GENERALIZED PELL NUMBERS AND POLYNOMIALS

A.G. Shannon and A.F. Horadam

1. INTRODUCTION

We define sequences of *generalized Pell numbers* with the notation introduced by Horadam [6]

$$\{P_{r,n}\} \equiv \{P_{r,n}(1, 2^r; 2^r, -1)\} \tag{1.1}$$

and by the second order recurrence relation

$$P_{r,n} = 2^r P_{r,n-1} + P_{r,n-2}, \quad n > 2 \tag{1.2}$$

with initial conditions $P_{r,1} = 1, P_{r,2} = 2^r$, (though we can allow for $n \leq 0$). For instance,

$$\{P_{0,n}(1, 1; 1, -1\} \equiv \{F_n\}, \tag{1.3}$$

$$\{P_{1,n}(1, 2; 2, -1\} \equiv \{P_n\}, \tag{1.4}$$

the ordinary Fibonacci and Pell sequences respectively. We also define an allied sequence

$$\{Q_{r,n}\} \equiv \{P_{r,n}(2^r, 2^r + 2; 2^r, -1)\}, \tag{1.5}$$

so that

$$\{Q_{0,n}(1, 3; 1, -1\} \equiv \{L_n\}, \tag{1.6}$$

the ordinary Lucas numbers. Note that $P_{r,0} = 0, Q_{r,0} = 2^r + 2 - 2^{2r}$, if we extend the definition to $n = 0$.

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It is the intention of this paper to explore the number theoretic and combinatorial properties of these numbers and related polynomials $p_r(x)$ and $p_{r,n}(x)$ defined below in (3.2) and (3.5). Further, it is shown that any polynomial can be expressed in terms of related generalized Pell polynomials.

The $\{P_{r,n}\}$ arose in [9] in the combinatorial matrix defined by

$$S_{p,q}(n;2) = [s_{i,j}(n)]_{n\times n} \tag{1.7}$$

where

$$s_{i,j}(n) = \binom{j-1}{n-i} p^{i+j-n-1} q^{n-i}, \tag{1.8}$$

and

$$S'_{2^r,-1} S_{2^r,-1} = S_{2^{r+1},-1}, r \geq 0, \tag{1.9}$$

where

$$S'_{2^r,-1} = S_{2^r,-1} E \tag{1.10}$$

in which E is the unit matrix with rows reversed, that is, the elementary (self-inverse) matrix

$$E = [e_{i,j}]_{n\times n}$$

$$e_{i,j} = \begin{cases} 1 & \text{if } j = n-i+1 \\ 0 & \text{otherwise.} \end{cases}$$

An example of (1.9) when $r = 1$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 12 & 4 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 1 & 4 & 12 \\ 1 & 2 & 4 & 8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 12 \\ 0 & 1 & 8 & 48 \\ 1 & 4 & 16 & 64 \end{pmatrix} \tag{1.11}$$

The falling (from left to right) diagonal sums, starting at the bottom, in the S matrices (when considered as infinite in extent) are generalized Pell numbers $\{P_{r,n}\}$. For instance, in (1.11) we have $\{1, 2, 5, 12, \ldots\}$, the ordinary Pell numbers, on the left, and $\{1, 4, 17, 72, \ldots\}$ on the right, which is $\{P_{2,n}\}$.

2. GENERAL TERMS

The auxiliary equation associated with the recurrence relation (1.2) is given by

$$\lambda^2 - 2^r\lambda - 1 = 0 \tag{2.1}$$

which has roots given by

$$\alpha = \frac{2^r + \Delta}{2}, \text{ and } \beta = \frac{2^r - \Delta}{2}, \tag{i}$$

in which

$$\Delta = \alpha - \beta = \sqrt{(4 + 2^{2r})}. \tag{ii}$$

We note that

$$\alpha + \beta = 2^r, \ \alpha\beta = -1. \tag{iii}$$

The Binet forms of the general terms are

$$P_{r,n} = \frac{\alpha^n - \beta^n}{\Delta}, \tag{iv}$$

$$Q_{r,n} = \alpha^n + \beta^n. \tag{v}$$

Using (i)-(v), we then get identities analogous to the well-known results for Fibonacci, Pell and Lucas numbers:

