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SOME NEW IDENTITIES CONCERNING GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract

In this paper we obtain some identities containing generalized Fibonacci and Lucas numbers. Some of them are new and some are well known. By using some of these identities we give some congruences concerning generalized Fibonacci and Lucas numbers such as

$$V_{2mn+r} \equiv (-(-t)^m)^n V_r \pmod{V_m},$$
$$U_{2mn+r} \equiv (-(-t)^m)^n U_r \pmod{V_m},$$

and

$$V_{2mn+r} \equiv (-t)^{mn} V_r \pmod{U_m},$$
$$U_{2mn+r} \equiv (-t)^{mn} U_r \pmod{U_m}.$$

Keywords: Generalized Fibonacci numbers; Generalized Lucas numbers

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1. Introduction

Let k and t be nonzero real numbers. Generalized Fibonacci sequence $\{U_n\}$ is defined by $U_0 = 0$, $U_1 = 1$, and $U_{n+1} = kU_n + tU_{n-1}$ for $n \geq 1$ and generalized Lucas sequence $\{V_n\}$ is defined by $V_0 = 2$, $V_1 = k$, and $V_{n+1} = kV_n + tV_{n-1}$ for $n \geq 1$. U_n and V_n are called generalized Fibonacci numbers and generalized Lucas numbers respectively.

For $k = t = 1$, we have classical Fibonacci and Lucas sequences $\{F_n\}$ and $\{L_n\}$. For $k = 2$ and $t = 1$, we have Pell and Pell-Lucas sequences $\{P_n\}$ and $\{Q_n\}$. For more

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information about generalized Fibonacci and Lucas numbers one can consult [1], [2], [3], and [4]. For $t = 1$, the sequence $\{U_n\}$ has been investigated in [5] and [6].

Generalized Fibonacci and Lucas numbers for negative subscript are defined as

$$U_{-n} = \frac{-U_n}{(-t)^n} \text{ and } V_{-n} = \frac{V_n}{(-t)^n} \tag{1.1}$$

respectively.

Now assume that $k^2 + 4t > 0$. Then it is well known that

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n \tag{1.2}$$

where $\alpha = (k + \sqrt{k^2 + 4t})/2$ and $\beta = (k - \sqrt{k^2 + 4t})/2$. The above identities are known as Binet formulae. Let α and β be the roots of the equations $x^2 - kx - t = 0$. Clearly $\alpha + \beta = k$, $\alpha - \beta = \sqrt{k^2 + 4t}$, and $\alpha\beta = -t$. Moreover, it can be seen that

$$V_n = U_{n+1} + tU_{n-1} = kU_n + 2tU_{n-1} \tag{1.3}$$

and

$$(k^2 + 4t)U_n = V_{n+1} + tV_{n-1} \tag{1.4}$$

for every $n \in \mathbb{Z}$

For $t = 1$, $\mp(U_n, V_n)$ are all the integer solutions of the equation $x^2 - (k^2 + 4)y^2 = \mp 4$ and for $t = -1$, $\mp(U_n, V_n)$ are all the integer solutions of the equation $x^2 - (k^2 - 4)y^2 = 4$. Also, for $t = 1$, $\mp(U_n, U_{n-1})$ are all the integer solutions of the equation $x^2 - kxy - y^2 = \mp 1$ and for $t = -1$, $\mp(U_n, U_{n-1})$ are all the integer solutions of the equation $x^2 - kxy + y^2 = 1$ (see[7],[8], and [9]).

Many identities concerning generalized Fibonacci and Lucas numbers can be proved by using Binet formulae, induction and matrices. In the literature, the matrices

$$\begin{bmatrix} 0 & 1 \\ t & k \end{bmatrix} \text{ and } \begin{bmatrix} k & t \\ 1 & 0 \end{bmatrix}$$

are used in order to produce identities (see[4],[10]). Since

$$\begin{bmatrix} k & t \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ t & k \end{bmatrix}$$

are similar matrices, they give the same identities.

In this study we will characterize all the 2×2 matrices X satisfying the relation $X^2 = kX + tI$. Then we will obtain different identities by using this property. In fact the matrices

$$\begin{bmatrix} k & t \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ t & k \end{bmatrix}$$

are special cases of the 2×2 matrices X satisfying $X^2 = kX + tI$.

