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Review Article

q-Genocchi Numbers and Polynomials Associated with q-Genocchi-Type l-Functions

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The main purpose of this paper is to study generating functions of the q-Genocchi numbers and polynomials. We prove a new relation for the generalized q-Genocchi numbers, which is related to the q-Genocchi numbers and q-Bernoulli numbers. By applying Mellin transformation and derivative operator to the generating functions, we define q-Genocchi zeta and l-functions, which are interpolated q-Genocchi numbers and polynomials at negative integers. We also give some applications of the generalized q-Genocchi numbers.

1. Introduction definitions and notations

In [1], Jang et al. gave new formulae on Genocchi numbers. They defined poly-Genocchi numbers to give the relation between Genocchi numbers, Euler numbers, and poly-Genocchi numbers. In [2], Kim et al. constructed new generating functions of the q-analogue Eulerian numbers and q-analogue Genocchi numbers. They gave relations between Bernoulli numbers, Euler numbers, and Genocchi numbers. They also defined Genocchi zeta functions which interpolate these numbers at negative integers. Kim [3] gave new concept of the q-extension of Genocchi numbers and gave some relations between q-Genocchi polynomials and q-Euler numbers. In this paper, by using generating function of this numbers, we study q-Genocchi zeta and l-functions. In [4], Kim constructed q-Genocchi numbers and polynomials. By using these numbers and polynomials, he proved the q-analogue of alternating sums of powers of consecutive integers due to Euler:

$$\sum_{j=0}^{k-1}\left[j:q^2\right](-1)^{j-1}[j]^{n-1}q^{(k-j)(n+1)/2}=\frac{G_{n,k,q}-G_{n,k,q}(k)}{(1+q)n} \tag{1.1}$$

(cf. [4]), where if $q\in\mathbb{C}$, $|q|<1$,

$$[x]=[x:q]=\frac{1-q^x}{1-q},\qquad \left[j:q^2\right]=\frac{1-q^{2j}}{1-q^2}, \tag{1.2}$$

and the numbers $G_{n,k,q}$ are called q-Genocchi numbers which are defined by

$$(1+q)t\sum_{j=0}^{\infty}q^{k-j}\left[j:q^2\right](-1)^{j-1}\exp\left(t\left[j,q^2\right]q^{(k-j)/2}\right)=\sum_{j=0}^{\infty}G_{n,k,q}\frac{t^n}{n!}. \tag{1.3}$$

Note that $\lim_{q\to1}[x]=x$, (cf. [3, 5–9]). The Euler numbers E_n are usually defined by means of the following generating function (cf. [10–16]):

$$\frac{2}{e^t+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!},\quad |t|<\pi. \tag{1.4}$$

The Genocchi numbers G_n are usually defined by means of the following generating function (cf. [12, 13]):

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!},\quad |t|<\pi. \tag{1.5}$$

These numbers are classical and important in number theory. In [12], Kim defined generating functions of the q-Genocchi numbers and q-Euler numbers as follows:

$$(1+q)e^{t/(1-q)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(1+q^{n+1})(1-q)^n}\frac{t^n}{n!}=\sum_{m=0}^{\infty}E_{m,q}\frac{t^m}{m!}, \tag{1.6}$$

where $E_{m,q}$ denotes q-Euler numbers,

$$G_q(t)=(1+q)t\sum_{m=0}^{\infty}(-1)^nq^ne^{[n]t}=\sum_{m=0}^{\infty}G_{m,q}\frac{t^m}{m!}, \tag{1.7}$$

where $G_{m,q}$ denotes q-Genocchi numbers. Genocchi zeta function is defined as follows (cf. [13, page 108]): for $s\in\mathbb{C}$,

$$\zeta_G(s)=2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^s}. \tag{1.8}$$

