



Bijections and the Riordan group

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Abstract

One of the cornerstone ideas in mathematics is to take a problem and to look at it in a bigger space. In this paper we examine combinatorial sequences in the context of the Riordan group. Various subgroups of the Riordan group each give us a different view of the original sequence. In many cases this leads to both a combinatorial interpretation and to ECO rewriting rules. In this paper we will concentrate on just four of the subgroups of the Riordan group to demonstrate some of the possibilities of this approach.

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1. Introduction

To define the Riordan group we need two generating functions, $g(z) = 1 + g_1z + g_2z^2 + \dots$ and $f(z) = f_1z + f_2z^2 + \dots$ with $f_1 \neq 0$. Now let $M = (m_{n,k})_{n,k \geq 0}$ be the infinite lower triangular matrix with nonzero entries on the main diagonal such that $m_{n,k} = [z^n](g(z)(f(z))^k)$. If these conditions apply we say that M is an element of the Riordan group and we use the notation $M = (g(z), f(z)) = (g, f)$.

Suppose we multiply $M = (g, f)$ by a column vector $(a_0, a_1, \dots)^T$ and the result is the column vector $(b_0, b_1, \dots)^T$. If the generating function for the sequence (a_0, a_1, \dots) is $A(z)$ and similarly, (b_0, b_1, \dots) has $B(z)$ as its generating function then we obtain

$$B(z) = g(z)A(f(z)). \tag{*}$$

If we consider the second matrix one column at a time this leads to the group multiplication rule for the Riordan group which is

$$(g(z), f(z))(h(z), l(z)) = (g(z)h(f(z)), l(f(z))).$$

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This shows us that the identity is $I=(1,z)$, the usual matrix identity, and that

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z) \right),$$

where $\bar{f}(z)$ is the compositional inverse of $f(z)$. Two examples of Riordan matrices besides the identity matrix are the Pascal matrix $P=(1/(1-z), z/(1-z))$ and the Fibonacci matrix

$$F := \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 2 & 1 & & \cdots & \\ 0 & 0 & 1 & 3 & 1 & & \\ 0 & 0 & 0 & 3 & 4 & & \\ & & \cdots & & & & \end{bmatrix} = (1, z(1+z)).$$

Many more examples will appear in this paper but for more examples of the Riordan group and its uses in proving and inverting binomial identities see [10,12].

Given an array of numbers a very natural thing to do is to figure out the row sums. For instance, the sum of the n th row of Pascal’s triangle is 2^n , one of the key facts in mathematics. For any Riordan group matrix we have the following observation. To find the row sum we can multiply by the column vector $(1, 1, 1, \dots)^T$. This sequence has the generating function $1/(1-z)$ and by (*) the generating function for the row sums is $g(z)/(1-f(z))$.

What we plan to do is to reverse the process. We start with a sequence that we want as row sums, call it the target sequence, and one of the subgroups. From there we first solve for g and f , and try to work this back to the bijective level.

For instance, one problem we discuss is finding a combinatorial meaning for the coefficients in the matrix below:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 4 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 6 & 0 & 7 & 0 & 1 & 0 & 0 & \\ 0 & 31 & 0 & 10 & 0 & 1 & 0 & \\ 53 & 0 & 65 & 0 & 13 & 0 & 1 & \\ & & \cdots & & & & & \end{bmatrix}.$$

Note that the matrix is in the checkerboard subgroup and that the row sums are indeed the Catalan numbers.

- In our examples we will make use of the following generating functions:
- $C(z) = C = 1 + zC^2 = (1 - \sqrt{1 - 4z})/2z = 1 + z + 2z^2 + 5z^3 + \dots = \sum_{n=0}^{\infty} [1/(n+1)] \binom{2n}{n} z^n$ is the generating function for the Catalan numbers.
 - $r(z) = r = 1 + z(r + r^2) = 1 + 2z + 6z^2 + 22z^3 + 90z^4 + \dots$ is the generating function for the big Schröder numbers.