2. Main Theorems

2.1. Theorem. *If X is a square matrix with $X^2 = kX + tI$, then $X^n = U_nX + tU_{n-1}I$ for every $n \in \mathbb{Z}$.*

Proof. If $n = 0$, then the proof is obvious. It can be shown by induction that $X^n = U_n X + tU_{n-1}I$ for every $n \in \mathbb{N}$. We now show that $X^{-n} = U_{-n}X + tU_{-n-1}I$ for every $n \in \mathbb{N}$. Let $Y = kI - X = -tX^{-1}$. Then

$$\begin{aligned} Y^2 &= (kI - X)^2 = k^2 I - 2kX + X^2 \\ &= k^2 I - 2kX + kX + tI = k(kI - X) + tI = kY + tI. \end{aligned}$$

Thus $Y^n = U_n Y + tU_{n-1}I$ and this shows that

$$\begin{aligned} (-t)^n X^{-n} &= U_n Y + tU_{n-1}I = U_n(kI - X) + tU_{n-1}I \\ &= (kU_n + tU_{n-1})I - U_n X = -U_n X + U_{n+1}I. \end{aligned}$$

Then we get $X^{-n} = \dfrac{-U_n X}{(-t)^n} + \dfrac{U_{n+1}I}{(-t)^n}$. This implies that $X^{-n} = U_{-n}X + tU_{-n-1}I$ by (1.1). This completes the proof. $\square$

2.2. Theorem. *Let X be an arbitrary 2×2 matrix. Then $X^2 = kX + tI$ if and only if X is of the form*

$$X = \begin{bmatrix} a & b \\ c & k-a \end{bmatrix} \text{ with } \det X = -t$$

or $X = \lambda I$ where $\lambda \in \{\alpha, \beta\}$, where $\alpha = (k + \sqrt{k^2 + 4t})/2$ and $\beta = (k - \sqrt{k^2 + 4t})/2$.

Proof. Assume that $X^2 = kX + tI$. Then the minimum polynomial of X must divides $x^2 - kx - t$. Therefore it must be $x - \alpha$ or $x - \beta$ or $x^2 - kx - t$. In the first case $X = \alpha I$, in the second case $X = \beta I$, and in the third case, since X is 2×2 matrix, its characteristic polynomial must be $x^2 - kx - t$, so its trace is k and its determinant is $-t$. The argument reverses. $\square$

2.3. Corollary. *If $X = \begin{bmatrix} a & b \\ c & k-a \end{bmatrix}$ is a matrix with $\det X = -t$, then $X^n = \begin{bmatrix} aU_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix}$.*

Proof. Since $X^2 = kX + tI$, the result follows from Theorem 2.1. $\square$

2.4. Corollary. *$\alpha^n = \alpha U_n + tU_{n-1}$ and $\beta^n = \beta U_n + tU_{n-1}$ for every $n \in \mathbb{Z}$.*

Proof. Take $X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ with $\det X = \alpha\beta = -t$. Then by Theorem 2.1, it follows that

$$X^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} = \begin{bmatrix} \alpha U_n + tU_{n-1} & 0 \\ 0 & \beta U_n + tU_{n-1} \end{bmatrix}.$$

This implies that $\alpha^n = \alpha U_n + tU_{n-1}$ and $\beta^n = \beta U_n + tU_{n-1}$. $\square$

2.5. Corollary. *$U_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $V_n = \alpha^n + \beta^n$ for every $n \in \mathbb{Z}$.*

Proof. The result follows from Corollary 2.4. $\square$

2.6. Corollary. *Let $S = \begin{bmatrix} k/2 & (k^2+4t)/2 \\ 1/2 & k/2 \end{bmatrix}$. Then $S^n = \begin{bmatrix} V_n/2 & (k^2+4t)U_n/2 \\ U_n/2 & V_n/2 \end{bmatrix}$ for every $n \in \mathbb{Z}$.*