Kim [17] defined the ferminoic and deformic expression of p-adic q-Volkenborn integral at $q = -1$ and $q = 1$. He constructed integral equation of the fermionic expression of p-adic q-Volkenborn integral at $q = -1$. By using this integral equation, he defined new generating functions of λ-Euler numbers and polynomials. By using derivative operator to this functions, he constructed new λ-zeta, λ-l-functions and p-adic λ-l-functions, which are interpolated λ-Euler numbers and polynomials. He also gave some applications which are the formulae of the trigonometric functions by applying ferminoic and deformic expression of p-adic q-Volkenborn integral at $q = -1$ and $q = 1$. Kim and Rim [18] defined two-variable L-function. They gave main properties of this function. In [6], Kim constructed the two-variable p-adic q-L-function which interpolates the generalized q-Bernoulli polynomials attached to Dirichlet character. In [19], Simsek et al. constructed the two-variable Dirichlet q-L-function and the two-variable multiple Dirichlet-type Changhee q-L-function. In [8, 20], Simsek defined generating functions, which are interpolates twisted Bernoulli numbers and polynomials, twisted Euler numbers and polynomials. He[21] also gave new generating functions which produce q-Genocchi zeta functions and q-l-series with attached to Dirichlet character. Therefore, by using these generating functions, he constructed new q-analogue of Hardy-Berndt sums. He gave relations between these sums, q-Genocchi zeta functions and q-l-series as well,

$$\zeta_G(s)\Gamma(s) = \int_0^\infty \frac{2x^{s-1}}{e^{-x}+1}dx \tag{1.9}$$

(cf. [21]), where $\Gamma(s)$ is Euler's gamma function and $\zeta_G(1-n) = -G_n/n,\ n > 1$ (cf. [1], [13, page 108, equation (2.43)]). The first author defined q-analogue of the Genocchi zeta functions as follows [21].

Definition 1.1. Let $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$. q-analogue of the Genocchi type zeta function is expressed by the formula

$$\Im_{G,q}(s) = (1+q)\sum_{n=1}^{\infty}\frac{(-1)^n q^{-n}}{\left(q^{-n}[n]\right)^s}. \tag{1.10}$$

Remark 1.2. If $q \to 1$, then (1.10) reduces to ordinary Genocchi zeta functions (see [13, page 108]). Cenkci et al. [22], defined different type of q-Genocchi zeta functions, which are defined as follows:

$$\zeta_q^{(G)}(s) = q(1+q)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^n}{[n]^s}. \tag{1.11}$$

Simsek [21] defined q-analogue of the Hurwitz-type Genocchi zeta function by applying the Mellin transformations as follows:

$$\Im_q(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\sum_{n=0}^{\infty}(-1)^n q^{-n}e^{-(q^{-n}[n]+x)t}\right)dt. \tag{1.12}$$

Definition 1.3 (see [21]). Let $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, and $0 < x \leq 1$. q-analogue of the Hurwitz-type Genocchi zeta function is expressed by the formula

$$\Im_{G,q}(s, x) := [2]\Im_q(s, x). \tag{1.13}$$

Observe that when $x = 1$, the $\Im_{G,q}(s, x)$ is reduced to $\Im_{G,q}(s)$ and if $q \to 1$, then $\Im_{G,q}(s, x) \to \Im_G(s, x)$. A function $\Im_G(s, x)$ is called an ordinary Hurwitz-type Genocchi zeta function if $\Im_G(s, x)$ is expressed by the formula

$$\Im(s, x) := 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}, \tag{1.14}$$

where $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, and $0 < x \leq 1$, cf. [13].

In [21], Simsek defined q-analogue (Genocchi-type) one- and two-variable l-functions as follows, respectively; let χ be a Dirichlet character; let $s \in \mathbb{C}$ and $\mathrm{Re}(s) > 1$;

$$l_{G,q}(s, \chi) = \frac{(1+q)}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n=1}^{\infty} (-1)^n \chi(n) q^{-n} e^{-q^{-n}[n]t} \right) dt, \tag{1.15}$$

$$l_{G,q}(s, x, \chi) = \frac{(1+q)}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\sum_{n=0}^{\infty} (-1)^n \chi(n) q^{-n} e^{-(q^{-n}[n]+x)t} \right) dt. \tag{1.16}$$

A function $l_G(s, \chi)$ is called an ordinary Genocchi-type l-function if $l_G(s, \chi)$ is expressed by the formula

$$l(s, x) := 2\sum_{n=0}^{\infty} \frac{(-1)^n \chi(n)}{(n+x)^s}, \tag{1.17}$$

where $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$ and $0 < x \leq 1$, cf. [13].

