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Applications of Two Dimensional Fractional Mellin Transform

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Abstract: *The Mellin transform is a basic tool for analyzing the behavior of many important functions in mathematics and mathematical physics. Mellin transform has many applications such as algorithms, correlators, navigation, vowel recognition ,cryptographic scheme quantum calculus, radar classification of ships, electromagnetic, stress distribution, agriculture, medical stream, statistics, probability, signal processing, optics, and pattern recognition.*

Keywords: *Fractional Mellin Transform, Generalize function, Mellin Transform, Testing Function space*

1. INTRODUCTION

The Fourier transform is invariant in modulus to translations in frequency, but not to dilations. Therefore, the Fourier transform is no longer the appropriate transform to change the representation space of these signals. It has to be replaced by a new a new transform, the Mellin transform, which is invariant in modulus to dilations and decomposes the signal on a basis of hyperbolic signals. Historically, Riemann (1876) first recognized the Mellin transform. In the last few decades, the interest of the scientific community towards the fractional calculus experienced an exceptional boost, so that its applications can now be found in a great variety of natural sciences.

Two-dimensional Mellin transform can convert the auditory images of vowel sounds from vocal tracts with different sizes into an invariant Mellin image (MI) and, thereby, facilitate the extraction and separation of the size and shape information associated with a given vowel type. In signal processing terms, the MI of a sound is the Mellin transform of a stabilised wavelet transform of the sound. Toshio Irino a, Roy D. Patterson discussed in their article that the MI provides a good model of auditory vowel normalization, and that this provides a good framework for auditory processing from cochlea to cortex [1]. Robert Frontczak, Rainer Schobel presented in their work that how to modify the approach to value American call options on dividend-paying stocks. He proposed a new integral equation to determine the price of an American call option and its free boundary using modified Mellin transforms and also show how to derive the pricing formula for perpetual American call options using the new framework [2]. We can compute the price of an option on a basket of stocks using Mellin transforms in several variables [3]. Salvatore Butera , Mario Di Paola discussed in their work that using Mellin transform solution of the multi-order differential equations are calculated [4]. They also provide in their article that the solution of a multi-order, multi-degree-of-freedom fractional differential equation is addressed by using the Mellin integral transform [5].

The two dimensional fractional Mellin transform with parameter θ of $f(x,y)$ denoted by $\mathrm{FRMT}\{f(x,y)\}$ performs a linear operation, given by the integral transform

$$FRMT\{f(x,y)\} = F_\theta(u,v) = \int_0^\infty\int_0^\infty f(x,y)K_\theta(x,y,u,v)dxdy \tag{1}$$

where the kernel - $K_\theta(x,y,u,v) = x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}[u^2+v^2+\log x^2+\log y^2]}$ $0<\theta\le\frac{\pi}{2}$. (2)

Where, $C_{1\theta} = \frac{2\pi i}{sin\theta}$, $C_{2\theta} = \frac{\pi}{tan\theta}$ (3)

This paper will discussed some applications two dimensional fractional Mellin transform and defined the testing function space and two dimensional fractional Mellin transform.

2. Preliminaries

2.1 The Test Function Space E

An infinitely differentiable complex valued function $\phi(x,y)$ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_\alpha$, $K \subset S_b$ where

$S_a = \{x : x \in R, |x| \le a, a > 0\}$ $S_b = \{y : y \in R, |y| \le b, b > 0\}$, $I, K \in R^n$

$$\gamma_{E,q,\lambda}[\phi(x,y)] = \sup_{\substack{x\in I\\ y\in K}} \left| D_{x,y}^{q,\lambda}\phi(x,y)\right| < \infty$$

Thus $E(R^n)$ will denote the space of all $\phi(x,y) \in E(R^n)$ with compact support contained in S_α & S_b.

Note that the space E is complete and therefore a Frechet space. Moreover, we say that $f(x,y)$ is a fractional Mellin transformable if it is a member of E^*, the dual of E.

2.2 Two Dimensional Fractional Mellin Transform (FRMT)

The two dimensional fractional Mellin transform of $f(x,y) \in E^*(R^n)$ can be defined by $FRMT\{f(x,y)\} = F_\theta(u,v) = \langle f(x,y), K_\theta(x,y,u,v)\rangle$

$$K_\theta(x,y,u,v) = x^{\frac{2\pi iu}{\sin\theta}-1} y^{\frac{2\pi iv}{\sin\theta}-1} e^{\frac{\pi i}{\tan\theta}[u^2+v^2+\log x^2+\log y^2]}$$

Right hand side of equation (4) has a meaning as the application of $f(x,y) \in E^*$ to $K_\theta(x,y,u,v) \in E$.