Proof. Since $S^2 = kS + tI$, the proof follows from Corollary 2.3. $\square$

2.7. Corollary. *Let $X = \begin{bmatrix} k & t \\ 1 & 0 \end{bmatrix}$. Then $X^n = \begin{bmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{bmatrix}$.*

Proof. Since $X^2 = kX + tI$, the proof follows from Corollary 2.3. □

2.8. Lemma. *Let a, b, and $ka+b$ be nonzero real numbers and let k^2+4t be not a perfect square. Then*

$$\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}U_{j+r} = -(-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}U_{j-r}$$

and

$$\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}V_{j+r} = (-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}V_{j-r}.$$

Proof. Let $\mathbb{Z}[\alpha] = \{a\alpha+b \mid a,\ b\in\mathbb{Z}\}$. Define $\varphi : \mathbb{Z}[\alpha] \to \mathbb{Z}[\alpha]$ by $\varphi(a\alpha+b) = a\beta+b = a(k-\alpha)+b = -a\alpha+ka+b$. Then it can be shown that φ is ring homomorphism. Moreover, it can be shown that φ is injective. On the other hand, we get

$$\begin{aligned}-\alpha U_n + U_{n+1} &= -\alpha U_n + kU_n + tU_{n-1} = \varphi(\alpha U_n + tU_{n-1})\\ &= \varphi(\alpha^n) = \beta^n = (-t)^n\alpha^{-n}.\end{aligned}$$

Then it is seen that

$$\begin{aligned}\varphi((a\alpha+b)^n\alpha^r) &= \varphi((a\alpha+b)^n)\varphi(\alpha^r) = (-a\alpha+ka+b)^n(-t)^r\alpha^{-r}\\ &= (-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a\alpha)^j(ka+b)^{n-j}\alpha^{-r}\\ &= (-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}\alpha^{j-r}\\ &= (-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}(\alpha U_{j-r}+tU_{j-r-1})\\ &= \alpha\left((-t)^r\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}U_{j-r}\right)\\ &+\left(-(-t)^{r+1}\sum_{j=0}^{n}\binom{n}{j}(-a)^j(ka+b)^{n-j}U_{j-r-1}\right)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\varphi((a\alpha+b)^n\alpha^r) &= \varphi\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}\alpha^{j+r}\right)\\
&= \varphi\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}(\alpha U_{j+r}+tU_{j+r-1})\right)\\
&= \alpha\left(-\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}U_{j+r}\right)\\
&+ \left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}(kU_{j+r}+tU_{j+r-1})\right)\\
&= \alpha\left(-\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}U_{j+r}\right)+\left(\sum_{j=0}^{n}\binom{n}{j}a^j b^{n-j}U_{j+r+1}\right).
\end{aligned}$$

Then the proof follows. □

2.9. Theorem. *Let m, $r \in \mathbb{Z}$ with $m \neq 0$ and $m \neq 1$. Then*

$$U_{mn+r}=\sum_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}U_{j+r}t^{n-j}$$

and

$$V_{mn+r}=\sum_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}V_{j+r}t^{n-j}.$$

Proof. From Corollary 2.6, it follows that

$$S^{mn+r}=\begin{bmatrix} \dfrac{V_{mn+r}}{2} & \dfrac{(k^2+4t)U_{mn+r}}{2}\\ \dfrac{U_{mn+r}}{2} & \dfrac{V_{mn+r}}{2}\end{bmatrix}.$$

On the other hand, $S^m = U_m S + tU_{m-1}I$ and therefore

$$\begin{aligned}
S^{mn+r} &= (S^m)^n S^r = (U_m S + tU_{m-1}I)^n S^r = \sum_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}t^{n-j}S^{j+r}\\
&= \begin{bmatrix} \frac{1}{2}\sum\limits_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}t^{n-j}V_{j+r} & \frac{(k^2+4t)}{2}\sum\limits_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}t^{n-j}U_{j+r}\\ \frac{1}{2}\sum\limits_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}t^{n-j}U_{j+r} & \frac{1}{2}\sum\limits_{j=0}^{n}\binom{n}{j}U_m^j U_{m-1}^{n-j}t^{n-j}V_{j+r}\end{bmatrix}.
\end{aligned}$$