Observe that when $\chi \equiv 1$, (1.15) reduces to (1.10):

$$l_q(s, 1) = \Im_q(s). \tag{1.18}$$

We summarize our work as follows. In Section 2, we study generating functions of the q-Genocchi numbers and polynomials. By using infinite and finite series, we give some definitions of the q-Genocchi numbers and polynomials. We find new relations between generalized q-Genocchi numbers with attached to χ, q-Genocchi numbers and Barnes' type Changhee q-Bernoulli numbers. In Section 3, by applying Mellin transformation and derivative operator to the generating functions of the q-Genocchi numbers, we construct q-Genocchi zeta and l-functions, which are interpolated q-Genocchi numbers and polynomials at negative integers. We also give some new relations related to these numbers and polynomials.

2. q-Genocchi number and polynomials

In this section, we give some new relations and identities related to q-Genocchi numbers and polynomials. Firstly we give some generating functions of the q-Genocchi numbers, which were defined by Kim [3, 10, 11]:

$$F_q(t) = e^{t/(1-q)} \sum_{j=0}^{\infty} \frac{(1+q)}{[2:q^{j+1}]} \left(\frac{1}{q-1} \right)^j \frac{t^j}{j!} = (1+q) \sum_{l=0}^{\infty} (-q)^l e^{[l]t}, \tag{2.1}$$

and let

$$F_q^*(t) = t(1+q)\sum_{l=0}^{\infty}(-q)^l e^{[l]t} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!} \tag{2.2}$$

(cf.[3, 10, 11, 23]), where $G_{n,q}$ denotes q-Genocchi numbers.

We note that q-Genocchi numbers, $G_{n,q}$, were defined by Kim [3, 10, 11].

By using the above generating functions, q-Genocchi polynomials, $G_{n,q}(x)$, are defined by means of the following generating function:

$$F_q^*(t,x) = F_q^*(t)e^{tx} = \sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!}. \tag{2.3}$$

Our generating function of $G_{n,q}(x)$ is similar to that of [3, 12, 21, 23]. By using Cauchy product in (2.3), we easily obtain

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} G_{n,q}\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{t^n x^n}{n!} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} G_{k,q}\frac{x^{n-k}}{k!(n-k)!}t^n. \tag{2.4}$$

Then by comparing coefficients of t^n on both sides of the above equation, for $n \geq 2$, we obtain the following result.

Theorem 2.1. *Let n be an integer with $n \geq 2$. Then one has*

$$G_{n,q}(x) = \sum_{k=0}^{\infty}\binom{n}{k} x^{n-k} G_{k,q}. \tag{2.5}$$

By using the same method in [3, 12, 21] in (2.3), we have

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = (1+q)t\sum_{n=0}^{\infty}(-1)^n q^n e^{[n]t+xt} = (1+q)t\sum_{n=0}^{\infty}(-1)^n q^n \sum_{k=0}^{\infty}\frac{([n]+x)^k t^k}{k!}, \tag{2.6}$$

and after some elementary calculations, we have

$$\sum_{k=0}^{\infty} G_{k,q}(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\left((1+q)\sum_{n=0}^{\infty}(-1)^n q^n([n]+x)^{k-1}k\right)\frac{t^k}{k!}. \tag{2.7}$$

By comparing coefficients of $t^k/k!$ on both sides of the above equation, we arrive at the following corollary.

Corollary 2.2. *Let $k \in \mathbb{N}$. Then one has*

$$G_{k,q}(x) = k(q+1)\sum_{n=0}^{\infty}\sum_{j=0}^{k-1}\sum_{d=0}^{j}\binom{k-1}{j}\binom{j}{d}\frac{(-1)^{n+d}q^{d(n+1)}x^{k-j-1}}{(1-q)^j}. \tag{2.8}$$

We give some of q-Genocchi polynomials as follows: $G_{o,q}(x) = 0$, $G_{1,q}(x) = 1, G_{2,q}(x) = 2x - 2q/(1 + q^2), \ldots$.