It can be extended to the complex space as an entire function given by $FRMT\{f(x,y)\} = F_\theta(p,k) = \langle f(x,y), K_\theta(x,y,p,k)\rangle$ The right hand side is meaningful because for each $p,k \in C^n$, $K_\theta(x,y,p,k) \in E$, as a function of *x,y*.

3. Applications

3.1 Prove that

$$F_\theta\{(1)\}(u,v) = itan\theta e^{-\pi itan\theta(u^2+v^2)}$$

Proof:

$$\{FRMT(1)\}(u,v) = \int_0^\infty\int_0^\infty (1)x^{\frac{2\pi iu}{sin\theta}-1}y^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}dxdy$$

Putting $log\,x = s,\quad log\,y = t$

$$x = e^s,\qquad y = e^t$$

$$dx = e^s ds,\ dy = e^t dt$$

$$\{FRMT(1)\}(u,v)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e^s)^{\frac{2\pi iu}{sin\theta}-1}(e^t)^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+s^2+t^2)}e^s e^t dsdt$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(e)^{ia(us+vt)}e^{ib(s^2+t^2)}dsdt$$

Where, $a = \frac{2\pi}{sin\theta}$, $\quad b = \frac{\pi}{tan\theta}$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)}\int_{-\infty}^{\infty}(e)^{ibs^2+iaus}ds\int_{-\infty}^{\infty}(e)^{ibt^2+iavt}dt$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)}\sqrt{\frac{\pi i}{b}}\,e^{\frac{-ia^2u^2}{4b}}\sqrt{\frac{\pi i}{b}}e^{\frac{-ia^2v^2}{4b}}$$

$$= \frac{\pi i}{b}e^{i\left\{\frac{\pi}{tan\theta}-\frac{a^2}{4b}\right\}(u^2+v^2)}$$

$$= \frac{\pi itan\theta}{\pi}e^{i\left\{\frac{\pi}{tan\theta}-\frac{4\pi^2 tan\theta}{4\pi sin^2\theta}\right\}(u^2+v^2)}$$

$$= itan\theta e^{\frac{-i\pi}{sin\theta}\left\{\frac{sin^2}{cos\theta}\right\}(u^2+v^2)}$$

$$= itan\theta e^{-i\pi tan\theta(u^2+v^2)}$$

3.2 Prove that

$$F_\theta\{\delta(x-a,y-b)\}(u,v) = a^{\frac{2\pi iu}{sin\theta}-1}b^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2a+log^2b)}$$

Proof:

$$\{FRMT(1)\}(u,v) = \int_0^\infty\int_0^\infty f(x,y)x^{\frac{2\pi iu}{sin\theta}-1}y^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}dxdy$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)}\int_0^\infty\int_0^\infty f(x,y)x^{\frac{2\pi iu}{sin\theta}-1}y^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(log^2x+log^2y)}dxdy$$

$$F_\theta\{\delta(x-a,y-b)\}(u,v) = e^{\frac{\pi i}{tan\theta}(u^2+v^2)}a^{\frac{2\pi iu}{sin\theta}-1}b^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(log^2a+log^2b)}$$

$$= a^{\frac{2\pi iu}{sin\theta}-1}b^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2a+log^2b)}$$

3.3 Prove that

$$F_\theta\{(x^{ia}y^{ib})\}(u,v) = itan\theta e^{\frac{i}{cos\theta}\left\{\frac{-sin\theta}{4\pi}[4\pi^2(u^2+v^2)+a^2+b^2]-(au+bv)\right\}}$$

Proof:

$$\{FRMT(x^{ia}y^{ib})\}(u,v) = \int_0^\infty\int_0^\infty(x^{ia}y^{ib})x^{\frac{2\pi iu}{sin\theta}-1}y^{\frac{2\pi iv}{sin\theta}-1}e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)}dxdy$$

$$= \int_0^\infty \int_0^\infty x^{\frac{2\pi i u}{\sin\theta}+ia-1} y^{\frac{2\pi i v}{\sin\theta}+ib-1} e^{\frac{\pi i}{\tan\theta}(u^2+v^2+\log^2 x+\log^2 y)} dxdy$$

