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Operational formulae for certain classical polynomials

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OPERATIONAL FORMULAE FOR CERTAIN CLASSICAL POLYNOMIALS

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1. Recently Srivastava [7, p. 43] has defined a set of polynomials $A_n^{(\alpha)}(x)$ related to the Laguerre polynomials by means of the relations

$$(1.1) \qquad \sum_{r=0}^{n} A_r^{(\alpha)}(x)\, L_{n-r}^{(\alpha+r)}(x) = 0, \quad n \geq 1,$$

$$A_0^{(\alpha)}(x) = 1.$$

He also gave the generating function, hypergeometric representation and the Rodrigues' formula for these polynomials [7, pp. 44-45] in the forms:

$$(1.2) \qquad (1+t)^{-1-\alpha}\, e^{xt} = \sum_{r=0}^{\infty} t^r A_r^{(\alpha)}(x),$$

$$(1.3) \qquad A_n^{(\alpha)}(x) = \frac{1}{(n)!} \sum_{r=0}^{\infty} \frac{(-n)_r (1+\alpha)_r}{(r)!}\, x^{n-r},$$

and

$$(1.4) \qquad A_n^{(\alpha)}(x) = \frac{x^{n+\alpha+1}}{(n)!}\, D^n \{x^{-\alpha-1} e^x\}, \qquad \left(D = \frac{d}{dx}\right),$$

respectively.

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In this paper we give some operational formulae for these as well as Laguerre polynomials and employ them to derive many interesting results.

The first operational formula to be proved is

$$(1.5)\qquad \prod_{j=1}^{n} (xD + x - \alpha - j) = (n)! \sum_{r=0}^{n} \frac{x^r}{(r)!} A_{n-r}^{(\alpha)}(x)\, D^r .$$

Note that the formula (1.5) corresponds to the one given by Carlitz [3, p. 219] in the case of Laguerre polynomials [8, p. 428].

To prove (1.5) we observe that if

$$\Omega_n = \prod_{j=1}^{n} (xD + x - \alpha - j), \qquad \Omega_0 = 1,$$

it can be proved very easily by the method of induction that

$$\Omega_n(y) = x^{n+\alpha+1} e^{-x} D^n \{e^x x^{-\alpha-1} y\},$$

where y is some differentiable function of x.

Next since

$$\begin{aligned} D^n \{e^x x^{-\alpha-1} \cdot y\} &= \sum_{r=0}^{n} \binom{n}{r} D^{n-r} \{e^x x^{-\alpha-1}\} D^r y \\ &= \sum_{r=0}^{n} x^{-n-\alpha-1} e^x (n)! \frac{x^r}{(r)!} A_{n-r}^{(\alpha)}(x)\, D^r y, \end{aligned}$$

(1.5) follows immediately.

In (1.5) if we take $y = 1$, we obtain

$$(1.6)\qquad \prod_{j=1}^{n} (xD + x - \alpha - j) \cdot 1 = (n)!\, A_n^{(\alpha)}(x) .$$

As an application of (1.5) and (1.6), let us consider

$$\begin{aligned} (m+n)!\, A_{m+n}^{(\alpha)}(x) &= \prod_{j=1}^{m} (xD + x - \alpha - n - j) \prod_{j=1}^{n} (xD + x - \alpha - j) \cdot 1 \\ &= (n)! \prod_{j=1}^{m} (xD + x - \alpha - n - j)\, A_n^{(\alpha)}(x) \\ &= (m)!\,(n)! \sum_{r=0}^{n} \frac{x^r}{(r)!} A_{m-r}^{(\alpha+n)}(x)\, D^r A_n^{(\alpha)}(x). \end{aligned}$$

But since

$$D^r A_n^{(\alpha)}(x) = A_{n-r}^{(\alpha)}(x),$$

we readily get

$$\binom{m+n}{n} A_{m+n}^{(\alpha)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{(r)!} A_{m-r}^{(\alpha+n)}(x)\, A_{n-r}^{(\alpha)}(x)\,. \tag{1.7}$$

Further, from (1.7) we have

$$\sum_{m=0}^{\infty} \binom{m+n}{n} t^m A_{m+n}^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(xt)^r}{(r)!} A_{n-r}^{(\alpha)}(x) \sum_{m=0}^{\infty} t^m A_m^{(\alpha+n)}(x)$$

$$= \sum_{r=0}^{n} \frac{(xt)^r}{(r)!} A_{n-r}^{(\alpha)}(x) \cdot e^{xt} (1+t)^{-(\alpha+n+1)},$$

and making use of the relation [7, p. 45]

$$A_n^{(\alpha)}(x+y) = \sum_{r=0}^{n} \frac{y^r}{(r)!} A_{n-r}^{(\alpha)}(x)\,,$$

we get the known formula [6, p. 7]

$$\sum_{m=0}^{\infty} \binom{m+n}{n} t^m A_{m+n}^{(\alpha)}(x) = e^{xt} (1+t)^{-(\alpha+n+1)} A_n^{(\alpha)} \{x(1+t)\}\,. \tag{1.8}$$