From the generating function $F_q^*(t)$, we have the following.

Corollary 2.3. *Let* $k \in \mathbb{N}$. *Then one has*

$$G_{k,q} = k(1+q)\sum_{n=0}^{\infty}(-1)^n q^n [n]^{k-1} = \frac{k(1-q^2)}{(1-q)^k}\sum_{j=0}^{k-1}\frac{\binom{k-1}{j}(-1)^j}{1+q^{j+1}}. \tag{2.9}$$

Proof of the Corollary 2.3 was given by Kim [3, 12]. We give some of q-Genocchi numbers as follows: $G_{o,q} = 0$, $G_{1,q} = 1$, $G_{2,q} = -2q/(1+q^2), \ldots$.

Observe that if $q \to 1$, then $G_{2,1} = -1$.

By using derivative operator to (2.6), we have

$$\frac{d}{dx}\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!} = \frac{d}{dx}\left((1+q)t\sum_{n=0}^{\infty}(-1)^n q^n e^{([n]+x)t}\right) = \sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^{n+1}}{n!}. \tag{2.10}$$

After some elementary calculations, we arrive at the following corollary.

Corollary 2.4. *Let n be a positive integer. Then one has*

$$\frac{d}{dx}G_{n,q}(x) = nG_{n-1,q}(x). \tag{2.11}$$

Corollary 2.5. *Let n be a positive integer. Then one has*

$$G_{q,n}(x+y) = \sum_{k=0}^{n}\binom{n}{k}G_{k,q}(x)y^{n-k}. \tag{2.12}$$

Proof. Proof of this corollary is easily obtained from (2.4). □

Generalized q-Genocchi numbers are defined by means of the following generating function (this generating function is similar to that of [3, 12, 21–24]):

$$F_{q,\chi}(t) = (1+q)t\sum_{n=0}^{\infty}\chi(n)q^n(-1)^n e^{[n]t} = \sum_{n=0}^{\infty}G_{n,\chi,q}\frac{t^n}{n!}, \tag{2.13}$$

where χ denotes the Dirichlet character with conductor $d \in \mathbb{Z}^+$, the set of positive integers.

Observe that when $\chi \equiv 1$, (2.13) reduces to (2.3).

By (2.13), we have

$$\sum_{m=0}^{\infty}G_{m,\chi,q}\frac{t^m}{m!} = (1+q)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\chi(n)q^n(-1)^n[n]^m t^{m+1}}{m!}. \tag{2.14}$$

After some elementary calculations and by comparing coefficients t^m on both sides of the above equation, we get

$$G_{m,\chi,q} = (1+q)m\sum_{n=0}^{\infty}(-1)^n q^n \chi(n)[n]^{m-1}. \tag{2.15}$$

By setting $n = a+dj$, where $(j = 0,1,2,\ldots,\infty; a = 1,2,\ldots,d)$, and $\chi(a+jd) = \chi(a)$, in the above equation, we obtain

$$\begin{aligned} G_{m,\chi,q} &= (1+q)m\sum_{j=0}^{\infty}\sum_{a=1}^{d}(-1)^{a+jd}q^{a+jd}\chi(a+jd)[a+jd]^{m-1} \\ &= (1+q)m\sum_{a=1}^{d}\sum_{i=0}^{m-1}(-1)^a\binom{m-1}{i}q^{a(m-i)}\chi(a)[a]^i[d]^{m-i-1}\sum_{j=0}^{\infty}(-1)^{dj}q^{dj}\left[j,q^d\right]^{m-i-1}. \end{aligned} \tag{2.16}$$

In [15], Srivastava et al. defined the following generalized Barnes-type Changhee q-Bernoulli numbers.