$$Q_{r,n} = P_{r,n-1} + P_{r,n+1}, \tag{2.2}$$
$$P_{r,2n} = P_{r,n}Q_{r,n}, \tag{2.3}$$
$$\Delta^2 P_{r,n} = Q_{r,n+1} + Q_{r,n-1}, \tag{2.4}$$
$$P_{r,n+1}P_{r,n-1} - P_{r,n}^2 = (-1)^n, \tag{2.5}$$

$$Q_{r,n+1}Q_{r,n-1} = (-1)^{n-1}\Delta^2. \tag{2.6}$$

Since the proofs are trivial, they will be omitted.

Combining (2.2) and (2.4), we may introduce the concept of interrelated associated sequences.

Definition: $P_{r,n}^{(k)}$ and $Q_{r,n}^{(k)}$, the k^{th} *associated sequences* of $P_{r,n}$ and $Q_{r,n}$ respectively, are defined by

$$P_{r,n}^{(k)} = P_{r,n+1}^{(k-1)} + P_{r,n-1}^{(k-1)}, \tag{2.7}$$

$$Q_{r,n}^{(k)} = Q_{r,n+1}^{(k-1)} + Q_{r,n-1}^{(k-1)}, \tag{2.8}$$

with $P_{r,n}^{(0)} \equiv P_{r,n}, \ Q_{r,n}^{(0)} = Q_{r,n}$ so that

$$P_{r,n}^{(1)} = Q_{r,n} \quad \text{by (2.2)}, \tag{2.9}$$

$$Q_{r,n}^{(1)} = \Delta^2 P_{r,n} \text{ by (2.4).} \tag{2.10}$$

Some leisurely substitutions using (2.2) and (2.4) lead readily to the conclusions that

$$P_{r,n}^{(2m)} = \Delta^{2m} P_{r,n}, \quad P_{r,n}^{(2m+1)} = \Delta^{2m} Q_{r,n}, \tag{2.11}$$

$$Q_{r,n}^{(2m)} = \Delta^{2m} Q_{r,n}, \quad Q_{r,n}^{(2m+1)} = \Delta^{2m+2} P_{r,n}. \tag{2.12}$$

Succinctly, we write

$$P_{r,n}^{(2m+1)} = Q_{r,n}^{(2m)}, \tag{2.13}$$

$$Q_{r,n}^{(2m+1)} = \Delta^2 P_{r,n}^{(2m)}. \tag{2.14}$$

3. GENERATING FUNCTIONS

For notational convenience we let

$$p_{r,n} = P_{r,n+1}. \tag{3.1}$$

Define formally

$$p_r(x) = \sum_{n=0}^{\infty} p_{r,n} x^n. \tag{3.2}$$

Then it can be shown from (1.2) that the generating function for $p_r(x)$ is

$$p_r(x) = \frac{1}{1 - 2^r x - x^2}. \tag{3.3}$$

Theorem 1:

$$p_r(x) = \exp\left(\sum_{m=1}^{\infty} Q_{r,m} \frac{x^m}{m} \right). \tag{3.4}$$

Proof:

$$\begin{aligned} \ln p_r(x) &= -\ln(1-\alpha x)(1-\beta x) \text{ using (i), (iii)} \\ &= -\ln(1-\alpha x) - \ln(1-\beta x) \\ &= \sum_{m=1}^{\infty} \frac{\alpha^m x^m}{m} + \sum_{m=1}^{\infty} \frac{\beta^m x^m}{m} \\ &= \sum_{m=1}^{\infty} (\alpha^m + \beta^m) \frac{x^m}{m} \\ &= \sum_{m=1}^{\infty} Q_{r,m} \frac{x^m}{m}, \text{ as required.} \end{aligned}$$

