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Some aspects of Fibonacci polynomial congruences

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Abstract

This paper formulates a definition of Fibonacci polynomials which is slightly different from the traditional definitions, but which is related to the classical polynomials of Bernoulli, Euler and Hermite. Some related congruence properties are developed and some unanswered questions are outlined.

Keywords: Congruences, recurrence relations, Fibonacci sequence, Lucas sequences, umbral calculus.

MSC: 11B39;11B50;11B68

1. Introduction

The purpose of this paper is to consider some congruences associated with a generalized Fibonacci polynomial which is defined in the next section in relation to two generalized arbitrary order ($r \geq 2$) Fibonacci sequences, $\{u_n\}$ and $\{v_n\}$:

$$\left.\begin{array}{ll} u_n = \sum_{j=1}^{r}(-1)^{j+1}P_j u_{n-j} & n > 0 \\ u_n = 1 & n = 0 \\ u_n = 0 & n < 0 \end{array}\right| \tag{1.1}$$

and

$$\left.\begin{array}{ll} v_n = \sum_{j=1}^{r}(-1)^{j+1}P_j v_{n-j} & n \geq r \\ v_n = \sum_{j=1}^{r} \alpha_j^n & 0 \leq n < r \\ v_n = 0 & n < 0 \end{array}\right| \tag{1.2}$$

where the P_j are arbitrary integers and the α_j are the roots, assumed distinct, of the auxiliary equation for the recurrence relations above, namely,

$$0 = x^r - \sum_{j=1}^{r}(-1)^{j+1}P_j x^{r-j}.$$

For example, when $r = 2$ we have $u_n = P_1 u_{n-1} - P_2 u_{n-2}$ with $u_0 = 1$, $u_1 = P_1$, $u_2 = P_1^2 - P_2$, and so on. These are referred to as the Lucas *fundamental* numbers (see [8]). When $r = 2$ the $\{v_n$ correspond to the Lucas *primordial* numbers with $v_0 = 2$, $v_1 = \alpha_1 + \alpha_2 = P_1$, $v_2 = \alpha_1^2 + \alpha_2^2 = P_1^2 - 2P_2$ and so on (see [5], Table 1).

n	0	1	2	3	$\cdots$
u_n	1	P_1	$P_1^2 - P_2$	$P_1^3 - 2P_1P_2 + P_3$	$\cdots$
v_n	r	P_1	$P_1^2 - 2P_2$	$P_1^3 - 3P_1P_2 + rP_3$	$\cdots$

Table 1: First four terms of $\{u_n\}$ and $\{v_n\}$

In [11] the ordinary generating function

$$\sum_{n=0}^{\infty} u_n x^n = \prod_{j=1}^{r}(1 - \alpha_j x)^{-1} \tag{1.3}$$

is used to show that

$$\sum_{n=0}^{\infty} u_n x^n = \exp\left(\sum_{m=1}^{\infty} v_m \frac{x^m}{m}\right) \tag{1.4}$$

thus suggesting a generalized Fibonacci polynomial $u_n(x)$ defined formally as

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right). \tag{1.5}$$

Then from (1.4) and (1.5) we get (1.6) and (1.7)

$$u_n(0) = u_n n! \tag{1.6}$$

and thus

$$\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} = e^{xt}\sum_{n=0}^{\infty} u_n(0)\frac{t^n}{n!} \tag{1.7}$$

by analogy with the polynomials of Bernoulli, Euler and Hermite (see [2, 9]). Other analogies with these polynomials can also be obtained in [12].

We also note that there are many other ways of defining Fibonacci polynomials and their generalizations in literature, (see [1, 3, 6]). The aim in this paper is to extend some of the results associated with (1.5) to congruences (see [7]). Some of these properties for Fibonacci numbers were explored in [13]. Daykin, Dresel and Hilton also obtained some similar results by combining the roots of the auxiliary equation to aid their study of the structure of a second order recursive sequence in a finite field (see [4]).