Another operational formula for the polynomials $A_n^{(\alpha)}(x)$ is

$$(1+D)^{-1-\alpha} x^n = (n)!\, A_n^{(\alpha)}(x)\,, \qquad \left(D = \frac{d}{dx}\right). \tag{1.9}$$

To prove it we note that

$$(1+D)^{-1-\alpha} x^n = \sum_{r=0}^{n} (-1)^r \frac{(1+\alpha)_r}{(r)!} D^r x^n = \sum_{r=0}^{n} \frac{(1+\alpha)_r (-n)_r}{(r)!} x^{n-r},$$

which evidently yields the formula (1.9).

From (1.9) we have

$$(n)!\, A_n^{(\alpha+\beta)}(x) = (1+D)^{-1-\alpha-\beta}\, x^n,$$

$$A_n^{(\alpha+\beta)}(x) = (1+D)^{-\beta}\, A_n^{(\alpha)}(x); \tag{1.10}$$

the last formula gives us

$$A_n^{(\alpha+\beta)}(x) = \sum_{r=0}^{n} (-1)^r \frac{(\beta)^r}{(r)!}\, A_{n-r}^{(\alpha)}(x). \tag{1.11}$$

Further let us operate on the identity

$$e^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{(n)!}\, x^n,$$

by $(1+D)^{-1-\alpha}$; the familiar shift rule then gives us

$$(1+D)^{-1-\alpha}\, e^{xt} = e^{xt}\,(1+t)^{-1-\alpha}\left\{1+\frac{D}{1+t}\right\}^{-1-\alpha}\cdot 1 = (1+t)^{-1-\alpha}\, e^{xt}.$$

On the other hand the second member yields

$$\sum_{n=0}^{\infty} t^n\, A_n^{(\alpha)}(x)$$

with the help of (1.9).

We thus arrive at the familiar generating function (1.2).

Now replace by tD_y, $\left(D_y = \dfrac{d}{dy}\right)$ in (1.2) and operate on both sides by $(1+D_y)^{-1-\beta}$. The left-hand side gives us

$$(1+D_y)^{-1-\beta}\, e^{xty}\,(1+ty) = e^{xty}\,\{1+xt+D_y\}^{-1-\beta}\,(1+yt)^{-1-\alpha}$$

$$= e^{xyt}\,(1+xt)^{-1-\beta} \sum_{r=0}^{\infty} (-1)^r \frac{(1+\beta)_r}{(r)!}\cdot\frac{D_y^r}{(1+xt)^r}\,(1+yt)^{-1-\alpha}$$

$$\simeq e^{xyl}\,(1+xt)^{-1-\beta}\,(1+yt)^{-1-\alpha}\, {}_2F_0\left[1+\alpha, 1+\beta; -; \frac{t}{(1+xt)(1+yt)}\right]$$

and the right-hand side yields

$$\sum_{n=0}^{\infty} (n)!\, t^n A_n^{(\alpha)}(x)\, A_n^{(\beta)}(y)\,.$$

Combining these two sides we finally get

$$(1.12)\qquad \sum_{n=0}^{\infty} (n)!\, t^n A_n^{(\alpha)}(x)\, A_n^{(\beta)}(y)$$

$$\simeq e^{xyt}(1+xt)^{-1-\beta}(1+yt)^{-1-\alpha}\, {}_2F_0\left[1+\alpha, 1+\beta; -; \frac{t}{(1+xt)(1+yt)}\right].$$

From the relation (1.12) we have

$$(1+xt)^{-1-\beta}(1+yt)^{-1-\alpha}\, {}_2F_0\left[1+\alpha, 1+\beta; -; \frac{t}{(1+xt)(1+yt)}\right]$$

$$= \sum_{r,s,k=0}^{\infty} \frac{(1+\alpha)_r (1+\beta)_r (1+\alpha+r)_s (1+\beta+r)_k}{(r)!\,(s)!\,(k)!} (-1)^{s+k} x^k y^s \cdot t^{r+s+k}$$

$$= \sum_{n=0}^{\infty} (-t)^n \sum_{k=0}^{n} \sum_{r=0}^{k} (-1)^r \frac{(1+\alpha)_{n+r-k}\,(1+\beta)_k}{(r)!\,(k-r)!\,(n-k)!} x^{k-r} y^{n-k}$$

$$= \sum_{n=0}^{\infty} (-t)^n \sum_{k=0}^{n} \frac{(1+\beta)_k (1+\alpha)_{n-k}}{(n-k)!} y^{n-k} \sum_{r=0}^{k} (-1)^r \frac{(1+\alpha+n-k)_r}{(r)!\,(k-r)!} x^{k-r}$$

$$= \sum_{n=0}^{\infty} (-t)^n \sum_{k=0}^{n} \frac{(1+\beta)_k (1+\alpha)_{n-k}}{(n-k)!} y^{n-k} A_k^{(\alpha+n-k)}(x).$$

