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Generalized Identities Involving Common Factors of Fibonacci and Lucas Numbers

Bijendra Singh [1], Pooja Bhadouria [2] and Omprakash Sikhwal [3]

[1] School of Studies in Mathematics, Vikram University Ujjain, India
bijendrasingh@yahoo.com

[2] School of Studies in Mathematics, Vikram University Ujjain, India
pooja.kajal@yahoo.co.in

[3] Department of Mathematics
Mandsaur Institute of Technology,Mandsaur, India
opbhsikhwal@rediffmail.com, opsikhwal@gmail.com

Abstract

The Fibonacci and Lucas numbers appear in numerous mathematical problems. In this paper we present some generalized identities involving common factors of Fibonacci and Lucas numbers. Binet's formula will employ to obtain the identities. Some identities involving Fibonacci and Lucas polynomials also explained.

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1. INTRODUCTION

Fibonacci numbers are a popular topic for mathematical enrichment and popularization. They are famous for a host of interesting and surprising properties, and show up in text books, magazine articles, and websites.

There are a lot of identities of Fibonacci and Lucas numbers described in [1]. M. Thongmoon [4] defined various identities of Fibonacci and Lucas numbers.

The Fibonacci sequence is defined as

$F_{n+1} = F_n + F_{n-1}$ where $n \geq 1$ with initial conditions $F_0 = 0, F_1 = 1$ [1.1]

Binet's formula is

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{, where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} \quad [1.2]$$

which gives $\alpha + \beta = 1$ and $\alpha.\beta = -1$ [1.3]

The Lucas sequence is defined as

$L_{n+1} = L_n + L_{n-1}$, where $n \geq 1$ with initial conditions $L_0 = 2, L_1 = 1$ [1.4]

Binet's formula for the Lucas sequence is

$$L_n = \frac{(1+\sqrt{5})^n + (1-\sqrt{5})^n}{2^n} = \alpha^n + \beta^n \text{, where } \alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} \quad [1.5]$$

which gives $\alpha + \beta = 1$ and $\alpha.\beta = -1$ [1.6]

There are some known identities involving Fibonacci and Lucas numbers in [5].

The fundamental identity

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad [1.7]$$

Conjugation relation

$$F_n = \frac{(L_{n-1} + L_{n+1})}{5} \quad [1.8]$$

Successor relation

$$F_{n+1} = \frac{(F_n + L_n)}{2} \quad [1.9]$$

Double angle formula

$F_{2n} = F_n L_n$ [1.10]

Now, we present generalized identities involving common factors of Fibonacci and Lucas numbers.

2. GENERALIZED IDENTITIES

Theorem 2.1: $F_{4n+p} - F_p = F_{2n} L_{2n+p}$ **where** $n \geq 1$ **and** $p \geq 0$ **.**

Proof: $F_{2n} L_{2n+p} = \left[\dfrac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta}\right]\left[\alpha^{2n+p} + \beta^{2n+p}\right]$ [By (1.2) and (1.5)]

$$= \left[\frac{\alpha^{4n+p} - \beta^{4n+p}}{\alpha - \beta}\right] - \left[\frac{\beta^{2n}\alpha^{2n+p} - \alpha^{2n}\beta^{2n+p}}{\alpha - \beta}\right]$$

$$= F_{4n+p} - \frac{1}{\alpha - \beta}\left[(-1)^{2n}\alpha^p - (-1)^{2n}\beta^p\right] \quad \text{[By (1.6)]}$$

$$= F_{4n+p} - \frac{1}{\alpha - \beta}\left(\alpha^p - \beta^p\right)$$

$$= F_{4n+p} - F_p \quad [2.1]$$

Corollary2.2: For different values of p, [2.1] can be expressed for even and odd numbers.