Then the proof follows. □

2.10. Corollary. *Let m, $r \in \mathbb{Z}$ with $m \neq 0$ and $m \neq 1$. If k^2+4t is not a perfect square, then*

$$U_{mn+r}=-(-t)^r\sum_{j=0}^{n}\binom{n}{j}(-U_m)^j U_{m+1}^{n-j}U_{j-r}$$

and

$$V_{mn+r}=(-t)^r\sum_{j=0}^{n}\binom{n}{j}(-U_m)^j U_{m+1}^{n-j}V_{j-r}.$$

Proof. The proof follows from Lemma 2.8 and Theorem 2.9 by taking $a = U_m$ and $b = tU_{m-1}$ □

2.11. Corollary. *$V_n^2 - (k^2+4t)U_n^2 = 4(-t)^n$ for every $n \in \mathbb{Z}$.*

Proof. From Theorem 2.9, it follows that

$$\det S^n = (\det S)^n = (-t)^n$$

and

$$\det S^n = \frac{V_n^2 - (k^2+4t)U_n^2}{4}.$$

Then the proof follows. □

2.12. Theorem. *Let $n \in \mathbb{N}$ and m be a nonzero integer. Then*

$$2^n V_{mn+r} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} (k^2+4t)^j V_r + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} (k^2+4t)^{j+1} U_r \tag{2.1}$$

and

$$2^n U_{mn+r} = \frac{1}{2^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} (k^2+4t)^j U_r + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} (k^2+4t)^j V_r \tag{2.2}$$

Proof. Let $K = S + tS^{-1} = \begin{bmatrix} 0 & k^2+4t \\ 1 & 0 \end{bmatrix}$. Then $K^{2j} = (k^2+4t)^j I$ and $K^{2j+1} = (k^2+4t)^j K$. Since

$$S^m = \frac{1}{2}(V_m I + U_m K),$$

it follows that

$$S^{mn+r} = (S^m)^n S^r = (\frac{1}{2}(V_m I + U_m K))^n S^r = \frac{1}{2^n}\left(\sum_{j=0}^{n} \binom{n}{j} U_m^j K^j V_m^{n-j}\right) S^r$$

and therefore

$$\begin{aligned} 2^n S^{mn+r} &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} K^{2j} S^r + \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} K^{2j+1} S^r \\ &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} V_m^{n-2j} (k^2+4t)^j S^r \\ &+ \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} V_m^{n-2j-1} (k^2+4t)^j K S^r \end{aligned}$$

Since

$$KS^r = \begin{bmatrix} \frac{(k^2+4t)U_r}{2} & \frac{(k^2+4t)V_r}{2} \\ \frac{V_r}{2} & \frac{(k^2+4t)U_r}{2} \end{bmatrix}$$

and

$$S^{mn+r} = \begin{bmatrix} \frac{V_{mn+r}}{2} & \frac{(k^2+4t)U_{mn+r}}{2} \\ \frac{U_{mn+r}}{2} & \frac{V_{mn+r}}{2} \end{bmatrix},$$

the proof follows. □

2.13. Theorem.

$$U_{m+n} = U_m U_{n+1} + tU_{m-1}U_n \tag{2.3}$$

and

$$(-t)^{n-1}U_{m-n} = U_{m-1}U_n - U_m U_{n-1} \tag{2.4}$$

for every m, $n \in \mathbb{Z}$.

Proof. Let $X = \begin{bmatrix} k & t \\ 1 & 0 \end{bmatrix}$. Then from Corollary 2.7, it follows that

$$X^{m+n} = X^m X^n = \begin{bmatrix} U_{m+1} & tU_m \\ U_m & tU_{m-1} \end{bmatrix} \begin{bmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{bmatrix}$$

and

$$\begin{aligned} X^{m-n} = X^m (X^n)^{-1} &= \begin{bmatrix} U_{m+1} & tU_m \\ U_m & tU_{m-1} \end{bmatrix} \begin{bmatrix} U_{n+1} & tU_n \\ U_n & tU_{n-1} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} U_{m+1} & tU_m \\ U_m & tU_{m-1} \end{bmatrix} \frac{1}{(-t)^n} \begin{bmatrix} tU_{n-1} & -tU_n \\ -U_n & U_{n+1} \end{bmatrix}. \end{aligned}$$

