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The Riordan group

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Abstract

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Introduction

The central concept in this article is a group which we call the Riordan group. With the recent death of John Riordan this seems appropriate to name after him. The group is quite easily developed but unifies many themes in enumeration. The three applications discussed give a sampling of what can be done. The three applications are Euler's problem of the King walks, binomial and inverse identities, and a Bessel-Neumann expansion.

In Section 1 the group is defined, in Section 2 examples are discussed, in Sections 3-5 various applications are developed, while in Section 6 various references to the literature are given. The material in Section 4 is not entirely new but is included both because of its utility and the structure it lends to the world of combinatorial identities.

In this paper we restrict ourselves to ordinary generating functions while in [14] we discuss some applications that arise for exponential generating functions.

Notation and abbreviations.

- GF: generating function,
- $\mathbb{R}$: real numbers,
- $\mathbb{C}$: complex numbers,
- $\mathbb{N}$: natural numbers $= \{0, 1, 2, \dots\}$,
- $\mathbb{C}[[x]]$: ring of all formal power series over $\mathbb{C}$,
- $A := B$: A is defined to equal B.

For sequences we will use lower case letters $(a_0, a_1, a_2, \dots)$ and the corresponding capital letter will be used for the GF as in

$$A = A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

The central binomial coefficients $\{1, 2, 6, 20, 70, 252, \dots\} = \{\binom{2n}{n}\}_{n \ge 0}$ satisfy

$$\sum_{n \ge 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}. \tag{1}$$

The Catalan numbers $1/(n+1)\binom{2n}{n}$ have as their GF

$$C(x) = 1 + x[C(x)]^2 = \frac{1 - \sqrt{1-4x}}{2x}. \tag{2}$$

1. The Riordan group

Consider infinite matrices $M = (m_{ij})_{i,j \ge 0}$ with entries in $\mathbb{C}$, the complex numbers. Let $C_i(x) = \sum_{n \ge 0}^{\infty} m_{n,i} x^n$ be the generating function (GF) of the ith column of M.

We now make the crucial special assumption that

$$C_i(x) = g(x)[f(x)]^i, \tag{3}$$

where

$$g(x) = 1 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (4)

$$f(x) = x + f_2 x^2 + f_3 x^3 + \cdots.$$

In this case we write $M = (g(x), f(x))$ and say that $(g(x), f(x))$ is a *Riordan* matrix.

Now multiply M on the right by a column vector $(a_0, a_1, a_2, \dots)^{\mathrm{T}}$ and note that the resulting column vector $(b_0, b_1, b_2, \dots)^{\mathrm{T}}$ has the GF

$$\begin{aligned} B(x) &= a_0 C_0(x) + a_1 C_1(x) + a_2 C_2(x) + \cdots \\ &= a_0 g(x) + a_1 g(x) f(x) + a_2 g(x)[f(x)]^2 + \cdots \\ &= g(x)[a_0 + a_1 f(x) + a_2 [f(x)]^2 + \cdots] \\ &= g(x) A(f(x)). \end{aligned} \tag{5}$$

Another way to denote this is by

$$(g(x), f(x)) * A(x) = g(x) A(f(x)) = B(x).$$

For some applications of (5) see Section 4.

What happens when two Riordan matrices are multiplied? The typical column of $(h(x), l(x))$ is $h(x)[l(x)]^i$ and using this as $A(x)$ in (5) quickly yields the matrix multiplication

$$(g(x), f(x)) * (h(x), l(x)) = (g(x) h(f(x)), l(f(x))). \tag{6}$$

This is a group multiplication with identity $I = (1, x)$ and group inverse

$(g(x), f(x))^{-1} = (1/g(\bar{f}(x)), \bar{f}(x))$ where $f(\bar{f}(x)) = \bar{f}(f(x)) = x$. The existence of a unique compositional inverse in $C[[x]]$ is guaranteed by $f(x) = x + f_2x^2 + f_3x^3 + \cdots$. This is the *Riordan group* which we denote as R.

This multiplication is found in Roman [11, p. 43] with the following modifications. (A) Exponential generating functions are used instead of ordinary ones, (B) $C_i(x) = g(x)[f(x)]^i/i!$, (C) here we have avoided umbral operators, shift operators, and adjoints [all well presented in Roman] and have a proof depending only on matrix multiplication and generating functions.