- $m(z) = 1 + zm + z^2m^2 = 1 + z + 2z^2 + 4z^3 + 9z^4 + \dots$ is the generating function for the Motzkin numbers.
- $F(z) = C/(1+zC) = 1 + z^2 + 2z^3 + 6z^4 + 18z^5 + \dots$ is the generating function for the Fine numbers. See [6] for a survey of the Fine numbers. The functions C, r , and m are discussed in many places including Stanley's Combinatorial Enumeration [13, Vol. 2, Chap. 6]. The Catalan numbers count many things. The two that we will mention are Dyck paths and ordered trees. A Dyck path of length $2n$ is a path from $(0,0)$ to $(2n,0)$ using as steps $(1,1)$ and $(1,-1)$ with the added restriction that the path never goes below the x -axis. If we add the possible step $(1,0)$ we get the Motzkin numbers while the addition of $(2,0)$ as a possible step yields the big Schröder numbers. The Catalan numbers count the number of permutations achievable with a stack and the Schröder numbers count the number of permutations achievable with an input (or output) restricted double ended queue (see [7]).

2. Subgroups

Here is a list of some important subgroups of the Riordan group.

1. The Appell subgroup = $\{(g(z), z)\}$.
 - The c -Appell subgroup = $\{(g(z), cz) | c \neq 0\}$. A c -Appell matrix times a d -Appell matrix is a cd -Appell matrix.
2. The associated subgroup = $\{(1, f(z))\}$.
3. The Bell subgroup = $\{(g(z), zg(z))\} = \{(f(z)/z, f(z))\}$.
 - The c -Bell subgroup = $\{(g(z), czg(z)) | c \neq 0\} = \{(f(z)/z, cf(z)) | c \neq 0\}$. A c -Bell matrix times a d -Bell matrix is a cd -Bell matrix.
4. The checkerboard subgroup

$$= \{(g(z), f(z)) : g \text{ an even function, } f \text{ an odd function}\}.$$

5. The hitting time subgroup = $\{(zf'(z)/f(z), f(z))\}$.
6. The stochastic subgroup which is the stabilizer of the column vector $(1, 1, 1, \dots)^T$. Alternately those matrices with row sums all equal to 1.

Note: The stabilizer subgroup of $(1, 0, 0, 0, \dots)^T$ is the associated subgroup.

Of these the only normal subgroup is the Appell subgroup. Since $(g(z), f(z)) = (g(z), z)(1, f(z))$ we see that the Riordan group is the semidirect product of the Appell and associated subgroups. Similarly $(g(z), f(z)) = (zg(z)/f(z), z)(f(z)/z, f(z))$ shows the Riordan group as the semidirect product of the Appell subgroup and the Bell subgroup.

There are two more concepts that we will use: the Stieltjes transform matrix and ECO succession rules.

Let L be a matrix and then let \bar{L} be the matrix obtained from L by removing the top row of L and then moving all the remaining rows of L up one row. Then the Stieltjes matrix, S_L , of L is the solution of the matrix equation $LS_L = \bar{L}$. If L is nonsingular then a unique solution exists, $S_L = L^{-1}\bar{L}$. Since for us L will always be a lower triangular matrix with nonzero entries on the main diagonal this will always be the case. As an

example the Pascal matrix P has as its Stieltjes transform matrix

$$S_P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 1 \\ & & \dots & & & \end{bmatrix}.$$

A generating tree can be described by means of an ECO-system [3] of the form

$$\left\{ \begin{array}{l} (r), \\ (k) \rightarrow (c_1)(c_2)\cdots(c_k) \end{array} \right\},$$

where $r, k, g_i \in \mathbb{N}$. This means first that the root (or axiom) has r sons and then that for all k , any node labeled k will have k descendants and they will have the labels c_1, c_2, \dots, c_k . We then count the number of nodes at each level in the tree where the root is at level 0. One example is

$$\left\{ \begin{array}{l} (2), \\ (k) \rightarrow (2)(3)\cdots(k)(k+1) \end{array} \right\}$$

level 0 has (2), level 1 contains (2) and (3), level 3 contains (2), (3), (2), (3) and (4) which we shorten to $(2)^2(3)^2(4)$. Level 4 consists of $(2)^5(3)^5(4)^3(5)$ and so on. There is a close connection with automata and regular languages that we will not pursue here but see [2,5] for more on this connection.

We will deal mostly with the first three subgroups listed and will connect them to certain bijections. We pick a target sequence that we want to learn more about and then pick a subgroup. We then see which element in the subgroup yields the target sequence as its row sums.