Then the proof follows. □

Now we give some identities, which we will use later. All the given identities can be shown by using the previously obtained formulae for S^n and X^n.

$$U_n V_{m+1} + tU_{n-1}V_m = V_{n+m} \tag{2.5}$$

$$V_m V_n - (k^2+4t)U_m U_n = 2(-t)^n V_{m-n} \tag{2.6}$$

$$U_m V_n - U_n V_m = 2(-t)^n U_{m-n} \tag{2.7}$$

$$V_m V_n = V_{m+n} + (-t)^n V_{m-n} \tag{2.8}$$

$$(k^2+4t)U_m U_n = V_{m+n} - (-t)^n V_{m-n} \tag{2.9}$$

$$U_m V_n = U_{m+n} + (-t)^n U_{m-n} \tag{2.10}$$

$$(-t)^n V_{m-n} = U_{m+1}V_n - V_{n+1}U_m \tag{2.11}$$

$$V_r V_{r+2} - V_{r+1}^2 = (-t)^r (k^2+4t) \tag{2.12}$$

2.14. Theorem. *Let m, n, $r \in \mathbb{Z}$ with $r \neq 0$. Then*

$$U_r U_{m+n+r} = U_{m+r}U_{n+r} - (-t)^r U_m U_n,$$

$$U_r U_{m+n-r} = U_m U_n - (-t)^r U_{m-r}U_{n-r},$$

and

$$U_r U_{m+n} = U_m U_{n+r} - (-t)^r U_{m-r}U_n.$$

Proof. Take $a = \dfrac{U_{r+1}}{U_r}$ and consider $A = \begin{bmatrix} a & b \\ c & k-a \end{bmatrix}$ with $\det A = -t$. Then by Corollary 2.3, we get

$$A^n = \begin{bmatrix} aU_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix} = \begin{bmatrix} \frac{U_{r+1}}{U_r}U_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - \frac{U_{r+1}}{U_r}U_n \end{bmatrix}.$$

Using (2.3) and (2.4) we see that

$$A^n = \begin{bmatrix} \dfrac{U_{n+r}}{U_r} & bU_n \\ cU_n & \dfrac{-(-t)^r U_{n-r}}{U_r} \end{bmatrix}.$$

Since $\det A = -t$ and $a = \dfrac{U_{r+1}}{U_r}$, it follows that

$$\begin{aligned} bc &= \frac{kU_rU_{r+1} + tU_r^2 - U_{r+1}^2}{U_r^2} = \frac{U_r(kU_{r+1} + tU_r) - U_{r+1}^2}{U_r^2} \\ &= \frac{U_rU_{r+2} - U_{r+1}^2}{U_r^2} = \frac{-(-t)^r}{U_r^2} \end{aligned}$$

by (2.4). If we consider the matrix multiplication $A^nA^m = A^{m+n}$, then we get the result. □

2.15. Corollary. *$U_{n+r}U_{n-r} - U_n^2 = -(-t)^{n-r}U_r^2$ for all n, $r \in \mathbb{Z}$.*

Proof. Since $\det A = -t$, $\det A^n = (\det A)^n = (-t)^n$. Moreover, since

$$\det A^n = -(-t)^r \frac{U_{n+r}}{U_r}\frac{U_{n-r}}{U_r} - bcU_n^2 = -(-t)^r \left(\frac{U_{n+r}U_{n-r} - U_n^2}{U_r^2} \right) = (-t)^n,$$

it can be seen that $U_{n+r}U_{n-r} - U_n^2 = -(-t)^{n-r}U_r^2$. □

2.16. Theorem. *Let m, n, $r \in \mathbb{Z}$. Then*

$$V_rV_{m+n+r} = V_{m+r}V_{n+r} + (-t)^r(k^2 + 4t)U_mU_n,$$

$$V_rV_{m+n-r} = (k^2 + 4t)U_mU_n + (-t)^rV_{m-r}V_{n-r},$$

and

$$V_rU_{m+n} = U_nV_{m+r} + (-t)^rV_{n-r}U_m.$$

Proof. Take $a = \dfrac{V_{r+1}}{V_r}$ and consider $B = \begin{bmatrix} a & b \\ c & k-a \end{bmatrix}$ with $\det B = -t$. Then by Corollary 2.3, we get

$$B^n = \begin{bmatrix} aU_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - aU_n \end{bmatrix} = \begin{bmatrix} \frac{V_{r+1}}{V_r}U_n + tU_{n-1} & bU_n \\ cU_n & U_{n+1} - \frac{V_{r+1}}{V_r}U_n \end{bmatrix}.$$

