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Double Sums of Binomial Coefficients

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Abstract. We investigate the representation of double sums of binomial coefficients in integral form using the properties of the Beta function. Some nice identities result.

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1. Introduction

Recently a number of authors, including Sury [5], Zhao and Wang [8], Sury Wang and Zhao [6] Yang and Zhao [7], Sofo [2], and [3], have considered the summation of the reciprocals of binomial coefficients. In particular Sury Wang and Zhao [6] have shown, among other results, that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)\cdots(n+j)} = \frac{1}{(j-1)(j-1)!}. \tag{1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\binom{n+j}{n}} = j2^{j-1}\left(\ln 2 - \sum_{r=1}^{j-1}\frac{1}{r}\right) - j\sum_{r=1}^{j-1}(-1)^r \binom{j-1}{r}\frac{2^{j-1-r}}{r},$$

for $j = 1, 2, \ldots$

Sofo [2] also obtained the result

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n+j}{n}} = \frac{2^{j-2}}{(j-1)!}\left[\ln 2 + \sum_{r=1}^{j-2}(-1)^r \binom{j-2}{r}\left(\frac{2^r-1}{r\cdot 2^r}\right)\right] \tag{1.2}$$

$$= j!\ {}_3F_2\left[\left.\begin{matrix} \frac{1}{2}, 1, 1, \\ \frac{1+j}{2}, \frac{2+j}{2} \end{matrix}\right| 1\right].$$

Identities (1.1) to (1.2) are reciprocal binomial identities of the form

$$W\left(a,j\right)=\frac{1}{j!}\sum_{n=0}^{\infty}\frac{t^{n}}{\left(\begin{array}{c}an+j\\ an\end{array}\right)}$$

for $j=1,2,3\ldots,\ a\in\mathbb{R}^{+}\backslash\left\{0\right\}$ in which case the particular series (1.1) is a special case when $a=1$, and (1.2) for $a=2$.

Binomial coefficients play an important role in many areas of mathematics, including number theory, statistics and probability. The binomial coefficient is defined as

$$\left(\begin{array}{c}x\\ k\end{array}\right)=\begin{cases}\frac{1}{k!}\prod\limits_{r=0}^{k-1}(x-r), & \text{if } k>0\\ 1, & \text{if } k=0\\ 0, & \text{if } k<0\end{cases}\qquad\text{for } k\in\mathbb{Z}.$$

In this paper we shall extend the range of identities by considering the sum of $W(a,j)$ both over n and j of the form

$$S\left(a,t\right)=\sum_{j=2}^{\infty}W\left(a,j\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{t^{n}}{\left(\begin{array}{c}an+j\\ j\end{array}\right)}$$

and develop integral identities for the sums such that it may be helpful to represent them in closed form. The main aim of this paper is indeed, to express the sum $S\left(a,t\right)$ and its generalisation, in integral form. As we shall see below the sum $S\left(a,t\right)$ can be written as a series of generalised hypergeometric functions and in some special cases of the parameters may be expressed in closed form. It is also important to note that some of these double sums, as they stand without their integral representation, do not yield an output by the standard computer algebra system of Maple or Mathematica. When $S\left(a,t\right)$ cannot be written in closed form the integral representation is extremely useful in allowing one to obtain bounds and convexity properties for $S\left(a,t\right)$, however these results will be reported in another paper, some related results may be seen, for example, in the papers [1] and [4]. Finally, we shall give a generalisation to the sum $S\left(a,t\right)$.

2. Identity Representations

Consider the following theorem

Theorem 1. *Let $a \in \mathbb{R}^+ \backslash \{0\}$, $j = 2, 3, 4, \ldots.$ and $|t| \leq 1$, then*

$$S(a,t) = \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{an}} = \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{j}} \tag{2.1}$$

$$= \sum_{j=2}^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{\prod_{k=1}^{j} (an+k)} \tag{2.2}$$

$$= \int_{x=0}^{1} \frac{e^{1-x} - 1}{1 - tx^a} dx. \tag{2.3}$$

When a is a positive integer we can write

$$S(a,t) = \sum_{j=2}^{\infty} \frac{1}{j!} \, {}_{a+1}F_a \left[\begin{matrix} 1, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{matrix} \middle| t \right]. \tag{2.4}$$

Proof. Consider (2.1)

$$\begin{aligned} S(a,t) &:= \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n}{\binom{an+j}{j}} \\ &= \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n \Gamma(j+1) \Gamma(an+1)}{\Gamma(an+j+1)} \\ &= \sum_{j=2}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} \frac{t^n \Gamma(j) \Gamma(an+1)}{\Gamma(an+j+1)} \\ &= \sum_{j=2}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} t^n B(j, an+1) \end{aligned}$$

Where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} = \int_{u=0}^{1} u^{\alpha - 1} (1-u)^{\beta - 1} du, \text{ for } \alpha > 0 \text{ and } \beta > 0$$

is the classical Beta function and $\Gamma(\bullet)$ is the Gamma function.