2. Examples of elements in R

(A) The first example is the Pascal matrix.

$$P = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & 0 & \\ 1 & 2 & 1 & & & \vdots \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \end{pmatrix} = \left(\frac{1}{1-x}, \frac{x}{1-x}\right). \tag{7}$$

It's obvious that $g(x) = 1/(1-x) = 1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + \cdots$. Recall that if $A(x) = a_0 + a_1x + a_2x^2 + \cdots$, then

$$A(x)\frac{1}{1-x} = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i\right)x^n. \tag{8}$$

The binomial identity $\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$ translates to

$$C_k(x)\frac{x}{1-x} = C_{k+1}(x)$$

and thus to

$$C_i(x) = \frac{1}{1-x}\cdot\left(\frac{x}{1-x}\right)^i.$$

(B) The next example is another version of Pascal's triangle

$$B = \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & 0 & \\ 2 & 0 & 1 & & & & \\ 0 & 3 & 0 & 1 & & & \vdots \\ 6 & 0 & 4 & 0 & 1 & & \\ 0 & 10 & 0 & 5 & 0 & 1 & \\ & & & \cdots & & & \end{pmatrix} = \left(\frac{1}{\sqrt{1-4x^2}}, \frac{1-\sqrt{1-4x^2}}{2x}\right). \tag{9}$$

We will show how $f(x)$ is computed if the rule of formation in B is known. The first column is given by the central binomial coefficients with x^2 for x so as to have alternating zeros.

The rule of formation in B is that each entry is the sum of the elements to the left and right in the row above. In other words

$$b_{n+1,j}=b_{n,j-1}+b_{n,j+1}, \quad j\geq 1. \tag{10}$$

Thus

$$C_k(x)=xC_{k-1}(x)+xC_{k+1}(x)$$

i.e.,

$$g(x)(f(x))^k=xg(x)[f(x)]^{k-1}+xg(x)[f(x)]^{k+1}$$

or

$$f(x)=x+x(f(x))^2.$$

Solving for $f(x)$, we get

$$f(x)=\frac{1-\sqrt{1-4x^2}}{2x}. \tag{11}$$

From $f(x)=x\{1+[f(x)]^2\}$ it follows immediately that $x=\bar{f}(x)\{1+x^2\}$ so

$$\bar{f}(x)=\frac{x}{1+x^2}$$

and

$$\frac{1}{g(\bar{f}(x))}=\sqrt{1-4\left(\frac{x}{1+x^2}\right)^2}=\frac{1-x^2}{1+x^2}=1-2x^2+2x^4-\cdots. \tag{12}$$

Therefore we get our next example,

$$B^{-1}=\begin{pmatrix} 1 & & & & & & & 0 \\ 0 & 1 & & & & & & \\ -2 & 0 & 1 & & & & & \\ 0 & -3 & 0 & 1 & & & & \vdots \\ 2 & 0 & -4 & 0 & 1 & & & \\ 0 & 5 & 0 & -5 & 0 & 1 & & \\ -2 & 0 & 9 & 0 & -6 & 0 & 1 & \\ & & & \cdots & & & & \end{pmatrix}=\left(\frac{1-x^2}{1+x^2},\frac{x}{1+x^2}\right). \tag{13}$$

Successive columns after the leftmost can be formed using (8) to obtain $m_{n,i}=m_{n-1,i-1}-m_{n-2,i}$.

3. Euler's problem of the King walks

A King walks down a chessboard as follows

$$\begin{array}{ccccccccccl}
 & & & & 1 & & & & & \leftarrow & \text{start} \\
 & & & 1 & 1 & 1 & & & & \leftarrow & \text{after 1 step} \\
 & & 1 & 2 & 3 & 2 & 1 & & & \leftarrow & \text{after 2 steps} \\
 & 1 & 3 & 6 & \underline{7} & 6 & 3 & 1 & & \leftarrow & \text{after 3 steps} \\
1 & 4 & 10 & 16 & \underline{19} & 16 & 10 & 4 & 1 & \leftarrow & \text{after 4 steps} \\
 & & & & \cdots & & & & & & \quad\cdots
\end{array}$$

What is the GF for the underlined elements which are the central trinomial coefficients. This problem dates back to Euler. We cut the array above in half to get a lower triangular matrix T. A little experimentation (getting lucky) yields the following:

$$T := \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & 0 & & \\ 3 & 2 & 1 & & & \vdots \\ 7 & 6 & 3 & 1 & & \\ 19 & 16 & 10 & 4 & 1 & \\ & & \cdots & & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & 0 & & \\ 1 & 2 & 1 & & & \vdots \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & 0 & & \\ 2 & 0 & 1 & & & \vdots \\ 0 & 3 & 0 & 1 & & \\ 6 & 0 & 4 & 0 & 1 & \\ & & \cdots & & & \end{pmatrix} = PB. \tag{14}$$

