3)n - 1$, or

$$2c + 3e = 6k + 1. \qquad (3)$$

Taking both sides modulo 2, we see that $e \equiv 1 \pmod{2}$, while taking both sides modulo 3, we see that $c \equiv 2 \pmod{3}$. Thus we can write $e = 2r + 1$, $c = 3s - 1$, and substitute in (3) to get $k = r + s$. Since $s \geq 1$, it follows that $0 \leq r \leq k - 1$, and this gives our set of $k$ special representations in (1).

If $m \geq 1$, then $n + 1 \leq mn + 1 = 2a + b + 3d$, so $n \leq 2a + b + 3d - 1$. However, we know that $(6k+3)n - 1 \geq a(2n-2) + b(2n-1) + d(3n-3) > (n-1)(2a + b + 3d)$. Hence $(6k+3)n > (n-1)(2a+b+3d)+1 > (n-1)(2a+b+3d-1) \geq (n-1)n$. Thus $6k + 3 > n - 1$. It follows that if $n > 6k + 3$, then this case cannot occur, so all the representations of $(6k+3)n - 1$ are accounted for by the $k$ special representations given in (1). □

We are now ready to prove Theorem 1 (a).

*Proof.* We already know from Lemma 2 that for $n > 6k + 3$, the number $N := (6k+3)n - 1$ has exactly $k$ representations. It now suffices to show that if $t$ has exactly $k$ representations, for $k \geq 1$, then $t \leq N$.

We do this by assuming $t$ has at least one representation, say $t = a(2n-2)+b(2n-1)+c(2n)+d(3n-3)+e(3n)$, for some 5-tuple of non-negative integers $(a, b, c, d, e)$. Assuming these integers are large enough (it suffices to assume $a, b, c, d, e \geq 3$), we may take advantage of the internal symmetries of $X_n$ to obtain additional representations with the following swaps.

(a) $3(2n) = 2(3n)$; hence

$$a(2n-2) + b(2n-1) + c(2n) + d(3n-3) + e(3n)$$
$$= a(2n-2) + b(2n-1) + (c+3)(2n) + d(3n-3) + (e-2)(3n).$$

(b) $3(2n-2) = 2(3n-3)$; hence

$$a(2n-2) + b(2n-1) + c(2n) + d(3n-3) + e(3n)$$
$$= (a+3)(2n-2) + b(2n-1) + c(2n) + (d-2)(3n-3) + e(3n).$$

(c) $2n-2+2n = 2(2n-1)$; hence

$$a(2n-2)+b(2n-1)+c(2n)+d(3n-3)+e(3n)$$
$$= (a+1)(2n-2)+(b-2)(2n-1)+(c+1)(2n)+d(3n-3)+e(3n).$$

(d) $2n-2+2n-1+2n = 3n-3+3n$; hence

$$a(2n-2)+b(2n-1)+c(2n)+d(3n-3)+e(3n)$$
$$= (a+1)(2n-2)+(b+1)(2n-1)+(c+1)(2n)+(d-1)(3n-3)+(e-1)(3n).$$

We now do two things for each possible swap: first, we show that the requirement that $t$ have exactly $k$ representations imposes upper bounds on the size of the coefficients. Second, we swap until we have a representation which can be conveniently bounded in terms of $k$.

(a) If $\lfloor \frac{e}{2} \rfloor + \lfloor \frac{c}{3} \rfloor \geq k$, we can find at least $k+1$ representations of $t$. Thus we can find a representation of $t$ with $c \leq 2$ and $e \leq 2k-1$.

(b) Similarly, if $\lfloor \frac{d}{2} \rfloor + \lfloor \frac{a}{3} \rfloor \geq k$, we can find at least $k+1$ representations of $t$. Thus we can find a representation of $t$ with $d \leq 2k-1$ and $a \leq 2$. Combining this with (a), we can find a representation with $a, c \leq 2$ and $d+e \leq 2k-1$.

(c) If $\lfloor \frac{b}{2} \rfloor + \min\{a, c\} \geq k$, we can find at least $k+1$ representations of $t$. Thus we can find a representation of $t$ with $|b - \min\{a, c\}| \leq 1$. If we start with the assumption $a, c \leq 2$, this ensures that $\min\{a, b, c\} \leq \lfloor \frac{a+b+c}{3} \rfloor \leq \min\{a, b, c\} + 1$ and $\max\{a, b, c\} - \min\{a, b, c\} \leq 3$.