Putting $\log x = m, \quad \log y = n$

$$x = e^m, \qquad y = e^n$$

$$dx = e^m dm,\ dy = e^n dn$$

$\{FRMT(x^{ia}y^{ib})\}(u,v)$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^m)^{\frac{2\pi i u}{\sin\theta}+ia} (e^n)^{\frac{2\pi i v}{\sin\theta}+ib} e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} e^{\frac{\pi i}{\tan\theta}(m^2+n^2)} dmdn$$

$$= e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (e^m)^{i\left(\frac{2\pi u}{\sin\theta}+a\right)} (e^n)^{i\left(\frac{2\pi v}{\sin\theta}+b\right)} e^{\frac{\pi i}{\tan\theta}(m^2+n^2)} dmdn$$

$$= e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iAm} e^{iBn} e^{iC(m^2+n^2)} dmdn$$

Where, $A = \frac{2\pi u}{\sin\theta} + a, \quad B = \frac{2\pi v}{\sin\theta} + b,\ C = \frac{\pi}{\tan\theta}$

$\{FRMT(x^{ia}y^{ib})\}(u,v)$

$$= e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \int_{-\infty}^{\infty} e^{iCm^2+iAm} dm \int_{-\infty}^{\infty} e^{iCm^2+iBn} dn$$

$$= e^{\frac{\pi i}{\tan\theta}(u^2+v^2)} \sqrt{\frac{\pi i}{C}}\, e^{\frac{-iA^2}{4C}} \sqrt{\frac{\pi i}{C}} e^{\frac{-iB^2}{4C}}$$

$$= \frac{\pi i \tan\theta}{\pi} e^{i\left\{\frac{\pi}{\tan\theta}(u^2+v^2) - \frac{\tan\theta}{4\pi}\left[\left(\frac{2\pi u}{\sin\theta}+a\right)^2 + \left(\frac{2\pi v}{\sin\theta}+b\right)^2\right]\right\}}$$

$$= i\tan\theta e^{i\left\{\left[\frac{\pi}{\tan\theta} - \frac{\pi}{\sin\theta\cos\theta}\right](u^2+v^2) - \frac{1}{\cos\theta}(au+bv) - \frac{\tan\theta}{4\pi}(a^2+b^2)\right\}}$$

$$= i\tan\theta e^{i\left\{\frac{-\pi\sin^2\theta}{\sin\theta\cos\theta}(u^2+v^2) - \frac{1}{\cos\theta}(au+bv) - \frac{\tan\theta}{4\pi}(a^2+b^2)\right\}}$$

$$= i\tan\theta e^{i\left\{-\pi\tan\theta(u^2+v^2) - \frac{1}{\cos\theta}(au+bv) - \frac{\tan\theta}{4\pi}(a^2+b^2)\right\}}$$

$$= i\tan\theta e^{i\left\{\frac{-\tan\theta}{4\pi}[4\pi^2(u^2+v^2)+a^2+b^2] - \frac{1}{\cos\theta}(au+bv)\right)\right\}}$$

$$= i\tan\theta e^{\frac{i}{\cos\theta}\left\{\frac{-\sin\theta}{4\pi}[4\pi^2(u^2+v^2)+a^2+b^2] - (au+bv)\right)\right\}}$$

3.4 Prove that

$$F_\theta\left\{e^{i[a(logx)^2+b(logy)^2]}\right\}(u,v) == \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)}\ e^{\frac{-i\pi^2}{sin\theta cos\theta}\left\{\frac{u^2}{\pi+atan\theta}+\frac{v^2}{\pi+btan\theta}\right\}}$$

Proof:

$$\left\{FRMT\left[e^{i[alog^2x+blog^2y]}\right]\right\}(u,v)$$

$$=\int_0^\infty\int_0^\infty \left\{e^{i[alog^2x+blog^2y]}\right\} x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}(u^2+v^2+log^2x+log^2y)} dxdy$$

$$=\int_0^\infty\int_0^\infty x^{\frac{2\pi iu}{sin\theta}-1} y^{\frac{2\pi iv}{sin\theta}-1} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} e^{i(alog^2x+blog^2y)}\ e^{\frac{\pi i}{tan\theta}(log^2x+log^2y)} dxdy$$