To actually prove this factorization one puts the binomial coefficients together and (14) is equivalent to

$$(t_{n,k}) = \sum_{j\geq 0} \binom{n}{j+k, j, n-2j-k}. \tag{15}$$

This is immediate since moving from the central to the kth column requires $j+k$ steps to the right, j to the left and the remaining $n-2j-k$ straight down. Thus

$$\begin{aligned}
T &= \left(\frac{1}{1-x}, \frac{x}{1-x}\right) * \left(\frac{1}{\sqrt{1-4x^2}}, \frac{1-\sqrt{1-4x^2}}{2x}\right) \\
&= \left(\frac{1}{1-x}\,\frac{1}{\sqrt{1-4\left(\frac{x}{1-x}\right)^2}}, \frac{1-\sqrt{1-4\left(\frac{x}{1-x}\right)^2}}{2\left(\frac{x}{1-x}\right)}\right) \\
&= \left(\frac{1}{\sqrt{1-2x-3x^2}}, \frac{1-x-\sqrt{1-2x-3x^2}}{2x}\right).
\end{aligned} \tag{16}$$

The kth column of T has

$$\frac{1}{\sqrt{1-2x-3x^2}}\cdot\left(\frac{1-x-\sqrt{1-2x-3x^2}}{2x}\right)^k$$

as its GF and with $k=0$ the generating function for the King walks is $(1-2x-3x^2)^{-1/2}$. Alternate proofs for this can be given by the calculus of residues [2] or Lagrange inversion [3,4]. One result of this is that the GF for random walks of the form "$\searrow\downarrow\swarrow$" (i.e., $m_{n+1,k}=m_{n,k-1}+m_{n,k}+m_{n,k+1}$) is

$$\begin{aligned} f(x)&=\frac{1-x-\sqrt{1-2x-3x^2}}{2x}\\ &=x(1+x+2x^2+4x^3+9x^4+21x^5+\cdots). \end{aligned}$$

This is the GF for the moderately famous Motzkin numbers [3–5].

4. Identities and inverse relations

We start with a typical example taken from [10, p. 74] of Riordan's classic "Combinatorial Identities".

Let $\{\beta_k\}_{k\geq 0}=\{1,1,3,7,19,51,\ldots\}$ be the central trinomial coefficients. Then

$$0=(1-\beta)^{2n+1},\qquad \beta^k=\beta_k,$$

and

$$\binom{2n}{n}=(1-\beta)^{2n},\qquad \beta^k=\beta_k.$$

We view this classical umbral notation as a suggestive mnemonic shorthand. Translating a typical case goes as follows:

$$\begin{aligned} (1-\beta)^5&=\beta^0-\binom{5}{1}\beta^1+\binom{5}{2}\beta^2-\binom{5}{3}\beta^3+\binom{5}{4}\beta^4-\beta^5\\ &\equiv\beta_0-5\beta_1+10\beta_2-10\beta_3+5\beta_4-\beta_5\\ &=1-5+30-70+95-51=0. \end{aligned}$$

Treating $(1-\beta)^k$, $k=0,1,2,3,4$, similarly and going to matrix notation yields

$$\begin{pmatrix} 1 & & & & & 0 & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \vdots \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ & & & \cdots & & & \end{pmatrix}\begin{pmatrix} \beta_0\\ -\beta_1\\ \beta_2\\ -\beta_3\\ \beta_4\\ -\beta_5\\ \vdots \end{pmatrix}=\begin{pmatrix} 1\\ 0\\ 2\\ 0\\ 6\\ 0\\ \vdots \end{pmatrix}. \tag{17}$$

Since we have already encountered all of the relevant GFs we quickly obtain

$$\left(\frac{1}{1-x}, \frac{x}{1-x}\right) * \frac{1}{\sqrt{1+2x-3x^2}} = \frac{1}{1-x} \cdot \frac{1}{\sqrt{1+2\dfrac{x}{1-x}-3\dfrac{x^2}{(1-x)^2}}}$$

$$= \frac{1}{\sqrt{1-2x+x^2+2x-2x^2-3x^2}}$$

$$= \frac{1}{\sqrt{1-4x^2}}.$$

When tackling an identity this procedure can actually be useful. Write out the first four or five cases. Set them up as a matrix times a column vector, identify the generating functions for the columns, figure out $f(x)$, and compute $g(x)A(f(x))$. Here are a selection of identities from Riordan's "Combinatorial Identities" [10] which admit similar proofs.