Using (2.5) and (2.11) we see that

$$B^n = \begin{bmatrix} \dfrac{V_{n+r}}{V_r} & bU_n \\ cU_n & \dfrac{(-t)^r V_{n-r}}{V_r} \end{bmatrix}.$$

Since $\det B = -t$ and $a = \dfrac{V_{r+1}}{V_r}$, it follows that

$$bc = \frac{kV_rV_{r+1} + tV_r^2 - V_{r+1}^2}{V_r^2} = \frac{V_r(kV_{r+1} + tV_r) - V_{r+1}^2}{V_r^2}$$
$$= \frac{V_rV_{r+2} - V_{r+1}^2}{V_r^2} = \frac{(-t)^r(k^2+4t)}{V_r^2}$$

by (2.12). If we consider the matrix multiplication $B^nB^m = B^{m+n}$, then we get the result. □

2.17. Corollary. *$V_{n+r}V_{n-r} - (k^2+4t)U_n^2 = (-t)^{n-r}V_r^2$ for all $n, r \in \mathbb{Z}$.*

Proof. Since $\det B = -t$, $\det B^n = (\det B)^n = (-t)^n$. Moreover, since

$$\det B^n = (-t)^r \frac{V_{n+r}}{V_r}\frac{V_{n-r}}{V_r} - bcU_n^2 = (-t)^r\left(\frac{V_{n+r}V_{n-r}}{V_r^2} - \frac{(k^2+4t)U_n^2}{V_r^2}\right) = (-t)^n,$$

it can be seen that $V_{n+r}V_{n-r} - (k^2+4t)U_n^2 = (-t)^{n-r}V_r^2$. □

3. Sums and Congruences

Now we will give some sums containing generalized Fibonacci and Lucas numbers. Then we will give some congruences concerning generalized Fibonacci and Lucas numbers. Firstly, we will prove a lemma to use in the following theorems. It can be seen that

$$\alpha^{2n} = \alpha^nV_n - (-t)^n \tag{3.1}$$

and

$$\alpha^{2n} = \alpha^nU_n\sqrt{k^2+4t} + (-t)^n \tag{3.2}$$

by (1.2). Now we can give the following lemma.

3.1. Lemma.

$$S^{2n} = S^nV_n - (-t)^nI \tag{3.3}$$

and

$$S^{2n} = U_nKS^n + (-t)^nI \tag{3.4}$$

for every $n \in \mathbb{N}$, where K is as in Theorem 2.12.

Proof. Let $\mathbb{Z}[\alpha] = \{a\alpha + b \mid a,\ b \in \mathbb{Z}\}$ and $\mathbb{Z}[S] = \{aS + b \mid a,\ b \in \mathbb{Z}\}$. We define a function $\varphi : \mathbb{Z}[\alpha] \to \mathbb{Z}[S]$, given by $\varphi(a\alpha + b) = aS + bI$. Then φ is ring homomorphism. Moreover it is clear that $\varphi(\alpha) = S$ and therefore we get $\varphi(\alpha^n) = (\varphi(\alpha))^n = S^n$. Thus from (3.1), we get

$$S^{2n} = (\varphi(\alpha))^{2n} = \varphi(\alpha^{2n}) = \varphi(\alpha^nV_n - (-t)^n) = S^nV_n - (-t)^nI.$$

That is, $S^{2n} = S^nV_n - (-t)^nI$. Also from (3.2), we get

$$S^{2n} = (\varphi(\alpha))^{2n} = \varphi(\alpha^{2n}) = \varphi(U_n\sqrt{k^2+4t}\alpha^n + (-t)^n) =$$
$$U_n\varphi\left(\sqrt{k^2+4t}\right)S^n + (-t)^nI.$$