Let χ be the Dirichlet character with conductor d. Then the generalized Barnes-type Changhee q-Bernoulli numbers with attached to χ are defined as follows:

$$F_{q,\chi}(t \mid w_1) = -w_1 t\sum_{n=0}^{\infty}\chi(n)q^{w_1 n}e^{[w_1 n]t} = \sum_{n=0}^{\infty}\frac{\beta_{n,\chi,q}(w_1)t^n}{n!}, \quad |t| < 2\pi \tag{2.17}$$

(cf. [15]). Substituting $\chi \equiv 1$ and $w_1 = 1$ into the above equation, we have

$$F_{q,1}(t \mid 1) = -t\sum_{n=0}^{\infty}q^n e^{[n]t} = \sum_{n=0}^{\infty}\frac{\beta_{n,q}t^n}{n!}. \tag{2.18}$$

By using derivative operator to the above, we obtain

$$\frac{d^m}{dt^m}F_{q,1}(t \mid 1)|_{t=0} = \beta_{m,q} = -m\sum_{n=0}^{\infty}q^n[n]^{m-1}. \tag{2.19}$$

By substituting (2.9) and (2.19) into (2.16), after some calculations, we arrive at the following theorem.

Theorem 2.6. *Let χ be the Dirichlet character with conductor d. If d is odd, then one has*

$$G_{m,\chi}(q) = \sum_{a=1}^{d}\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^a q^{a(m-i)}\chi(a)[a]^i[d]^{m-i-1}G_{m-i}\left(q^d\right), \tag{2.20}$$

if d is even, then one has

$$G_{m,\chi}(q) = \sum_{a=1}^{d}\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{a+1}\frac{m}{m-i}q^{a(m-i)}\chi(a)[a]^i[d]^{m-i-1}\beta_{m-i,q^d}, \tag{2.21}$$

where β_{m-i,q^d} is defined in (2.19).

Remark 2.7. In Theorem 2.6, we give new relations between generalized q-Genocchi numbers, $G_{m,\chi}(q)$ with attached to χ, q-Genocchi numbers, $G_m(q)$, and Barnes-type Changhee q-Bernoulli numbers. For detailed information about generalized Barnes-type Changhee q-Bernoulli numbers with attached to χ see [15].

Generalized Genocchi polynomials are defined by means of the following generating function:

$$F_{q,\chi}(t,x) = F_{q,\chi}(t)e^{tx} = \sum_{n=0}^{\infty} G_{n,\chi,q}(x)\frac{t^n}{n!}. \tag{2.22}$$

Theorem 2.8. *Let χ be the Dirichlet character with conductor d. Then one has*

$$G_{n,\chi,q}(x) = \sum_{n=0}^{\infty} \binom{n}{k} G_{n,\chi,q}x^{n-k}. \tag{2.23}$$

Remark 2.9. Generating functions of $G_{n,q}(x)$ and $G_{n,\chi,q}(x)$ are different from those of [3, 12, 22, 23]. Kim defined generating function of $G_{n,q}(x)$, as follows [12]:

$$F_q(t,x) = (1+q)t\sum_{m=0}^{\infty} q^m(-1)^m e^{[m+x]t} = \sum_{m=0}^{\infty} G_{n,q}(x)\frac{t^m}{m!}. \tag{2.24}$$

In [21], Simsek defined generating function of $G_{n,q}(x)$ by

$$F_q(t,x) = \sum_{n=0}^{\infty} (-1)^n q^{-n} \exp(-(q^{-n}[n]+x)t). \tag{2.25}$$

3. q-Genocchi zeta and l-functions

In recent years, many mathematicians and physicians have investigated zeta functions, multiple zeta functions, l-series, q-Genocchi zeta, and l-functions, and q-Bernoulli, Euler, and Genocchi numbers and polynomials mainly because of their interest and importance. These functions and numbers are not only used in complex analysis, but also used in p-adic analysis and other areas. In particular, multiple zeta functions occur within the context of Knot theory, quantum field theory, applied analysis and number theory, (cf. [15]). In this section, we define q-Genocchi zeta and l-functions, which are interpolated q-Genocchi polynomials and generalized q-Genocchi numbers at negative integers. By applying the Mellin transformation to (2.3), we obtain

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2}F_q^*(-t,x)dt = \frac{-(1+q)}{\Gamma(s)}\int_0^{\infty} t^{s-1}\sum_{n=0}^{\infty}(-1)^n q^n e^{-([n]+x)t}dt = (1+q)\sum_{n=0}^{\infty}\frac{(-1)^{n+1}q^n}{([n]+x)^s}, \tag{3.1}$$

where $\operatorname{Re} s > 1$, $0 < x \leq 1$, and $|q| < 1$.