Thus (1.12) is equivalent to

$$(1.13)\qquad \sum_{r=0}^{n} \frac{(r)!}{(n-r)!} (-1)^r (xy)^{n-r} A_n^{(\alpha)}(x)\, A_n^{(\beta)}(y)$$

$$= \sum_{k=0}^{n} \frac{(1+\beta)_k (1+\alpha)_{n-k}}{(n-k)!} y^{n-k} A_k^{(\alpha+n-k)}(x).$$

2. Making use of the relation [7, p. 45]

$$L_n^{-(\alpha+n+1)}(-x) = A_n^{(\alpha)}(x)$$

and our formula (1.8), we get

$$(2.1) \qquad (1-D)^{\alpha+n}(-x)^n = (n)!\, L_n^{(\alpha)}(x),$$

which yields the following interesting result

$$(2.2) \qquad L_n^{(\alpha+\beta)}(x) = (1-D)^{\beta} L_n^{(\alpha)}(x).$$

Now consider

$$L_n^{(\alpha+\beta+1)}(x) = (1-D)^{\beta+1} L_n^{(\alpha)}(x) = \left[1 + \frac{D}{1-D}\right]^{-\beta-1} L_n^{(\alpha)}(x)$$

$$= \sum_{r=0}^{n} (-1)^r \frac{(\beta+1)_r}{(r)!} (1-D)^{-r} D^r L_n^{(\alpha)}(x),$$

and it follows that

$$(2.3) \qquad L_n^{(\alpha+\beta+1)}(x) = \sum_{r=0}^{n} \frac{(\beta+1)_r}{(r)!} L_{n-r}^{(\alpha)}(x).$$

Formula (2.3) was proved earlier in a different way by Al-Salam [1, p. 131], and our proof differs markedly with that of Rainville [5, p. 209].

Next let us consider the expression,

$$(-1)^n e^x D^n [e^{-x} L_n^{(\alpha)}(x)]$$

which by the usual shift rule gives us

$$(-1)^n e^x D^n [e^{-x} L_n^{(\alpha)}(x)] = (1-D)^n L_n^{(\alpha)}(x).$$

On making use of the relation (2.2) we obtain

$$(-1)^n e^x D^n [e^{-x} L_n^{(\alpha)}(x)] = L_n^{(\alpha+n)}(x),$$

which may be put in the form

$$R_n(1+\alpha, x) = (-1)^n e^x D^n[e^{-x} L_n^{(\alpha)}(x)], \tag{2.4}$$

where $R_n(a, x)$ is the pseudo Laguerre set defined by Shively [5, p. 298] as

$$R_n(a, x) = \frac{(a)_{2n}}{(n)!\,(a)_n}\, {}_1F_1(-n; a+n; x).$$

The formula (2.4) has been proved recently by Khandekar in a different way (see [4], p. 2).

Further, from (2.1) we have

$$\frac{(-x)^n}{(n)!} = (1-D)^{-\alpha-n} L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(\alpha+n)_r}{(r)!} D^r L_n^{(\alpha)}(x)$$

which gives

$$\frac{(-x)^n}{(n)!} = \sum_{r=0}^{n} (-1)^r \frac{(\alpha+n)_r}{(r)!} L_{n-r}^{(\alpha+r)}(x). \tag{2.5}$$

Next consider the identity

$$e^{-xt} = \sum_{n=0}^{\infty} (-x)^n \frac{t^n}{(n)!},$$

operate on both sides by $(1-D)^\alpha$, make use of (2.1) and proceed as in the cases of (1.12) and (1.13). We then get the generating function

$$(1+t)^\alpha e^{-xt} = \sum_{n=0}^{\infty} t^n L_n^{(\alpha-n)}(x), \tag{2.6}$$

due to Erdélyi, and the known formula [2, p. 151]

$$\sum_{n=0}^{\infty} (n)!\, t^n L_n^{(\alpha-n)}(x)\, L_n^{(\beta-n)}(y) \tag{2.7}$$

$$= \begin{cases} e^{xyt}(1-yt)^{\alpha-\beta} t^\beta (\beta)!\, L_\beta^{(\alpha-\beta)}\left(-\dfrac{(1-xt)(1-yt)}{t}\right) \\[2ex] e^{xyt}(1-xt)^{\beta-\alpha} t^\alpha (\alpha)!\, L_\alpha^{(\beta-\alpha)}\left(-\dfrac{(1-xt)(1-yt)}{t}\right). \end{cases}$$

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