2.1. The Appell subgroup

As an example let the target sequence be the Catalan numbers 1, 1, 2, 5, 14, 42, ... and we look at the Appell subgroup decomposition:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 9 & 3 & 1 & 0 & 1 & 0 & 0 & \dots \\ 28 & 9 & 3 & 1 & 0 & 1 & 0 \\ 88 & 28 & 9 & 3 & 1 & 0 & 1 \\ & & \dots & & & & & \end{bmatrix}.$$

While computing easily produces the first few rows we need the Riordan group to compute the generating functions. In these computations we interchange freely between a sequence and its generating function. We find that

$$(g(z), z) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} = (g(z), z) \frac{1}{1-z} = \frac{g(z)}{1-z} = C(z),$$

So that

$$\begin{aligned} g(z) &= C(z) - zC(z) = 1 + zC^2 - zC \\ &= 1 + zC(C - 1) = 1 + zC(zC^2). \end{aligned}$$

Thus

$$A = (C - zC, z) = (1 + z^2C^3, z). \tag{2.1}$$

We can immediately find a combinatorial interpretation for the coefficients. In the i th column, $i=0, 1, 2, \dots$ the entries count the number of Dyck paths starting with exactly i hills. A hill is a consecutive up and down step, starting and ending on the x -axis. When $i=0$ we have g as the generating function and we are counting Dyck paths that do not start with a hill. If we prefer an interpretation in terms of ordered trees then the i th column counts trees such that exactly the first i subtrees at the root, say from the left, consist of a single edge.

We can replace the Catalan numbers by any sequence. We will call any such sequence a target sequence and we denote its generating function as $T(z)$. Repeating the manipulations we've just done leads to

$$g(z) = T(z) - zT(z).$$

If our target sequence is the sequence of big Schröder numbers starting 1, 2, 6, 22, 90, 394, ... then $g(z) = r - zr = 1 + zr^2 = 1 + z + 4z^2 + 16z^3 + 68z^4 + 304z^5 + \dots$. This follows immediately from the defining equation for the large Schröder generating function, $r = 1 + zr + zr^2$. The standard setting for the Schröder number is that of paths from $(0, 0)$ to $(2n, 0)$, never going below the x -axis, with possible steps, $\{U=(1, 1), D=(1, -1), L=(2, 0)\}$. Then $g(z)$ is the generating function for those paths that do not start with an initial L step. (Or those paths which do not start UD .)

Here is a table of some other Appell decompositions

T -sequence	$\mathbf{T}(z)$	$\mathbf{g} = \mathbf{T} - z\mathbf{T}$	Comments
1, 1, 2, 5, 14, ...	$C = \frac{1 - \sqrt{1 - 4z}}{2z}$	$C - zC = 1 + z^2 C^3$	Catalan
1, 2, 6, 20, 70, ...	$B = \frac{1}{\sqrt{1 - 4z}}$	$B - zB = \frac{1 - z}{\sqrt{1 - 4z}}$	Central binomial
1, 1, 3, 11, 45, ...	$\frac{r+1}{2}$	$1 + \frac{z(r^2 - 1)}{2}$	Little Schröder
1, 2, 6, 22, 90, ...	$r = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}$	$1 + zr^2$	Big Schröder
1, 3, 11, 45, 197, ...	$\frac{r-1}{2z}$	$\frac{r^2 + 1}{2}$	Bigger Schröder
1, 2, 5, 14, 42, ...	C^2	$C - zC = 1 + zC^3$	Big Catalan
1, 1, 2, 4, 9, 21, ...	m	$1 + zm^2$	Motzkin

For all these cases the i th column counts those paths starting with exactly i hills or level steps or whatever else starts and ends on the x -axis and is marked by z . We can find a generating tree for a given target sequence, T , by first finding the element, $L = (g, z)$ in the Appell subgroup with the target sequence T . Then find the Stieltjes matrix, S_L . For instance for the big Schröder numbers we have

$$\begin{aligned}
 L \cdot S_L &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & \dots \\ 16 & 4 & 1 & 1 & 0 \\ 68 & 16 & 4 & 1 & 1 \\ & & \dots & & & \dots \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 1 & 0 & \dots \\ 31 & 0 & 0 & 0 & 1 \\ 121 & 0 & 0 & 0 & 0 \\ & & \dots & & & \dots \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 \\ 16 & 4 & 1 & 1 & 0 & \dots \\ 68 & 16 & 4 & 1 & 1 \\ 304 & 68 & 16 & 4 & 1 & 1 \\ & & \dots & & & \dots \end{bmatrix} = \bar{L}.
 \end{aligned}$$

Next, we look at the row sums of S_L and call them r_1, r_2, r_3, \dots . We can then set up an ECO-system generating tree as follows:

$$\Omega: \begin{cases} (r_1), & \text{root or axiom,} \\ (r_k) \rightarrow (r_1)^{r_k - 1} (r_{k+1}). \end{cases}$$