If p=0, then $F_{4n} = F_{2n} L_{2n}$ where $n \geq 1$ [2.2]

If p=1, then $F_{4n+1} - 1 = F_{2n} L_{2n+1}$ where $n \geq 1$ [2.3]

If p=2, then $F_{4n+2} - 1 = F_{2n} L_{2n+2}$ where $n \geq 1$ and so on. [2.4]

Theorem 2.3: $F_{4n+p} + F_p = F_{2n+p} L_{2n}$ **where** $n \geq 1$ **and** $p \geq 0$ **.**

Proof: $F_{2n+p}L_{2n} = \left[\dfrac{\alpha^{2n+p}-\beta^{2n+p}}{\alpha-\beta}\right]\left[\alpha^{2n}+\beta^{2n}\right]$ [By (1.2) and (1.5)]

$$= \left[\frac{\alpha^{4n+p}-\beta^{4n+p}}{\alpha-\beta}\right]+\left[\frac{\alpha^{2n+p}\beta^{2n}-\beta^{2n+p}\alpha^{2n}}{\alpha-\beta}\right]$$

$$= F_{4n+p}+\frac{1}{\alpha-\beta}\left[(-1)^{2n}\alpha^{p}-(-1)^{2n}\beta^{p}\right] \quad \text{[By (1.6)]}$$

$$= F_{4n+p}+\frac{1}{\alpha-\beta}\left(\alpha^{p}-\beta^{p}\right)$$

$$= F_{4n+p}+F_{p} \qquad [2.5]$$

Corollary 2.4: For different values of p, [2.5] can be expressed for even and odd numbers.

If p=0, then $F_{4n} = F_{2n}L_{2n}$ where $n \ge 1$ [2.6]

If p=1, then $F_{4n+1}+1 = F_{2n+1}L_{2n}$ where $n \ge 1$ [2.7]

If p=2, then $F_{4n+2}+1 = F_{2n+2}L_{2n}$ where $n \ge 1$ and so on. [2.8]

Theorem 2.5: $L_{4n+p}-L_{p} = 5F_{2n}F_{2n+p}$ **where $n \ge 1$ and $p \ge 0$.**

Proof: $5F_{2n}F_{2n+p} = 5\left[\dfrac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta}\right]\left[\dfrac{\alpha^{2n+p}-\beta^{2n+p}}{\alpha-\beta}\right]$ [By (1.2)]

$$= \frac{5}{(\alpha-\beta)^2}\left[\left(\alpha^{4n+p}+\beta^{4n+p}\right)-\left(\alpha^{2n}\beta^{2n+p}+\beta^{2n}\alpha^{2n+p}\right)\right]$$

$$= L_{4n+p}-\left[(-1)^{2n}\beta^{p}+(-1)^{2n}\alpha^{p}\right] \quad \text{[By (1.6)]}$$

$$= L_{4n+p}-\left(\alpha^{p}+\beta^{p}\right)$$

$$= L_{4n+p}-L_{p} \qquad [2.9]$$

Corollary 2.6: For different values of p, [2.9] can be expressed for even and odd numbers.

If p=0, then $L_{4n} - 2 = 5F_{2n}F_{2n}$ where $n \geq 1$ [2.10]

If p=1, then $L_{4n+1} - 1 = 5F_{2n}F_{2n+1}$ where $n \geq 1$ [2.11]

If p=2, then $L_{4n+2} - 3 = 5F_{2n}F_{2n+2}$ where $n \geq 1$ and so on. [2.12]

Theorem 2.7: $L_{4n+p} + L_p = L_{2n}L_{2n+p}$ **where n ≥ 1 and p ≥ 0 .**

Proof: $L_{2n}L_{2n+p} = \left[\alpha^{2n} + \beta^{2n}\right]\left[\alpha^{2n+p} + \beta^{2n+p}\right]$ [By (1.5)]

$$L_{2n}L_{2n+p} = \left[\left(\alpha^{4n+p} + \beta^{4n+p}\right) + \left(\alpha^{2n}\beta^{2n+p} + \beta^{2n}\alpha^{2n+p}\right)\right]$$

$$= L_{4n+p} + \left[(-1)^{2n}\beta^{p} + (-1)^{2n}\alpha^{p}\right] \quad \text{[By (1.6)]}$$

$$= L_{4n+p} + \left(\alpha^{p} + \beta^{p}\right)$$

$$= L_{4n+p} + L_p \quad [2.13]$$

Corollary 2.8: For different values of p, [2.13] can be expressed for even and odd numbers.