Thus, Hurwitz-type q-Genocchi zeta function is defined by the following definition.

Definition 3.1. Let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ and let $q \in \mathbb{C}$ with $|q| < 1$. Then one defines

$$\zeta_{G,q}(s,x) = (1+q)\sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^n}{([n]+x)^s}. \tag{3.2}$$

Observe that when $x = 1$ in (3.2), then we obtain Riemann-type q-Genocchi zeta function:

$$\zeta_{G,q}(s) = (1+q)\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^n}{[n]^s}. \tag{3.3}$$

Hurwitz-type q-Genocchi zeta function interpolates q-Genocchi polynomials at negative integers. For $s = 1-k$, $k \in \mathbb{Z}^+$, and by applying Cauchy residue theorem to (3.1), we can obtain the following theorem.

Theorem 3.2. *For $s = 1-k$, $k > 0$, then one has*

$$\zeta_{G,q}(1-k,x) = -\frac{G_{k,q}(x)}{k}. \tag{3.4}$$

Remark 3.3. The second proof of Theorem 3.2 can be obtained by using $(d^k/dt^k)|_{t=0}$ derivative operator to (2.3) as follows:

$$\begin{aligned} \frac{d^k}{dt^k} F_q^*(t,x)\bigg|_{t=0} &= (1+q)\frac{d^k}{dt^k}\left(t\sum_{n=0}^{\infty}(-1)^n q^n e^{([n]+x)t}\right)\bigg|_{t=0}, \\ \frac{-G_{k,q}(x)}{k} &= (1+q)\sum_{n=0}^{\infty}(-1)^{n+1} q^n ([n]+x)^{k-1}. \end{aligned} \tag{3.5}$$

Thus we obtained the desired result.

By applying Mellin transformation to (2.13), we obtain

$$l_{q,G}(s,\chi) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2} F_{q,x}(-t)\,dt = (1+q)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\chi(n) q^n}{[n]^s}. \tag{3.6}$$

Thus we can define Dirichlet-type q-Genocchi l-function as follows.

Definition 3.4. Let χ be the Dirichlet character with conductor d. Let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. One defines

$$l_{q,G}(s,\chi) = (1+q)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\chi(n) q^n}{[n]^s}. \tag{3.7}$$

Relation between $l_{q,G}(s,\chi)$ and $\zeta_{q,G}(s,x)$ is given by the following theorem.

Theorem 3.5. *Let χ be the Dirichlet character with conductor d. Then one has*

$$l_{q,G}(s,\chi)=\frac{(1+q)}{(1+q^d)[d]^s}\sum_{a=0}^{d-1}\chi(a)q^{a(1-s)}(-1)^a\zeta_{q^d,G}\left(s,\frac{q^{-a}[a]}{[d]}\right). \tag{3.8}$$

Proof. By setting $n=a+dk$, where $(k=0,1,2,\dots,\infty;\ a=1,2,3,\dots,d)$ in (3.7), we obtain,

$$\begin{aligned}
l_{q,G}(s,\chi)&=(1+q)\sum_{a=0}^{d}\sum_{k=1}^{\infty}\frac{\chi(a+kd)q^{a+kd}(-1)^{a+kd+1}}{[a+kd]^s}\\
&=(1+q)\sum_{a=0}^{d}\sum_{k=0}^{\infty}\frac{\chi(a+kd)q^{a+kd}(-1)^{a+kd+1}}{([a]+q^a[d][k:q^d])^s}\\
&=\frac{(1+q)}{(1+q^d)}\sum_{a=0}^{d-1}\frac{\chi(a)q^{a(1-s)}(-1)^a}{[d]^s}\sum_{k=0}^{\infty}\frac{(1+q^d)q^{kd}(-1)^{kd+1}}{([k:q^d]+q^{-a}[a]/[d])^s}.
\end{aligned} \tag{3.9}$$

After some elementary calculations, we arrive at the desired result of the theorem. □

The function $l_{q,G}(s,\chi)$ interpolates generalized q-Genocchi numbers, which are given by the following theorem.