We next define a type of *generalized Pell polynomial*, $p_{r,n}(x)$, by means of an exponential generating function which has the form of a Sheffer generating function [4]:

$$\sum_{n=0}^{\infty} p_{r,n}(x)\frac{t^n}{n!} = e^{xt}p_r(t). \tag{3.5}$$

So

$$p_{r,n} = p_{r,n}(0)/n!. \tag{3.6}$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} p_{r,n}(x)\frac{t^n}{n!} &= e^{xt}\sum_{n=0}^{\infty} p_{r,n}t^n \\ &= e^{xt}\sum_{n=0}^{\infty} p_{r,n}(0)\frac{t^n}{n!}\end{aligned} \tag{3.7}$$

analogous to the classical polynomials

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2xt}\sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!}, \tag{3.8}$$

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = e^{xt}\sum_{n=0}^{\infty} B_n(0)\frac{t^n}{n!}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = e^{xt}\sum_{n=0}^{\infty} E_n(0)\frac{t^n}{n!}, \tag{3.10}$$

of Hermite, Bernoulli and Euler respectively (Andrews et al, 1999).

4. POLYNOMIAL PROPERTIES

The Bernoulli polynomials can be expressed in the *umbral calculus* [7] by

$$B_n(x) = (x + B(0))^n$$

in which, after expansion of the binomial, a superscript is replaced by a subscript (and where $B_n(0) = B_n$). Similarly,

$$p_{r,n}(x) = (x + p_r(0))^n. \tag{4.1}$$

Theorem 2:

$$p_{r,n}(x) = \sum_{k=0}^{n} \binom{n}{k} p_{r,n-k}(0) x^k. \tag{4.2}$$

Proof:

$$\sum_{n=0}^{\infty} p_{r,n}(x)\frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \sum_{j=0}^{\infty} p_{r,j} t^j \text{ from (3.2), (3.5)}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!}{k!} p_{r,n-k} \frac{x^k t^n}{n!},$$

so

$$p_{r,n}(x) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(n-k)! p_{r,n-k}(0) x^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} p_{r,n-k}(0) x^k, \text{ by (3.6), (3.7) as required (as in (4.1)).}$$

The first few expressions for $p_{r,n}(x)$ are set out in Table 1. The coefficients for non-zero $p_{r,n}(x)$ are elements of sequences which have entries in Sloane and Plouffe [12].

n	$p_{r,n}(x)$
0	$p_{r,0}$
1	$p_{r,0}x + p_{r,1}$
2	$p_{r,0}x^2 + 2p_{r,1}x + 2p_{r,2}$
3	$p_{r,0}x^3 + 3p_{r,1}x^2 + 6p_{r,2}x + 6p_{r,3}.$

Table 1. The first few expressions for $p_{r,n}(x)$

On the assumption of continuity and uniform convergence in the appropriate closed intervals, $p_{r,n}(x)$ is an *Appell polynomial* because

$$\frac{\partial}{\partial x}\sum_{n=0}^{\infty} p_{r,n}(x)\frac{t^n}{n!} = te^{xt}p_r(t)$$

$$= t\sum_{n=0}^{\infty} p_{r,n}(x)\frac{t^n}{n!}$$

$$= \sum_{n=1}^{\infty} np_{r,n-1}(x)\frac{t^n}{n!}$$

which yields the Appel set criterion:

$$p'_{r,n}(x) = np_{r,n-1}(x), \ n = 1,2,3,\ldots. \tag{4.3}$$

Differentiating t times, we obtain

$$p^{(t)}_{r,n}(x) = \frac{n!}{(n-t)!}p_{r,n-t}(x). \tag{4.4}$$

The *differential equation* for $p_{r,n}(x)$ is now readily obtained.

Theorem 3:

$$p''_{r,n}(x) - (n-1)p'_{r,n-1}(x) = (n-1)p_{r,n-2}(x) = 0. \tag{4.5}$$

Proof:

From (4.3) we have that

$$\begin{aligned} p''_{r,n}(x) &= n(n-1)p_{r,n-2}(x) \\ &= (n-1)[(n-1)p_{r,n-2}(x) + p_{r,n-2}(x)] \\ &= (n-1)p'_{r,n-1}(x) + (n-1)p_{r,n-2}(x) \text{ as required.} \end{aligned}$$

Similarly from (4.3) we can obtain an *integration formula*

$$\int_0^x p_{r,n}(x) = \frac{p_{r,n+1}(x) - p_{r,n+1}(0)}{n+1}. \tag{4.6}$$

The $p_{r,n}(x)$ are not orthogonal since Shohat [10] has proved that the only system of orthogonal polynomials which is an Appell polynomial sequence is that which is reducible to the Hermite polynomials by a linear transformation. The $p_{r,n}(x)$ are related to the Hermite polynomials by the result in (4.7).