2. Fibonacci polynomials

We emphasize that the concern here is with the formal aspects of the theory and in the term-by-term differentiation of series we assume that conditions of continuity and uniform convergence are satisfied in the appropriate closed intervals. Thus a result we shall find useful is a recurrence relation for these Fibonacci polynomials

$$u_{n+1}(x) = xu_n(x) + \sum_{j=0}^{n} n^{\underline{j}} v_{j+1} u_{n-j}(x) \tag{2.1}$$

in which $n^{\underline{j}}$ is the falling factorial coefficient.

Proof of (2.1). Since

$$\sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!} = \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right)$$

and

$$\frac{\partial}{\partial t} \sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} u_{n+1}(x) \frac{t^n}{n!}$$

and

$$\frac{\partial}{\partial t}\left(\exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right)\right) = \left(x + \sum_{m=0}^{\infty} v_{m+1} t^m\right) \exp\left(xt + \sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right),$$

we have that

$$\begin{aligned} \sum_{n=0}^{\infty} u_{n+1}(x) \frac{t^n}{n!} &= \left(x + \sum_{m=0}^{\infty} v_{m+1} t^m\right) \sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} xu_n(x) \frac{t^n}{n!} + \left(\sum_{m=0}^{\infty} v_{m+1} t^m\right)\left(\sum_{n=0}^{\infty} u_n(x) \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} xu_n(x) \frac{t^n}{n!} + \sum_{n=0}^{\infty} \sum_{j=0}^{n} n^{\underline{j}} v_{j+1} u_{n-j}(x) \frac{t^n}{n!} \end{aligned}$$

which yields the required result on equating coefficients of t. □

When $x = 0$ this becomes

$$(n+1)\,u_{n+1} = \sum_{j=0}^{n} v_{j+1} u_{n-j} \tag{2.2}$$

since $n! = n^{\underline{j}}(n-j)!$. When $r = 2$ and $P_1 = -P_2 = 1$, equation (2.2) becomes the known (see [5])

$$nF_{n+1} = \sum_{j=0}^{n-1} L_{j+1} F_{n-j}.$$

Now from (1.5) it follows that

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x)\frac{t^n}{n!} &= \exp(xt)\exp\left(\sum_{m=1}^{\infty} v_m \frac{t^m}{m}\right) \\ &= \sum_{k=0}^{\infty} x^k \frac{t^k}{k!} \sum_{j=0}^{\infty} u_j t^j \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!} u_{n-k} x^k \frac{t^n}{n!}.\end{aligned}$$

So that on equating coefficients of t we get

$$u_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} u_{n-k} x^k \tag{2.3}$$

and with (1.6)

$$u_n(x) = \sum_{k=0}^{n} \frac{n!}{k!} \frac{u_{n-k}(0)}{(n-k)!} x^k$$

so that

$$u_n(x) = \sum_{k=0}^{n} \binom{n}{k} u_{n-k}(0) x^k. \tag{2.4}$$

Then

$$u_0(x) = u_0 = 1.$$

It is of interest to note another connection between these Fibonacci polynomials and the classical polynomials. We can write equation (2.4) in the suggestive form

$$u_n(x) = (x + u_n(0))^n \tag{2.5}$$

which is analogous to the well-known

$$B_n(x) = (x + B_n(0))^n \tag{2.6}$$

for the Bernoulli polynomials, and in which it is understood that after the expansion of the right hand sides of (2.1)and (2.2), terms of the form a^k are replaced by a_k as in the umbral calculus (see [10]).

3. Fibonacci polynomial congruences

We now use induction on t and n to prove that

$$u_{n+tm}(x) \equiv u_n(x)\,(u_m(x))^t \pmod{m} \tag{3.1}$$

Proof of (3.1). When $t = 0$, the result is obvious for all n. When $t = 1$ and $n = 1$, we note from (2.1) that $u_1(x) = x + v_1$, and

$$\begin{aligned} u_{m+1}(x) &= (x+v_1)u_m(x) + \sum_{j=1}^{m} m^{\underline{j}} v_{j+1} u_{m-j}(x) \\ &\equiv (x+v_1)u_m(x) \pmod{m} \\ &\equiv u_1(x)u_m(x) \pmod{m}. \end{aligned}$$