If p=0, then $L_{4n} + 2 = L_{2n}L_{2n}$ where $n \geq 1$ [2.14]

If p=1, then $L_{4n+1} + 1 = L_{2n}L_{2n+1}$ where $n \geq 1$ [2.15]

If p=2, then $L_{4n+2} + 3 = L_{2n}L_{2n+2}$ where $n \geq 1$ and so on. [2.16]

Theorem 2.9: $F_{4n+p} - F_p = F_nL_nL_{2n+p}$ **where n ≥ 1 and p ≥ 0**

Theorem 2.10: $L_{4n+p} - L_p = 5F_nL_nF_{2n+p}$ **where n ≥ 1 and p ≥ 0**

Using [1.10] in Theorem [2.1] and Theorem [2.5] respectively, to get above two identities.

Theorem 2.11: $F_{3n+p} - (-1)^n F_{n+p} = F_n L_{2n+p}$ **where n ≥ 1 and p ≥ 0**

Proof: $$F_n L_{2n+p} = \left[\frac{\alpha^n - \beta^n}{\alpha - \beta}\right]\left[\alpha^{2n+p} + \beta^{2n+p}\right] \qquad \text{[By (1.2) and (1.5)]}$$

$$= \left[\frac{\alpha^{3n+p} - \beta^{3n+p}}{\alpha - \beta}\right] - \left[\frac{\beta^n \alpha^{2n+p} - \alpha^n \beta^{2n+p}}{\alpha - \beta}\right]$$

$$F_n L_{2n+p} = F_{3n+p} - \frac{1}{\alpha - \beta}\left[(-1)^n \alpha^{n+p} - (-1)^n \beta^{n+p}\right] \quad \text{[By (1.6)]}$$

$$= F_{3n+p} - \frac{(-1)^n}{\alpha - \beta}\left(\alpha^{n+p} - \beta^{n+p}\right)$$

$$= F_{3n+p} - (-1)^n F_{n+p} \qquad [2.17]$$

In theorem [2.1] to [2.11], values of p give various generalized identities of Fibonacci and Lucas numbers.

Now we state some identities. Their proof can be given same as theorem [2.1].

Theorem 2.12: $F_{3n+p} + (-1)^n F_{n+p} = F_{2n+p} L_n$**, where n ≥ 1 and p ≥ 0**

Theorem 2.13: $L_{3n+p} - (-1)^n L_{n+p} = 5 F_n F_{2n+p}$ **, where n ≥ 1 and p ≥ 0**

Theorem 2.14: $L_{3n+p} + (-1)^n L_{n+p} = L_n L_{2n+p}$**, where n ≥ 1 and p ≥ 0 .**

3. SOME IDENTITIES WITH FIBONACCI AND LUCAS POLYNOMIALS

Lupas Alexandru [3] has defined Fibonacci polynomials by the recurrence relation

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x), n \geq 2 \text{ with } f_1(x) = 1, f_2(x) = x \qquad [3.1]$$

The Binet's form is given by

$$f_n(x) = \frac{\left(x+\sqrt{x^2+4}\right)^n - \left(x-\sqrt{x^2+4}\right)^n}{2^n\sqrt{x^2+4}} = \frac{\lambda_1{}^n - \lambda_2{}^n}{\lambda_1 - \lambda_2} \qquad [3.2]$$

where $\lambda_1 = \dfrac{x+\sqrt{x^2+4}}{2}$ and $\lambda_2 = \dfrac{x-\sqrt{x^2+4}}{2}$ [3.3]

which gives $\lambda_1.\lambda_2 = -1$ [3.4]