Theorem 4:

$$\sum_{m=0}^{\infty} H_m(x) p_{r,n}(y) \frac{t^m}{m!} = \exp(2xyt - y^2t^2) \sum_{n=0}^{\infty} H_n(x - yt) p_{r,n}(0) \frac{t^n}{n!}. \qquad (4.7)$$

Proof: We use the known result [1]:

$$\sum_{n=0}^{\infty} H_{m+n}(x) \frac{y^n}{n!} = \exp(2xy - y^2) H_m(x - y)$$

so that the right hand side of (4.2)

$$\begin{aligned} e^{2xyt-y^2t^2} \sum_{n=0}^{\infty} H_n(x-yt) p_{r,n}(0) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{m+n}(x) p_{r,n}(0) \frac{y^m t^{m+n}}{m!n!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} H_{m+n}(x) p_{r,n}(0) \binom{m+n}{n} \frac{y^m t^{m+n}}{(m+n)!} \\ &= \sum_{m=0}^{\infty} H_m(x) \left(\sum_{n=0}^{m} \binom{m}{n} p_{r,n}(0) y^{m-n} \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} H_m(x) p_{r,m}(y) \frac{t^m}{m!}, \text{ as required (by Theorem 2).} \end{aligned}$$

The $p_{r,n}(x)$ are not of binomial type [8] because

$$\begin{aligned} p_{r,n}(x+y) &= (x + y + p_r(0))^n \\ &\neq \sum_{i=0}^{n} \binom{n}{i} (x + p_r(0))^i (y + p_r(0))^{n-i} \\ &= \sum_{i=0}^{n} \binom{n}{i} p_{r,i}(x) p_{r,n-i}(y). \end{aligned}$$

We can also obtain an *addition formula.*

Theorem 5:

$$p_{r,n}(x+y) = \sum_{k=0}^{n}\sum_{j=0}^{k} p_{r,n-k} n! \frac{x^j}{j!}\frac{y^{k-j}}{(k-j)!}. \tag{4.8}$$

Proof: From (3.2) and (3.5)

$$\sum_{n=0}^{\infty} p_{r,n}(x+y)\frac{t^n}{n!} = \left[1 + (x+y)\frac{t}{1!} + (x+y)^2\frac{t^2}{2!} + (x+y)^3\frac{t^3}{3!} + \dots\right]\sum_{n=0}^{\infty} p_{r,n}t^n$$

$$= \sum_{m=0}^{\infty}(x+y)^m\frac{t^m}{m!}\sum_{n=0}^{\infty} p_{r,n}t^n$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} p_{r,n}\frac{t^{n+m}}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\binom{m}{k}x^k y^{m-k} p_{r,n-m}\frac{t^{n}}{m!} \text{ by changing the order of summation,}$$

so

$$p_{r,n}(x+y) = \sum_{m=0}^{n}\sum_{k=0}^{m}\frac{n!}{k!(m-k)!}x^k y^{m-k} p_{r,n-m}, \text{ as required, on equating coefficients of } t.$$

Special Cases:

$$(i)\quad p_{r,n}(2x) = \sum_{m=0}^{n}\sum_{k=0}^{m} n! p_{r,n-m}\frac{x^k}{k!}\frac{y^{m-k}}{(m-k)!} (\textit{duplication formula}),$$

$$(ii)\quad p_{r,n}(x+1) = \sum_{m=0}^{n}\sum_{k=0}^{m}\frac{n!}{(m-k)!} p_{r,n-m}\frac{x^k}{k!},$$

$$(iii)\quad p_{r,n}(0) = \sum_{m=0}^{n} n! p_{r,n-m}(x)\frac{x^m}{m!}.$$

More generally,

$$p_{r,n}(tx) = \sum_{m=0}^{n} \sum_{k=0}^{m} n! \binom{m}{k} p_{r,n-m}(t-1) \frac{x^m}{m!} \text{ (\textit{multiplication formula}).}$$

Further investigations could be made of properties analogous to those of other classical polynomials, such as Jacobi and Laguerre polynomials [13].