For the big Schröder example we have been following the r_i -sequence starts 2, 4, 10, 32, ... and we have the following starting values:

Level	
0	(2)
1	(2)(4)
2	(2) ³⁺¹ (4)(10)
3	(2) ⁴⁺³⁺⁹ (4) ⁴ (10) ¹ (32) ¹

The exponents at level k are the entries in the k th row of L . To find the r_i we need a generating function for the $r_i - 1$. If we call this generating function R and again make use of (*) we get the equation $(g, z)R = gR = (g - 1)/z$ or $R = (g - 1)/zg$. In the example we have been following $g = r - zr = 1 + zr^2$ and

$$R = \frac{g - 1}{zg} = \frac{zr^2}{z(r - zr)} = \frac{r}{1 - z}$$

and indeed the sequence 1, 3, 9, 29, ... is the sequence of partial sums of the sequence of big Schröder numbers 1, 2, 6, 22, 90, 394,

2.2. The associated subgroup

The associated subgroup consists of the elements of the form $(1, f(z))$. If the target sequence has the generating function $T(z)$ then we have

$$(1, f(z)) \left(\frac{1}{1 - z} \right) = 1 \cdot \frac{1}{1 - f(z)} = T(z).$$

Thus $T(z) = 1 + f(z)T(z)$ or $T(z) = 1/(1 - f(z))$ and we have, in the case of random walks, a combinatorial interpretation; $f(z)$ is the generating function for the first return to the x -axis or for elevated paths. More generally, $T(z) = 1/(1 - f(z))$ is the ordinary generating function equivalent of the exponential formula where $f(z)$ is the generating function for the connected part.

T -sequence	$\mathbf{T}(z)$	\mathbf{f}	Comments
1, 1, 2, 5, 14, ...	$C = \frac{1 - \sqrt{1 - 4z}}{2z}$	zC	Catalan
1, 2, 6, 20, 70, ...	$B = \frac{1}{\sqrt{1 - 4z}}$	$2zC$	Central binomial
1, 1, 3, 11, 45, ...	$\frac{r+1}{2}$	zr	Little Schröder
1, 2, 6, 22, 90, ...	$r = \frac{1 - z - \sqrt{1 - 6z + z^2}}{2z}$	$z(1 + r)$	Big Schröder
1, 3, 11, 45, 197, ...	$\frac{r-1}{2z}$	$z(2 + r)$	Bigger Schröder
1, 1, 2, 4, 9, 21, ...	$m = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}$	$z(1 + mz)$	Motzkin
1, 1, 2, 3, 5, 8, ...	$F = \frac{1}{1 - z - z^2}$	$z(1 + z)$	Fibonacci

Each entry in this table can be expanded as a Riordan group matrix and often there are interesting things that can be said. For instance, the Motzkin example gives us the matrix

$$M = (m(n, k))_{n, k \geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 3 & 1 & 0 & 0 \dots \\ 0 & 4 & 6 & 6 & 4 & 1 & 0 \\ 0 & 9 & 13 & 13 & 10 & 5 & 1 \\ & & & \dots & & & \end{bmatrix}.$$

The entry $m(n, k)$ counts the number of single source directed animals consisting of n points of which k are on the x -axis. There is an interesting recursion

$$m(n+1, k) = m(n, k-1) + \sum_{j \geq 0} m(n-1, k-1+j).$$

This recursion can be called the Aigner hook formula (see [1]).

2.3. The Bell subgroup

There is an obvious isomorphism between the associated subgroup and the Bell subgroup. Take a matrix in the associated subgroup, remove the leftmost column and the top row, and you have an element in the Bell subgroup. Despite this, the combinatorics involved is different. Again letting $T(z) = T$ be the generating function for the target sequence we have

$$\begin{aligned} (g, zg) \left(\frac{1}{1-z} \right) &= T, \\ \frac{g}{1-zg} &= T, \\ g &= T(1-zg), \\ g(1+zT) &= T, \\ g &= \frac{T}{1+zT}. \end{aligned}$$

Here are some examples.