(1) $\sum_{k=0}^{n} (-1)^{n-k} 2^{2k} \binom{n+k}{2k} = 2n+1$ (p. 6),
(2) $\binom{2n}{n} = (3-\beta)^n$, $\beta^k \equiv \beta_k$, the central trinomial coefficients (p. 74),
(3) $c_n = c(4-c)^{n-1}$, $c^k \equiv c_k$, the Catalan numbers (p. 156),
(4) $c_{n+1} = \sum_{k\geq 0} \binom{n}{2k} 2^{n-2k} c_k$ (p. 157, also [13,6]),
(5) $\sum_{k\geq 0} \binom{n-k}{k} 6^k = (3^{n+1} - (-2)^{n+1})/5$ (p. 76),
(6) $f_n + 2f_{n-1} + \cdots + (n+1)f_0 = f_{n+4} - (n+4)$ where $\sum_{n\geq 0} f_n x^n = 1/(1-x-x^2)$, the f_n being the Fibonacci numbers (p. 157).

Obviously, the greater of store of generating functions available the better. We have not mentioned exponential generating functions but they can be done similarly if $C_k(x)$ becomes $g(x)[f(x)]^k/k!$. The book of Roman [11] treats the exponential case from a quite different viewpoint and includes a wealth of further examples. There might be a way to classify many identities either by the function f or by the relationship of g and f.

Another major topic in "Combinatorial Identities" is inverse relations. Most of these can be derived directly by multiplying $(g(x), f(x)) * A(x) = B(x)$ on the left by $(g(x), f(x))^{-1}$. For instance the pair [10, p. 62, #5]

$$a_n = \sum_{k\geq 0} \binom{n-k}{k} b_{n-k} \tag{18}$$

$$\Leftrightarrow \quad b_n = \sum_{k\geq 0} (-1)^k \left[\binom{n+k-1}{k} - \binom{n+k-1}{k-1}\right] a_{n-k}$$

can be verified as follows. Rewrite (18) as

$$\begin{bmatrix} 1 & & & & & 0 & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 2 & 1 & & & \vdots \\ 0 & 0 & 1 & 3 & 1 & & \\ 0 & 0 & 0 & 3 & 4 & 1 & \\ & & & \cdots & & & \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} \tag{19}$$

or $(1,x(1+x))*B(x)=A(x)$. The first key is to find $\bar{f}(x)$ when $f(x)=x(1+x)$. Then $x=\bar{f}(x)+[\bar{f}(x)]^2$. Since $C(x)=1+x[C(x)]^2$ we find that $\bar{f}(x)=xC(-x)=x-x^2[C(-x)]^2=x-[\bar{f}(x)]^2$. Therefore $(1,x+x^2)^{-1}=(1,xC(-x))$ and

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & & 0 & \\ 0 & 1 & & & & & \\ 0 & -1 & 1 & & & & \\ 0 & 2 & -2 & 1 & & & \vdots \\ 0 & -5 & 5 & -3 & 1 & & \\ 0 & 14 & 14 & 9 & -4 & 1 & \\ & & & \cdots & & & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}. \tag{20}$$

The entries in this matrix are the ballot numbers if we disregard sign so after rearranging

$$b_n=\sum_{k\geq 0}^{n-1}(-1)^k\left[\binom{n+k-1}{k}-\binom{n+k-1}{k-1}\right]a_{n-k}$$

$$=\sum_{k\geq 0}(-1)^k\frac{n-k}{n+k}\binom{n+k}{k}a_{n-k}.$$

To do one quick example using exponential GFs [10, p. 114, #3] we have $a_n=\sum_{k\geq 0}\binom{n}{2k}b_{n-2k} \Leftrightarrow b_n=\sum_{k\geq 0}\binom{n}{2k}E_{2k}a_{n-2k}$ where $\operatorname{sech} x=\sum_{n\geq 0}E_{2n}x^{2n}/(2n)!$. At first glance this is unintuitive. Why the secant numbers in this idyllic setting? The matrices clarify this quickly.

$$\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & 0 & \\ 1 & 0 & 1 & & & \vdots \\ 0 & 3 & 0 & 1 & & \\ 1 & 0 & 6 & 0 & 1 & \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}. \tag{21}$$

Here $g(x)=1+1\cdot(x^2/4\,!)+1\cdot(x^4/6\,!)+1\cdot(x^6/4\,!)+\cdots=\cosh x$ and $f(x)=x$.

Thus, $(g(x),x)^{-1}=(\cosh x,x)^{-1}=(\operatorname{sech} x,x)$ and the inverse pair is verified.

Any invertible infinite lower triangular matrix leads to an inverse pair and when this matrix is in the Riordan group the computations reduce to finding inverse functions.