5. COMBINATORIAL PROPERTIES

A composition of the positive integer n is a vector $(a_1, a_2, \ldots, a_k)$, the components of which are the positive integers such that $a_1 + a_2 + \cdots + a_k = n$ [3]. If the vector has order k, then the composition is a *k-part composition.* In what follows $\gamma(n)$ indicates summation over all the compositions $(a_1, a_2, \ldots, a_k)$ of n, the number of components being variable [5]. Let

$$R_n = \sum_{\gamma(n)} \frac{(-1)^{k-1}}{k} p_{r,a_1} \cdots p_{r,a_k}. \tag{5.1}$$

Then formally

$$\begin{aligned}
\sum_{n=1}^{\infty} R_n x^n &= \sum_{n=1}^{\infty} \sum \gamma(n) \frac{(-1)^{k-1}}{k} p_{r,a_1} \cdots p_{r,a_k} x^n \\
&= \sum_{k=1}^{\infty} - \left(- \sum_{n=1}^{\infty} p_{r,n} x^n \right)^k / k \\
&= \ln \left(1 + \sum_{n=1}^{\infty} p_{r,n} x^n \right) \\
&= \ln \left(\sum_{n=1}^{\infty} p_{r,n} x^n \right),
\end{aligned}$$

that is,

Theorem 6:

$$\sum_{n=1}^{\infty} p_{r,n} x^n = \exp \left(\sum_{n=1}^{\infty} R_n x^n \right). \tag{5.2}$$

This, from (3.2) and (3.4), is satisfied by

$$R_n = \frac{1}{n} Q_{r,n}. \tag{5.3}$$

Thus,

Theorem 7:

$$Q_{r,n} = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} p_{r,a_1} \dots p_{r,a_k}. \tag{5.4}$$

When $r = 0$, we find for the Fibonacci and Lucas numbers that

$$L_n = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} f_{a_1} \dots f_{a_k}, \text{using (1.3), (1.4)} \tag{5.5}$$

in which $f_n = F_{n+1}$. For instance, when $n = 3$,

$$\sum_{\gamma(3)} (-1)^{k-1} \frac{3}{k} f_{a_1} \dots f_{a_k} = -\frac{3}{2} f_1 f_2 - \frac{3}{2} f_2 f_1 + \frac{3}{1} f_3 + \frac{3}{3} f_1 f_1 f_1$$

$$= -3 - 3 + 9 + 1 = 4 = L_3.$$

6. CONCLUDING COMMENTS

Theorem 8: Any polynomial can be expressed in terms of the generalized Pell polynomials.

Proof: From (3.4) and (3.5) we have that

$$\exp xt = \exp\left(-\sum_{m=1}^{\infty} Q_{r,m} \frac{x^m}{m}\right) \sum_{n=0}^{\infty} p_{r,n}(x) \frac{t^n}{n!}$$

$$= \frac{1}{p_r(x)} \sum_{n=0}^{\infty} p_{r,n}(x) \frac{t^n}{n!}$$

$$= (1 - 2^r x - x^2) \sum_{n=0}^{\infty} p_{r,n}(x) \frac{t^n}{n!};$$

on equating coefficients of t^n we get

$$x^n - p_{r,n}(x) - 2^r n p_{r,n-1}(x) - n(n-1)p_{r,n-2}(x). \tag{6.1}$$

For example, when $n = 2$, from Table 1,

$$\begin{aligned} p_{r,2}(x) - 2^{r+1}p_{r,1}(x) - 2p_{r,0}(x) &= (x^2 + 2^{r+1}x + 2^{2r+1} + 2) - (2^{r+1}x + 2^{2r+1}) - 2 \\ &= x^2. \end{aligned}$$

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