(d) If $\min\{a, b, c\} + \min\{d, e\} \geq k$ we can find at least $k+1$ representations of $t$. When this swap is followed by (a) or (b) (if necessary) we can find a representation with $d+e \leq 2k-1$, $a+b+c \leq 3$ and $a, c \leq 2$.

Putting this all together, we see that $t \leq (2n-1) + 2(2n) + (2k-1)(3n) = (6k+3)n - 1$, as desired. □

In order to prove Theorem 1 (b), we need a lemma.

**Lemma 3** *The integers $k(n-1), k(n-1)+1, \ldots, kn$ are representable for $k=2$ and $k \geq 4$ and for $n \geq 4$.*

*Proof.* We prove the result by induction on $k$. The base cases are $k = 2, 4$, and we have the representations given below:

$$\begin{aligned} 4n-4 &= 2(2n-2) \\ 4n-3 &= (2n-2)+(2n-1) \\ 4n-2 &= 2(2n-1) \\ 4n-1 &= (2n-1)+(2n) \\ 4n &= 2(2n). \end{aligned}$$

Now suppose $ln-m$ is representable for $4 \le l < k$ and $0 \le m \le l$. We want to show that $kn-t$ is representable for $0 \le t \le k$. There are three cases, depending on $k \pmod 3$.

If $k \equiv 0 \pmod 3$, and $k \ge 4$, then $(k-2)n-t = kn-t-2n$ is representable if $t \le k-2$; otherwise $(k-2)n-t+2 = kn-t-(2n-2)$ is representable. By adding $2n$ or $2n+2$, respectively, we get a representation for $kn-t$.

If $k \equiv 1 \pmod 3$, and $k \ge 4$, or if $k \equiv 2 \pmod 3$, then $(k-3)n-t = kn-t-3n$ is representable if $t \le k-3$; otherwise $(k-3)n-t+3 = kn-t-(3n-3)$ is representable. By adding $3n$ or $3n+3$, respectively, we get a representation for $kn-t$. □

Now we prove Theorem 1 (b).

*Proof.* First, let's show that every integer $> n^2-3n+1$ is representable. Since if $t$ has a representation, so does $t+2n-2$, it suffices to show that the $2n-2$ numbers $n^2-3n+2, n^2-3n+3, \ldots, n^2-n-1$ are representable.

We use Lemma 3 with $k = n-2$ to see that the numbers $(n-2)(n-1) = n^2-3n+2, \ldots, (n-2)n = n^2-2n$ are all representable. Now use Lemma 3 again with $k = n-1$ to see that the numbers $(n-1)(n-1) = n^2-2n+1, \ldots, (n-1)n = n^2-n$ are all representable. We therefore conclude that every integer $> n^2-3n+1$ has a representation.

Finally, we show that $n^2-3n+1$ does not have a representation. Suppose, to get a contradiction, that it does:

$$n^2-3n+1 = a(2n-2)+b(2n-1)+c(2n)+d(3n-3)+e(3n).$$

Reducing modulo $n$ gives $1 \equiv -2a-b-3d \pmod n$, so there exists an integer $m$ such that $2a+b+3d = mn-1$. Since $a, b, d$ are non-negative, we must have $m \ge 1$.

Now $n^2-3n+1 \ge a(2n-2)+b(2n-1)+d(3n-3) > (n-1)(2a+b+3d)$. Thus

$$n^2-3n+1 \ge (n-1)(mn-1) = mn^2-(m+1)n+1. \tag{4}$$

If $m=1$, we get $n^2-3n+1 \ge n^2-2n+1$, a contradiction. Hence $m \ge 2$. From (4) we get $(m-1)n^2-(m-2)n \le 0$. Since $n \ge 1$, we get $(m-1)n-(m-2) \le 0$, a contradiction. □

## 3. Additional Remarks

One might object to our examples because the numbers are not pairwise relatively prime. But there also exist reasonable 5-tuples with $g_0 > g_1$ for which all pairs are relatively prime: for example, $g_0(9, 10, 11, 13, 17) = 25$, but $g_1(9, 10, 11, 13, 17) = 24$. More generally one can use the techniques in this paper to show that

$$g_0(10n - 1, 15n - 1, 20n - 1, 25n, 30n - 1) = 50n^2 - 1$$

and

$$g_1(10n - 1, 15n - 1, 20n - 1, 25n, 30n - 1) = 50n^2 - 5n$$

for $n \geq 1$, so that $g_0 - g_1 \to \infty$ as $n \to \infty$.

