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ON THE UNIFICATION OF GENERALIZED HERMITE AND LAGUERRE POLYNOMIALS

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The authors summarize their results on the polynomials $J_n^{(a)}(x, r, p, q)$ whose study was initiated by them over a decade ago. These polynomials are defined by (1.1) below; see also the remarks surrounding eqn. (1.3) which connects these polynomials with a certain general sequence of functions introduced by Srivastava and Panda (1975).

1. INTRODUCTION

With a view to unifying the study of the polynomials introduced by Chak (1956), Palas (1959), Chatterjea (1964), Singh and Srivastava (1963), Gould and Hopper (1962) and Krall and Frink (1949), the authors, initiated the study of the class of polynomials $\left\{ J_n^{(a)}(x, r, p, q) \right\}$ defined by (see Joshi and Singhal 1968).

$$J_n^{(a)}(x, r, p, q) = C(q, n)x^{-a} \exp(px^r)\, D_x^n \left\{ x^{a+nq} \exp(-px^r) \right\} \qquad ...(1.1)$$

where q is a non-negative integer and

$$C(q, n) = \frac{(-1)^{1/2n\,(q-1)\,(q-2)}}{2^{1/2nq\,(q-1)}\,(1)_{nq(2-q)}} . \qquad ...(1.2)$$

It is easy to see that when $q = 0$, $J_n^{(a)}(x, r, p, q)$ reduce to the polynomials $H_n^r(x, a, p)$ studied by Gould and Hopper (1962), and when $q = 1$, we get the polynomials $L_n^{(a)}(x, r, p)$ of Singh and Srivastava (1963), whereas the case $q = 2$, corresponds to the Bessel polynomials of Krall and Frink (1949).

Our work pertaining to these polynomials was presented by us at the Spring 1969 meeting of the Allegheny Mountain Section of the Mathematical Association of America. Subsequently, a part of this work dealing with operational formulae associated with $J_n^{(a)}(x, r, p, q)$ appeared in Joshi and Singhal (1968) but, unfortunately, the more important part, viz. the introduction of the polynomial, its explicit representation, generating function and some other miscellaneous results, has thus far remained in unpublished form. Meanwhile, several further generalizations of Laguerre, Hermite and Bessel polynomials and their various generalizations including $J_n^{(a)}(x, r, p, q)$ have appeared in the literature. To quote a few such instances, we mention here the polynomials introduced by Srivastava and Singhal (1972), and subsequently by Srivastava and Panda (1975). In terms of the functions introduced by Srivastava and Panda (1975), which they denoted by $S_n^{(\alpha,\beta)}[x, a, b, c, d; \gamma, \epsilon; w(x)]$, our polynomials $J_n^{(a)}(x, r, p, q)$ become the special case:

$$S_n^{(a;0)}[x, 1, 0, 0, 1; q, \epsilon; w(x)] = \{n!\, C(q,n)\}^{-1} J_n^{(a)}(x, r, p, q). \qquad \ldots(1.3)$$

Despite the introduction of these more general polynomials, we still feel that, for the sake of completeness and future reference, our work on $J_n^{(a)}(x, r, p, q)$, which we give in a condensed form in the next section of this paper, must get recorded.

2. The Results on $J_n^{(a)}(x, r, p, q)$

First we observe that the definition (1.1) may be transformed to yield the explicit representation

$$J_n^{(a)}(x, r, p, q) = C(q,n)\,(a+qn-n+1)_n\, x^{qn-n} \exp(px^r) \times {}_rF_r\left[\begin{matrix}\Delta(r, a+qn+1) & ; \\ \Delta(r, a+qn-n+1) & ;\end{matrix} -px^r\right] \qquad \ldots(2.1)$$

where $\Delta(r, a)$ stands for the set of r parameters

$a/r, (a+1)/r, \ldots (a+r-1)/r.$

Further, letting $\delta = x d/dx$ and making use of the properties

$$F(\delta)\, x^n = F(n) x^n \qquad \ldots(2.2)$$

and

$$F(\delta)\,[\exp\{g(x)\} f(x)] = \exp\{g(x)\}\, F(\delta+xg')\, f(x) \qquad \ldots(2.3)$$

and the explicit representation (2.1), it can be shown fairly easily that $J_n^{(a)}(x, r, p, q)$ satisfies the differential equation:

$$[(\delta-prx^r-qn-n)\,(\delta-prx^r+a+1-r)_r+prx^r\,(\delta-prx^r+n+a+1)_r]\, Y=0. \qquad \ldots(2.4)$$

On the other hand if we combine the definition (1.1) with the Lagrange formula (Polya and Szegö 1925, p. 133)

$$\frac{f(y)}{1-tg'(y)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n [f(x)\{g(x)\}^n], \quad y = x+tg(y) \qquad \ldots(2.5)$$

we are led to the generating relation

$$\sum_{n=0}^{\infty} \frac{t^n}{n!\, C(q,n)}\, J_n^{(a)}(x, r, p, q) = \frac{(1+v)^{a+1}}{1-(q-1)v} \exp\{px^r[1-(1+v)^r]\} \qquad \ldots(2.6)$$

where

$$\nu = x^{q-1}t(\nu+1)^q. \qquad \ldots(2.7)$$

Finally, we note that the definition (1.1) when used appropriately would yield the recurrence relations (2.8), (2.9), (2.10), and the doubly additive addition formula (2.11) given below:

$$J_{n+1}^{(a)}(x, r, p, q) = \frac{C(q,n+1)}{C(q,n)} \Big\{(a+nq+q)x^{q-1} J_n^{(a+q-1)}(x, r, p, q) - prx^{q+r-1} J_n^{(a+q+r-1)}(x, r, p, q)\Big\} \qquad \ldots(2.8)$$

$$(xD - prx^r + a)\, J_n^{(a)}(x, r, p, q) = \frac{x^{1-q}C(q,n)}{C(q,n)}\, J_{n+1}^{(a-q)}(x, r, p, q) \qquad \ldots(2.9)$$

$$J_n^{(a+1)}(x, r, p, q) - J_n^{(a)}(x, r, p, q) = \frac{nC(q,n)}{C(q,n-1)}\, x^q J_{n-1}^{(a+q)}(x, r, p, q) \qquad \ldots(2.10)$$

$$J_n^{(a+b)}(x, r, p_1+p_2, q) = \sum_{k=0}^{n} \binom{n}{k} \frac{C(q,n)}{C(q,k)C(q,n-k)}\, J_{n-k}^{(a+qk)}(x, r, p_1, q) \times J_k^{(b-qk)}(x, r, p_2, q). \qquad \ldots(2.11)$$

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