Now

$$\begin{aligned} S(a,t) &= \sum_{j=2}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} t^n \int_{x=0}^{1} x^{an} (1-x)^{j-1} dx \\ &= \sum_{j=2}^{\infty} \frac{(1-x)^{j-1}}{(j-1)!} \int_{x=0}^{1} \sum_{n=0}^{\infty} (tx^a)^n dx \\ &= \int_{x=0}^{1} \frac{1}{1 - tx^a} \sum_{j=2}^{\infty} \frac{(1-x)^{j-1}}{(j-1)!} dx \end{aligned}$$

by an allowable change of order of summation and integration, hence,

$$S(a,t)=\int_{x=0}^{1}\frac{e^{1-x}-1}{1-tx^{a}}dx$$

which is the result (2.3). From (2.1)

$$S(a,t)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}V_n(a,t)$$

where $V_n(a,t)=\dfrac{t^n}{\binom{an+j}{an}}$.

From the ratio of successive terms, over n, of (2.1)

$$\frac{V_{n+1}}{V_n}=\frac{t\ (an+1)_j}{(an+a+1)_j}$$

where

$$(p)_\alpha=\frac{\Gamma(p+\alpha)}{\Gamma(p)}=\begin{cases} p(p+1)\cdots(p+\alpha-1)\,,\text{for } \alpha>0\\ 1,\ \text{for } \alpha=0\end{cases}$$

is Pochhammer's symbol

we arrive at the result (2.4). □

In the case when a is a positive integer, by known properties of the hypergeometric function we may state that

$$_{a+1}F_a\left[\begin{matrix}1,\frac{1}{a},\frac{2}{a},\frac{3}{a},\ldots,\frac{a}{a}\\ \frac{1+j}{a},\frac{2+j}{a},\frac{3+j}{a},\ldots,\frac{a+j}{a}\end{matrix}\middle|\, t\right]$$

$$={}_{j+1}F_j\left[\begin{matrix}1,\frac{1}{a},\frac{2}{a},\frac{3}{a},\ldots,\frac{j}{a}\\ \frac{1+a}{a},\frac{2+a}{a},\frac{3+a}{a},\ldots,\frac{a+j}{a}\end{matrix}\middle|\, t\right].$$

A number of illustrative examples are given,

1.

$$S(1,1)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{n+j}{j}}=\sum_{j=2}^{\infty}\frac{1}{(j-1)(j-1)!}$$

$$=\int_{x=0}^{1}\frac{e^{1-x}-1}{1-x}dx=\sum_{j=2}^{\infty}\frac{1}{j!}\,{}_2F_1\left[\begin{matrix}1,1\\ 1+j\end{matrix}\middle|\,1\right]$$

$$=-\gamma+E_i(1)$$

where γ = Euler's constant$=\lim\limits_{p\to\infty}\left[\sum\limits_{k=1}^{\infty}\frac{1}{k}-\log(p)\right]\approx .5772$, and $E_i(z)$ = Exponential Integral $=\ -PV\int\limits_{-z}^{\infty}\frac{e^{-t}}{t}dt$.

2.

$$S\left(1,-\frac{1}{2}\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})^n}{\binom{n+j}{j}}=\int_{x=0}^{1}\frac{e^{1-x}-1}{1+\frac{x}{2}}dx$$

$$=\sum_{j=2}^{\infty}\frac{1}{j!}\,{}_2F_1\left[\begin{array}{c}1,1\\1+j\end{array}\middle|-\frac{1}{2}\right]=2e^3\left[E_i(-3)-E_i(-2)\right]-\log\left(\frac{9}{4}\right)$$

3.

$$S\left(2,\frac{1}{4}\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{(\frac{1}{4})^n}{\binom{2n+j}{j}}=\int_{x=0}^{1}\frac{e^{1-x}-1}{1-\left(\frac{x}{2}\right)^2}dx$$

$$=\sum_{j=2}^{\infty}\frac{1}{j!}\,{}_3F_2\left[\begin{array}{c}1,1,\frac{1}{2}\\\frac{1+j}{2},\frac{2+j}{2}\end{array}\middle|\frac{1}{4}\right]$$

$$=e^3\left[E_i(-3)-E_i(-2)\right]+\frac{1}{e}\left[E_i(2)-E_i(1)\right]-\log(3)$$