5. The Bessel connection

We return to example B^{-1} from Section 2 and then ignore the negative signs. Call this matrix

$$B^*=\begin{bmatrix} 1 & & & & & & 0 & \\ 0 & 1 & & & & & & \\ 2 & 0 & 1 & & & & & \\ 0 & 3 & 0 & 1 & & & & \\ 2 & 0 & 4 & 0 & 1 & & & \vdots \\ 0 & 5 & 0 & 5 & 0 & 1 & & \\ 2 & 0 & 9 & 0 & 6 & 0 & 1 & \\ & & & & \cdots & & & \end{bmatrix} = \left(\frac{1+x^2}{1-x^2}, \frac{x}{1-x^2}\right). \tag{22}$$

Multiplying B^* by $(1,1,1,\dots)^{\mathrm{T}}$ yields the row sums $1,1,3,4,7,11,18,29,\dots = \{L_n\}_{n\geq 0}$. It is easy to see that $L_{n+1}=L_n+L_{n-1}$ for $n\geq 2$. These are the *Lucas numbers* (except that here we use $L_0=1$).

Since

$$B^*\begin{bmatrix}1\\1\\1\\\vdots\end{bmatrix} = \begin{bmatrix}L_0\\L_1\\L_2\\L_3\\\vdots\end{bmatrix}$$

multiplication by $(B^*)^{-1}$ yields

$$\begin{bmatrix}1\\1\\1\\\vdots\end{bmatrix} = B^{*-1}\begin{bmatrix}L_0\\L_1\\L_2\\\vdots\end{bmatrix}. \tag{23}$$

Note that

$$(B^*)^{-1}=\begin{bmatrix} 1 & & & & & \\ 0 & 1 & & 0 & & \\ -2 & 0 & 1 & & & \vdots \\ 0 & -3 & 0 & 1 & & \\ 6 & 0 & -4 & 0 & 1 & \\ & & \cdots & & & \end{bmatrix} = \left(\frac{1}{\sqrt{1+4x^2}}, \frac{1-\sqrt{1+4x^2}}{2x}\right). \tag{24}$$

Now reinterpret (23) in terms of exponential generating functions. This yields the surprising result that $e^x=\sum_{n=0}^{\infty} L_nJ_n(2x)$. Here J_n is the Bessel function of order n. This follows directly since

$$J_n(2x) = \sum_{m\geq 0} (-1)^m \binom{n+2m}{m} \frac{x^{n+2m}}{(n+2m)\,!}.$$

We conclude by outlining a short new proof of a result due to Touchard [15]. Let $\mu(n)$ denote the nth reduced menage number. Then Touchard's theorem [15] states that

$$\sum_{n=0}^{\infty} \mu(n) I_n(2x) = \frac{e^{-2x}}{1-x} \tag{25}$$

when I_n is the modified Bessel function of order n.

This proof starts with the result of Kaplansky and Touchard that

$$\mu(n) = \sum_{k\geq 0} \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)\,!\,(-1)^k.$$

Replacing k by $n-k$ and summing down from $k=n$ to 0, we get

$$\begin{pmatrix} \mu(0) \\ \mu(1) \\ \mu(2) \\ \mu(3) \\ \mu(4) \\ \mu(5) \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 2 \\ 13 \\ \vdots \end{pmatrix} = M \begin{pmatrix} 0! \\ 1! \\ 2! \\ 3! \\ 4! \\ 5! \\ \vdots \end{pmatrix}; \quad M = B^{-1}P^{-2} \tag{26}$$

where

$$M = (m_{nk})_{n,k\geq 0}, \quad m_{nk} = \frac{2n}{n+k} \binom{n+k}{n-k} (-1)^{n-k}.$$

To finish multiply (26) on the left by B and interpret in terms of exponential generating functions.

6. Comments

The most important influence for this work is the book of Roman and several papers of Rota. We were working through Roman's book when we realized we could give a quick proof from first principles of the group multiplication law (6).

Another early set of papers involving the Bell subgroup $\{g(x), x(g(x))\}$ are by Jabotinsky [7–9]. His main interest was in the analytic behavior of $g(x)$ and its iterates.

Since many of our examples involved either generating functions or triangular arrays we realized we could abandon linear operators, adjoints, Scheffer sequences, shift invariant operators, Steffensen's formula, etc. and be left with a nice theory using only a few ideas from group theory and linear algebra. The full theory is needed for differential operators and polynomial sequences.

There are many other topics that can be developed just using the Riordan group. Among them are divisibility properties, determinental sequences, Stieltjes transform, and adaptation to GFs of the form $A(x)=\sum_{n=0}^{\infty} a_n x^n/c_n$ for various sequences c_n.

If $c_n=n!$, then Roman [11] is a rich source of examples and is a well written introduction to the theory alluded to two paragraphs ago.

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