Since

$$\varphi\left(\sqrt{k^2+4t}\right) = \varphi(2\alpha - k) = 2S - kI = \begin{bmatrix} 0 & k^2+4t \\ 1 & 0 \end{bmatrix} = K,$$

we get $S^{2n} = U_nKS^n + (-t)^nI$. □

3.2. Theorem. *Let m, $r \in \mathbb{Z}$. Then*

$$U_{2mn+r} = (-(-t)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j U_{mj+r} (-(-t)^m)^{-j}$$

and

$$V_{2mn+r} = (-(-t)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j V_{mj+r} (-(-t)^m)^{-j}$$

for every $n \in \mathbb{N}$.

Proof. It is known that

$$S^{2m} = S^m V_m - (-t)^m I \tag{3.5}$$

by (3.3). Taking the n-th power of (3.5), we get

$$S^{2mn} = (S^m V_m - (-t)^m I)^n = \sum_{j=0}^{n} \binom{n}{j} V_m^j (-(-t)^m)^{n-j} S^{mj}.$$

Multiplying both sides of this equation by S^r, we obtain

$$S^{2mn+r} = (-(-t)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j (-(-t)^m)^{-j} S^{mj+r}.$$

Thus it follows that

$$U_{2mn+r} = (-(-t)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j U_{mj+r} (-(-t)^m)^{-j}$$

and

$$V_{2mn+r} = (-(-t)^m)^n \sum_{j=0}^{n} \binom{n}{j} V_m^j V_{mj+r} (-(-t)^m)^{-j}$$

by Corollary 2.6. $\square$

3.3. Corollary. *Let k and t be integers. Then for all $n, m \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$ such that $mn + r \geq 0$ if $t \neq \pm 1$, we get*

$$U_{2mn+r} \equiv (-(-t)^m)^n U_r \pmod{V_m}$$

and

$$V_{2mn+r} \equiv (-(-t)^m)^n V_r \pmod{V_m}.$$

3.4. Theorem. *Let m, $r \in \mathbb{Z}$ and m be nonzero integer. Then*

$$\begin{aligned} U_{2mn+r} &= (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} U_{2mj+r} D^j t^{-2mj} \\ &\quad + (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} V_{2mj+m+r} D^j (-t)^{m(-2j-1)} \end{aligned}$$

and

$$V_{2mn+r} = (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} V_{2mj+r} D^j t^{-2mj}$$
$$+ (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} U_{2mj+m+r} D^{j+1} (-t)^{m(-2j-1)}$$

for every $n \in \mathbb{N}$, where $D = k^2 + 4t$.

Proof. It is known that

$$S^{2m} = U_m K S^m + (-t)^m I$$

by (3.4). It is clear that

$$S^{2mn+r} = (U_m K S^m + (-t)^m I)^n S^r = \sum_{j=0}^{n} \binom{n}{j} U_m^j K^j ((-t)^m)^{n-j} S^{mj+r}.$$

On the other hand, it can be seen that $K^{2j} = D^j I$ and $K^{2j+1} = D^j K$. Therefore, we get

$$S^{2mn+r} = (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} K^{2j} t^{-2mj} S^{2mj+r}$$
$$+ (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} K^{2j+1} (-t)^{m(-2j-1)} S^{2mj+m+r}$$
$$= (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} U_m^{2j} D^j t^{-2mj} S^{2mj+r}$$
$$+ (-t)^{mn} \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} U_m^{2j+1} D^j (-t)^{m(-2j-1)} K S^{2mj+m+r}.$$

The proof follows from Corollary 2.6. □

3.5. Corollary. *Let k and t be integers. Then for all $n, m \in \mathbb{N}$ and $r \in \mathbb{Z}$ such that $mn + r \geq 0$ if $t \neq \pm 1$, we get*

$$U_{2mn+r} \equiv (-t)^{mn} U_r \pmod{U_m}$$

and

$$V_{2mn+r} \equiv (-t)^{mn} V_r \pmod{U_m}.$$

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