Assume the result is true for $t = 1$, and $n = 1, 2, \cdots, s$; that is,

$$u_{m+n}(x) \equiv u_m(x)u_n(x) \pmod{m}, \qquad n = 1, 2, \cdots, s.$$

Then

$$\begin{aligned} u_{m+s+1}(x) &= (x+v_1)u_{m+s}(x) + \sum_{j=1}^{m+s} (m+s)^{\underline{j}} v_{j+1} u_{m+s-j}(x) \\ &\equiv (x+v_1)u_{m+s}(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{m+s-j}(x) \pmod{m} \end{aligned}$$

since

$$\begin{aligned} (m+s)^{\underline{j}} &= (m+s)(m+s-1)\cdots(m+s-j+1) \\ &\equiv s(s-1)\cdots(s-j+1) \pmod{m}. \end{aligned}$$

Thus

$$\begin{aligned} u_{m+s+1}(x) &\equiv (x+v_1)u_m(x)u_s(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{s-j}(x)u_m(x) \pmod{m} \\ &= u_m(x)\left((x+v_1)u_s(x) + \sum_{j=1}^{s} s^{\underline{j}} v_{j+1} u_{s-j}(x)\right) \pmod{m} \\ &= u_m(x)u_{s+1}(x) \pmod{m}. \end{aligned}$$

So when $t = 1$, for all n,

$$u_{n+m}(x) \equiv u_n(x)\,(u_m(x))^1 \pmod{m},$$

when $t = 2$, for all n,

$$u_{n+2m}(x) \equiv u_n(x)\,(u_m(x))^2 \pmod{m}.$$

Assume the result holds for $t = 3, 4, \cdots, k$:

$$\begin{aligned} u_{n+(k+1)m}(x) &\equiv u_{n+km}(x)u_m(x) \pmod{m} \\ &\equiv \left(u_n(x)\left(u_m(x)\right)^k\right)u_m(x) \pmod{m} \\ &\equiv u_n(x)\left(u_m(x)\right)^{k+1} \pmod{m} \end{aligned}$$

and this completes the proof of (3.1). □

As a simple illustration of (3.1), if $r = 2$, $m = 2$, $n = 3$, and $t = 1$, then from (2.3)

$$\begin{aligned} u_5(x) &= \sum_{k=0}^{5} \frac{5!}{k!} u_{5-k} x^k \\ &\equiv \frac{5!}{4!} u_1 x^4 + \frac{5!}{5!} u_0 x^5 \pmod{2} \\ &\equiv 5x^4 + x^5 \pmod{2} \\ &\equiv x^4 + x^5 \pmod{2} \end{aligned}$$

and similarly,

$$\begin{aligned} u_3(x) &\equiv 3x^2 + x^3 \pmod{2} \\ &\equiv x^2 + x^3 \pmod{2} \\ u_2(x) &\equiv x^2 \pmod{2} \end{aligned}$$

or

$$u_5(x) \equiv u_3(x)\left(u_2(x)\right) \pmod{2}.$$

It follows that for $n = 2, 3, \cdots$,

$$u_n(x)\left(u_m(x)\right)^t - u_{n+tm}(x) = \sum_{j=-ntm}^{tn} B_j(n)u_{n+j}(x) \tag{3.2}$$

in which the $B_j(n) = B_j(n; t, m)$ are also polynomials in n with integer coefficients modulo m. We may also assume that in the summation $B_j(n) = 0$ $(-ntm \leq j < -n)$.

4. Conclusion

The $\{u_s(0)\}$ satisfy recurrence relations with variable coefficients:

$$\begin{aligned} u_n(0) &= n!u_n \\ &= n!\sum_{j=1}^{r}(-1)^{j+1}P_j u_{n-j} \end{aligned}$$

$$= \sum_{j=1}^{r} (-1)^{j+1} P_j \frac{n!}{(n-j)!} u_{n-j}(0).$$

This may be worthy of further separate investigation, as may two-dimensional polynomials of the form $\{u_{m,n}(x)\}$ to correspond with horizontal and vertical tilings of Fibonacci numbers.

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