T -sequence	$T(z)$	g	Comments
1, 1, 2, 5, 14, ...	$C = \frac{1-\sqrt{1-4z}}{2z}$	$F = \frac{1}{z} \cdot \frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$	Catalan, Fine
1, 2, 6, 20, 70, ...	$B = \frac{1}{\sqrt{1-4z}}$	$\frac{\sqrt{1-4z}-z}{1-4z-z^2}$	Central binomial
1, 1, 3, 11, 45, ...	$\frac{r+1}{2}$	$1+zr$	Little Schröder
1, 2, 6, 22, 90, ...	$r = \frac{1-z-\sqrt{1-6z+z^2}}{2z}$	$\frac{1+r}{2}$	Big Schröder
1, 3, 11, 45, 197, ...	$\frac{r-1}{2z}$	$z(2+r)$	Bigger Schröder
1, 2, 4, 8, ..., 2^n , ...	$\frac{1}{1-2z}$	$\frac{1}{1-z}$	Pascal's triangle
1, 1, 2, 4, 9, ...	m	$\gamma = \frac{1+zm}{1+z}$	Motzkin, gamma

In all these examples the i th column counts the number of paths with exactly i hills or exactly i level steps.

Let us examine the example involving the Catalan and Fine number sequences:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 & 0 \\ 6 & 4 & 3 & 0 & 1 & 0 & 0 \\ 18 & 13 & 6 & 4 & 0 & 1 & 0 \\ 57 & 40 & 21 & 8 & 5 & 0 & 1 \\ & & & \dots & & & \end{bmatrix} = (F, zF).$$

We get the total number of hills by multiplying by the column vector $(0, 1, 2, 3, 4, \dots)^T$ which has as its generating function $z/(1 - z)^2$. Our Riordan group calculation is

$$\begin{aligned} (F, zF)z/(1 - z)^2 &= F \frac{zF}{(1 - zF)^2} = z \left(\frac{F}{1 - zF} \right)^2 \\ &= zC^2 = C - 1. \end{aligned}$$

Thus the total number of hills is

$C_n = [1/(n + 1)]\binom{2n}{n}$ for $n \geq 1$. Another way to say this is that the average number of hills over all Dyck paths of length $2n$ is 1 if each Dyck path is equally likely to be chosen. There is an interesting connection between this matrix (F, zF) and ECO succession rules that are discussed in [4]. Both examples from [4] are in the Bell subgroup.

Let us also look at an example involving the c -Bell subgroup. The 3-Bell matrix that has the very big Schröder numbers as its target is

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 9 & 0 & 0 & 0 \\ 6 & 12 & 0 & 27 & 0 & 0 \\ 26 & 36 & 54 & 0 & 81 & 0 \\ 114 & 168 & 162 & 216 & 0 & 243 \\ & & \dots & & & \end{bmatrix}.$$

The generating function for the left most column is $g = (1 - r)/z(1 - 3r)$.

By way of contrast we can take the 1-Bell decomposition of the same target sequence and we get.

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 & 0 \\ 22 & 16 & 6 & 1 & 0 & 0 \\ 90 & 68 & 30 & 8 & 1 & 0 \\ 394 & 304 & 146 & 48 & 10 & 1 \\ & & \dots & & & \end{bmatrix} = (r, zr).$$

Solving for X in the matrix equation $WX = V$ yields

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 0 \\ 4 & -12 & 9 & 0 & 0 & 0 \\ -8 & 36 & -54 & 27 & 0 & 0 \\ 16 & -96 & 216 & -216 & 81 & 0 \\ -32 & 240 & -720 & 1080 & -810 & 243 \\ & & & \dots & & \end{bmatrix} \dots = \left(\frac{1}{1+2z}, \frac{3z}{1+2z} \right).$$

The matrix X is interesting since it is the unique matrix which is in the stochastic subgroup and is also a 3-Bell matrix.

We can interpret the target sequence as the number of Dyck paths such that each peak at height 1 can be colored in any of 3 colors while all higher peaks are bicolored. The function g counts such paths with no peaks at height one.

3. The checkerboard example

For the checkerboard example mentioned in the introduction, we write T as a sum of its even and odd parts so that $T = E + \mathcal{O}$. We have that $g(z)$ is an even function and that $f(z)$ is an odd function. Thus $(g, f)1/(1-z) = g/(1-f) = E + \mathcal{O}$. Equating even and odd parts yields $g = (E - f\mathcal{O}) + (\mathcal{O} - fE)$ and $f = \mathcal{O}/E$ while $g = (E^2 - \mathcal{O}^2)/E$. For the Catalan numbers we have $T = C$ and after some manipulation we find

$$E = \frac{1}{2} \left(\frac{1 - \sqrt{1-4z}}{2z} + \frac{1 - \sqrt{1+4z}}{-2z} \right), \mathcal{O} = \frac{1}{2} \left(\frac{1 - \sqrt{1-4z}}{2z} - \frac{1 - \sqrt{1+4z}}{-2z} \right),$$

$$f = \frac{2 - (\sqrt{1-4z} + \sqrt{1+4z})}{\sqrt{1+4z} - \sqrt{1-4z}}, \text{ and } g = \frac{1}{z} \frac{(1 - \sqrt{(1+4z)})(1 - \sqrt{(1-4z)})}{\sqrt{(1-4z)} - \sqrt{(1+4z)}}.$$