For $k \geq 2$, let $f(k)$ be the least non-negative integer $i$ such that there exists a reasonable $k$-tuple $X$ with $g_i(X) > g_{i+1}(X)$. A priori $f(k)$ may not exist. For example, if $k = 2$, then we have $g_i(x_1, x_2) = (i + 1)x_1x_2 - x_1 - x_2$, so $g_i(x_1, x_2) < g_{i+1}(x_1, x_2)$ for all $i$. Thus $f(2)$ does not exist. In this paper, we have shown that $f(5) = 0$.

This raises the obvious question of other values of $f$.

**Theorem 4** *We have $f(i) = 0$ for $i \geq 4$.*

*Proof.* As mentioned in the Introduction, the example $(8, 9, 11, 14, 15)$ shows that $f(5) = 0$.

For $i = 4$, we have the example $g_0(24, 26, 36, 39) = 181$ and $g_1(24, 26, 36, 39) = 175$, so $f(4) = 0$. (This is the reasonable quadruple with $g_0 > g_1$ that minimizes the largest element.)

We now provide a class of examples for $i \geq 6$. For $n \geq 6$ define $X_n$ as follows:

$$X_n = (n + 1, n + 4, n + 5, [n + 7..2n + 1], 2n + 3, 2n + 4),$$

where by $[a..b]$ we mean the list $a, a + 1, a + 2, \ldots, b$.

For example, $X_8 = (9, 12, 13, 15, 16, 17, 19, 20)$. Note that $X_n$ is of cardinality $n$. We make the following three claims for $n \geq 6$.

(a) $X_n$ is reasonable.

(b) $g_0(X_n) = 2n + 7$.

(c) $g_1(X_n) = 2n + 6$.

(a): To see that $X_n$ is reasonable, assume that some element $x$ is in the $\mathbb{N}$-span of the other elements. Then either $x = ky$ for some $k \geq 2$, where $y$ is the smallest element of $X_n$, or $x \geq y + z$, where $y, z$ are the two smallest elements of $X_n$. It is easy to see both of these lead to contradictions.

(b) and (c): Clearly $2n+7$ is not representable, and $2n+6$ has the single representation $(n+1)+(n+5)$. It now suffices to show that every integer $\geq 2n+8$ has at least two representations. And to show this, it suffices to show that all integers in the range $[2n+8..3n+8]$ have at least two representations.

Choosing $(n+4)+[n+7..2n+1]$ and $(n+5)+[n+7..2n+1]$ gives two distinct representations for all numbers in the interval $[2n+12..3n+5]$. So it suffices to handle the remaining cases $2n+8, 2n+9, 2n+10, 2n+11, 3n+6, 3n+7, 3n+8$. This is done as follows:

$$\begin{aligned}
2n+8 = (n+1)+(n+7) &= 2(n+4) \\
2n+9 = (n+4)+(n+5) &= \begin{cases} 3(n+1), & \text{if } n=6; \\ (n+1)+(n+8), & \text{if } n\geq 7. \end{cases} \\
2n+10 = 2(n+5) &= \begin{cases} (n+1)+(2n+3), & \text{if } n=6; \\ 3(n+1), & \text{if } n=7; \\ (n+1)+(n+9), & \text{if } n\geq 8. \end{cases} \\
2n+11 = (n+4)+(n+7) &= \begin{cases} (n+1)+(2n+4), & \text{if } n=6; \\ (n+1)+(2n+3), & \text{if } n=7; \\ 3(n+1), & \text{if } n=8; \\ (n+1)+(n+10), & \text{if } n\geq 9. \end{cases} \\
3n+6 = 2(n+1)+(n+4) &= (n+5)+(2n+1) \\
3n+7 = 2(n+1)+(n+5) &= (n+4)+(2n+3) \\
3n+8 = (n+5)+(2n+3) &= (n+4)+(2n+4).
\end{aligned}$$

□

We do not know the value of $f(3)$. The example

$$g_{14}(8,9,15) = 172$$
$$g_{15}(8,9,15) = 169$$

shows that $f(3) \leq 14$.

**Conjecture 5** $f(3) = 14$.

We have checked all triples with largest element $\leq 200$, but have not found any counterexamples.

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