Now, Lupas Alexandru [3] has defined Lucas polynomials by the recurrence relation

$$l_{n+1}(x) = xl_n(x) + l_{n-1}(x), n \geq 2 \text{ with } l_0(x) = 2, l_1(x) = x \qquad [3.5]$$

The Binet's form is given by

$$l_n(x) = \frac{\left(x+\sqrt{x^2+4}\right)^n + \left(x-\sqrt{x^2+4}\right)^n}{2^n} = \lambda_1{}^n + \lambda_2{}^n \qquad [3.6]$$

where $\lambda_1 = \dfrac{x+\sqrt{x^2+4}}{2}$ and $\lambda_2 = \dfrac{x-\sqrt{x^2+4}}{2}$ [3.7]

which gives $\lambda_1.\lambda_2 = -1$ [3.8]

Theorem 3.1: $f_{4n+s}(x) - f_s(x) = f_{2n}(x)l_{2n+s}(x)$ **where n≥1 and s≥0.**

Proof: $$f_{2n}(x)l_{2n+s}(x) = \left[\frac{\lambda_1{}^{2n} - \lambda_2{}^{2n}}{\lambda_1 - \lambda_2}\right]\left[\lambda_1{}^{2n+s} + \lambda_2{}^{2n+s}\right] \qquad \text{[By (3.2) and(3.6)]}$$

$$= \left[\frac{\lambda_1{}^{4n+s} - \lambda_2{}^{4n+s}}{\lambda_1 - \lambda_2}\right] - \frac{1}{(\lambda_1 - \lambda_2)}\left[\lambda_1{}^{2n+s}\lambda_2{}^{2n} - \lambda_2{}^{2n+s}\lambda_1{}^{2n}\right]$$

$$= f_{4n+s}(x) - \frac{(\lambda_1.\lambda_2)^{2n}}{(\lambda_1 - \lambda_2)}\left[\lambda_1^{\,s} - \lambda_2^{\,s}\right]$$

$$= f_{4n+s}(x) - \frac{(-1)^{2n}}{(\lambda_1 - \lambda_2)}\left[\lambda_1^{\,s} - \lambda_2^{\,s}\right] \qquad \text{[By(3.8)]}$$

$$= f_{4n+s}(x) - f_s(x) \qquad [3.9]$$

Corollary 3.2: For different values of s, [3.9] can be expressed for even and odd number of Fibonacci and Lucas polynomials.

If s=0, then $f_{4n}(x) = f_{2n}(x) l_{2n}(x)$ where $n \geq 1$ [3.10]

If s=1, then $f_{4n+1}(x) - 1 = f_{2n}(x) l_{2n+1}(x)$ where $n \geq 1$ [3.11]

If s=2, then $f_{4n+2}(x) - x = f_{2n}(x) l_{2n+2}(x)$ where $n \geq 1$ and so on. [3.12]

Now we state some identities. Their proof can be given same as theorem [3.1].

Theorem 3.1: $f_{4n+s}(x) + f_s(x) = f_{2n+s}(x) l_{2n}(x)$ **where n≥1 and s≥0.**

Theorem 3.1: $l_{4n+s}(x) - l_s(x) = (x^2 + 4) f_{2n}(x) f_{2n+s}(x)$ **where n≥1 and s≥0.**

Theorem 3.1: $l_{4n+s}(x) + l_s(x) = l_{2n}(x) l_{2n+s}(x)$ **where n≥1 and s≥0.**

4. CONCLUSION

This paper explained generalized identities involving common factors of Fibonacci and Lucas numbers. Mainly Binet's formula employ for the identities. Some identities involving Fibonacci and Lucas polynomials also developed and derived. The concept can be executed for generalized Fibonacci and Lucas sequences as well as polynomials.

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