The coefficients of $g(z)$ are the number of ordered trees such that every subtree at the root has an even number of edges. In fact the k th column with generating function $g(z)(f(z))^k$ $k=0, 1, 2, \dots$ counts the number of ordered trees with exactly k of the subtrees at the root with an odd number of edges. It also can be shown that

$$(g, f)^{-1} = \left(\frac{1}{1+z^2}, \frac{z(1-z^2)}{(1+z^2)^2} \right)$$

and this leads to a new ECO rewriting rule for the Catalan numbers.

$$\Omega \begin{cases} (1) & \text{root or axiom} \\ (1) & \rightarrow (2) \\ (2) & \rightarrow (1)(4) \\ (4) & \rightarrow (2)^3(6) \\ (4k+2) & \rightarrow (1)^2(4)^4(8)^4 \dots (4k-4)^4(4k)^3(4k+4) \\ (4k) & \rightarrow (2)^4(6)^4 \dots (4k-6)^4(4k-2)^3(4k+2). \end{cases}$$

Here is an intriguing coincidence. Consider the following game. At turn one you throw 4 balls numbered 1 through 4 into a basket. The gnome living in the basket throws 2 of the 4 out on the lawn. At turn two you throw ball 5,6,7, and 8 into the basket and the gnome throws 2 of the 6 balls now available to him out on the lawn. At each turn you throw in 4 balls and then the gnome throws out 2. The question is after n turns how many different arrangements of balls on the lawn are possible. It is easy to see that after one turn there are 6 possibilities and with a bit of work you find that after two turns there are 53 possibilities. Indeed after three turns there are 554 possibilities. These are the nonzero terms in g . This can be proven since the generating functions agree as shown in [9]. There are many other nice results in [8,9] can also be consulted for the case where 2 balls are thrown in at each turn and the gnome throws out 1. It remains a tantalizing question to find a bijection between the balls on the lawn in the (4,2) case and the number of trees with all subtrees having an even number of edges.

References

- [1] M. Aigner, Motzkin numbers, *European J. Combin.* 19 (1999) 663–675.
- [2] E. Barucci, A. Del Lungo, A. Frosini, S. Rinaldi, From rational functions to regular languages, Proc. 12th FPSAC, Moscow, 2000.
- [3] E. Barucci, A. Del Lungo, E. Pergola, R. Pinzani, ECO, A Methodology for the enumeration of combinatorial objects, *J. Difference Equations Appl.* 5 (1999) 40–52.
- [4] E. Barucci, E. Pergola, R. Pinzani, S. Rinaldi, ECO-systems for Dyck and Schröder paths, *PU.M.A.* 11 (2000) 401–407.
- [5] E. Barucci, S. Rinaldi, Some linear recurrences and their combinatorial interpretations by means of regular languages, *Theoret. Comput. Sci.* 255 (2001) 679–686.
- [6] E. Deutsch, L. Shapiro, A survey of the Fine numbers, *Discrete Math.* 241 (2001) 241–265.
- [7] D. Knuth, *The Art of Computer Programming*, Vol. 1, Addison-Wesley, Reading, MA, 1973.
- [8] C. Mallows, L. Shapiro, Balls on the lawn, *J. Integer Sequences* (electronic, www.research.att.com/~njas/sequences/JIS/) article 99.1.5.
- [9] D. Merlini, D. Rogers, R. Sprugnoli, M.C. Verri, The tennis ball problem, *J. Combin. Theory, Ser. A* 99 (2002) 73–88.
- [10] L.W. Shapiro, S. Getu, W.-J. Woan, L. Woodson, The Riordan Group, *Discrete Appl. Math.* 34 (1991) 229–239.
- [11] N.J.A. Sloane, S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, 1995. Also on line at www.research.att.com/~njas/sequences/index..
- [12] R. Sprugnoli, Riordan arrays and combinatorial sums, *Discrete Math.* 132 (1994) 267–290.
- [13] R.P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.