Theorem 3.6. *Let $n\in\mathbb{Z}^+$. Let χ be the Dirichlet character with conductor d. Then one has*

$$l_{q,G}(1-n,\chi)=-\frac{G_{n,\chi}(q)}{n}. \tag{3.10}$$

Proof. Proof of this theorem is similar to that of Theorem 3.2. So we omit the proof. □

We give some applications. Setting $s=1-n, n\in\mathbb{Z}^+$ and using Theorem 3.2 in Theorem 3.5, we get

$$l_{q,G}(1-n,\chi)=\frac{(1+q)[d]^{n-1}}{n(1+q^d)}\sum_{a=1}^{d}(-1)^{a+1}\chi(a)q^aG_{n,q^d}\left(\frac{q^{-a}[a]}{[d]}\right). \tag{3.11}$$

By comparing both sides of the above equation and Theorem 3.6, we obtain distributions relation of the generalized Genocchi numbers as follows.

Corollary 3.7. *Let χ be the Dirichlet character with conductor d. Then one has*

$$G_{n,\chi}(q)=\frac{(1+q)[d]^{n-1}}{(1+q^d)}\sum_{a=1}^{d}(-1)^{a+1}\chi(a)q^aG_{n,q^d}\left(\frac{q^{-a}[a]}{[d]}\right), \tag{3.12}$$

where $n\geq 0$, and $G_{n,q^d}(q^{-a}[a]/[d])$ is the q-Genocchi polynomial.

By substituting (2.5) into (3.12), we have the following corollary.

Corollary 3.8. *Let χ be the Dirichlet character with conductor d. Then one has*

$$\begin{aligned} G_{n,\chi}(q) &= \frac{(1+q)[d]^{n-1}}{(1+q^d)} \sum_{a=1}^{d} (-1)^a \chi(a) q^a \sum_{k=0}^{n} \binom{n}{k} \left(\frac{q^{-a}[a]}{[d]} \right)^{n-k} G_{k,q^d} \\ &= \frac{(1+q)}{(1+q^d)[d]} \sum_{a=1}^{d} (-1)^a \chi(a) q^a [a]^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{q^a[d]}{[a]} \right)^{k} G_{k,q^d}. \end{aligned} \tag{3.13}$$

If we substitute (2.7) into (3.12), we get a new relation for the distribution relation of q-Genocchi numbers:

$$\begin{aligned} G_{n,\chi}(q) &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{a=1}^{d} (-1)^{a+1} \chi(a) q^a \sum_{j=0}^{\infty} (-1)^j q^j \left([j] + \frac{q^{-a}[a]}{[d]} \right)^{n-1} \\ &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^{d} (-1)^{a+j+1} \chi(a) q^{a+j} \sum_{m=0}^{n-1} \binom{n-1}{m} \left(\frac{q^{-a}[a]}{[d]} \right)^{m} [j]^{n-1-m} \\ &= \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^{d} \sum_{m=0}^{n-1} (-1)^{a+j+1} \binom{n-1}{m} \chi(a) q^{a+j} \left(\frac{q^{-a}[a]}{[d]} \right)^{m} [j]^{n-m-1}. \end{aligned} \tag{3.14}$$

Thus we arrive at the following corollary.

Corollary 3.9. *Let χ be the Dirichlet character with conductor d. Then one has*

$$G_{n,\chi}(q) = \frac{n(1+q)[d]^{n-1}}{(1+q^d)} \sum_{j=0}^{\infty} \sum_{a=1}^{d} \sum_{m=0}^{n-1} (-1)^{a+j+1} \binom{n-1}{m} \chi(a) q^{a+j} \left(\frac{q^{-a}[a]}{[d]} \right)^{m} [j]^{n-m-1}. \tag{3.15}$$

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