Putting $log\,x = p, \quad log\,y = q$

$$x = e^p, \qquad y = e^q$$

$$dx = e^p dp,\ dy = e^q dq$$

$$\left\{FRMT\left[e^{i[alog^2x+blog^2y]}\right]\right\}(u,v)$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \int_{-\infty}^\infty\int_{-\infty}^\infty (e^p)^{\frac{2\pi iu}{sin\theta}-1}(e^q)^{\frac{2\pi iv}{sin\theta}-1} e^{i\left[\left(a+\frac{\pi}{tan\theta}\right)p^2+\left(b+\frac{\pi}{tan\theta}\right)q^2\right]} e^p e^q dpdq$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \int_{-\infty}^\infty\int_{-\infty}^\infty (e^p)^{\frac{2\pi iup}{sin\theta}}(e^q)^{\frac{2\pi ivq}{sin\theta}} e^{i\left[\left(a+\frac{\pi}{tan\theta}\right)p^2+\left(b+\frac{\pi}{tan\theta}\right)q^2\right]} dpdq$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \int_{-\infty}^\infty\int_{-\infty}^\infty (e^p)^{\frac{2\pi i}{sin\theta}(up+vq)} e^{i\left[\left(a+\frac{\pi}{tan\theta}\right)p^2+\left(b+\frac{\pi}{tan\theta}\right)q^2\right]} dpdq$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \int_{-\infty}^\infty e^{iAp+iCp^2} dp \int_{-\infty}^\infty e^{iBq+iDq^2} dq$$

Where, $A=\frac{2\pi u}{sin\theta},\ B=\frac{2\pi v}{sin\theta},\ \ C=a+\frac{\pi}{tan\theta},\ \ D=b+\frac{\pi}{tan\theta}$

$$\left\{FRMT\left[e^{i[alog^2x+blog^2y]}\right]\right\}(u,v)$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \int_{-\infty}^\infty e^{iCp^2+iAp} dp \int_{-\infty}^\infty e^{iDq^2+iBq} dq$$

$$= e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \sqrt{\frac{\pi i}{C}}\ e^{\frac{-iA^2}{4C}} \sqrt{\frac{\pi i}{D}} e^{\frac{-iB^2}{4D}}$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i}{4}\left(\frac{A^2}{C}+\frac{B^2}{D}\right)}$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i}{4}\left\{\frac{4\pi^2u^2}{sin^2\theta}\Big/\left(a+\frac{\pi}{tan\theta}\right)+\frac{4\pi^2v^2}{sin^2\theta}\Big/\left(b+\frac{\pi}{tan\theta}\right)\right\}}$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i}{4}\left\{\frac{4\pi^2u^2}{sin^2\theta}\Big/\left(\frac{atan\theta+\pi}{tan\theta}\right)+\frac{4\pi^2v^2}{sin^2\theta}\Big/\left(\frac{btan\theta+\pi}{tan\theta}\right)\right\}}$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i}{4}\left\{\frac{4\pi^2u^2}{sin^2\theta}\left(\frac{tan\theta}{\pi+atan\theta}\right)+\frac{4\pi^2v^2}{sin^2\theta}\left(\frac{tan\theta}{\pi+btan\theta}\right)\right\}}$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i\pi^2}{sin^2\theta}tan\theta\left\{\frac{u^2}{\pi+atan\theta}+\frac{v^2}{\pi+btan\theta}\right\}}$$

$$F_\theta\left\{e^{i[a(logx)^2+b(logy)^2]}\right\}(u,v)$$

$$= \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i\pi^2}{sin\theta cos\theta}\left\{\frac{u^2}{\pi+atan\theta}+\frac{v^2}{\pi+btan\theta}\right\}} \; \frac{i\pi tan\theta}{\sqrt{(atan\theta+\pi)(btan\theta+\pi)}}$$

$$F_\theta\left\{e^{i[a(logx)^2+b(logy)^2]}\right\}(u,v) = \frac{\pi i}{\sqrt{CD}} e^{\frac{\pi i}{tan\theta}(u^2+v^2)} \; e^{\frac{-i\pi^2}{sin\theta cos\theta}\left\{\frac{u^2}{\pi+atan\theta}+\frac{v^2}{\pi+btan\theta}\right\}}$$

Hence proved

4. Conclusion

In this paper we have defined Distributional two dimensional fractional Mellin transform with compact support. Some applications for two dimensional fractional Mellin transform is proved.

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