4.

$$S\left(\frac{1}{2},1\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{1}{\binom{\frac{n}{2}+j}{j}}=\int_{x=0}^{1}\frac{e^{1-x}-1}{1-\sqrt{x}}dx$$

$$=\sum_{j=2}^{\infty}\frac{1}{j!}\,{}_2F_1\left[\begin{array}{c}1,2\\1+2j\end{array}\middle|1\right]$$

$$=-\gamma+E_i(1)+\frac{2}{3}\,{}_2F_2\left[\begin{array}{c}1,1\\2,\frac{5}{2}\end{array}\middle|1\right]$$

5.

$$S\left(1,\frac{1}{2}\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})^n}{\binom{n+j}{j}}=\int_{x=0}^{1}\frac{e^{1-x}-1}{1-\frac{x}{2}}dx=\sum_{j=2}^{\infty}\frac{1}{j!}\,{}_2F_1\left[\begin{array}{c}1,1\\1+j\end{array}\middle|\frac{1}{2}\right]$$

$$=\sum_{j=2}^{\infty}\frac{1}{(j-1)!}\left\{\psi\left(\frac{j+1}{2}\right)-\psi\left(\frac{j}{2}\right)\right\}=\frac{2}{e}\left[E_i(2)-E_i(1)\right]-2\log(2),$$

where $\psi(z)$ is the Digamma function.

We can make the following remark.

Remark 1. *The series (2.1), $S(a,-1)$ can be expressed in terms of the Lerch transcendent.*

In particular, from (2.2)

$$S(a,-1) := \sum_{j=2}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{\prod_{k=1}^{j}(an+k)} = \sum_{j=2}^{\infty}\sum_{n=0}^{\infty}(-1)^n\sum_{k=1}^{j}\frac{A_{j,k}}{(an+k)},$$

where

$$A_{j,k} = \lim_{n\to\left(-\frac{k}{a}\right)}\left\{\frac{an+k}{\prod_{k=1}^{j}(an+k)}\right\} = \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!}.$$

Hence

$$S(a,-1) = \sum_{j=2}^{\infty}\sum_{n=0}^{\infty}(-1)^n\sum_{k=1}^{j}\frac{(-1)^{k+1}}{(k-1)!\,(j-k)!\,(an+k)} = \sum_{j=2}^{\infty}\sum_{k=1}^{j}\frac{(-1)^{k+1}}{(k-1)!\,(j-k)!}\sum_{n=0}^{\infty}\frac{(-1)^n}{(an+k)} = \frac{1}{2a}\sum_{j=2}^{\infty}\sum_{k=1}^{j}\frac{(-1)^{k+1}}{(k-1)!\,(j-k)!}\left\{\psi\left(\frac{1}{2}+\frac{k}{2a}\right)-\psi\left(\frac{k}{2a}\right)\right\}, \tag{2.5}$$

where $\psi(z)$ is the Psi, or digamma function.

The Lerch transcendent, $\phi(z,s,\alpha)$ is defined as

$$\phi(z,s,\alpha) = \sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s},$$

where the $(n+\alpha)=0$ term is excluded from the sum.

The polygamma functions $\psi^{(k)}(z)$, $k\in\mathbb{N}$ are defined by

$$\psi^{(k)}(z) := \frac{d^{k+1}}{dz^{k+1}}\log\Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right), \quad k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\},$$
$$= -\int_0^1 \frac{(\ln(t))^k\ t^{z-1}}{1-t}dt.$$

where $\psi^{(0)}(z) = \psi(z)$, denotes the Psi, or digamma function, defined by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \log\Gamma(z) = \int_1^z \psi(t)\,dt.$$

From (2.5)

$$S(a,-1) = \sum_{j=2}^{\infty}\sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!} \cdot \frac{1}{a}\sum_{n=0}^{\infty} \frac{(-1)^n}{\left(n+\frac{k}{a}\right)}$$
$$= \sum_{j=2}^{\infty}\sum_{k=1}^{j} \frac{(-1)^{k+1}}{(k-1)!\,(j-k)!a}\phi\left(-1,1,\frac{k}{a}\right).$$

In the next theorem we consider a generalisation of Theorem 1

Theorem 2. *Let* $a \in \mathbb{R}^+ \setminus \{0\}$, $m \geq 1$, $j \geq m+1$ *and* $|t| \leq 1$, *then*

$$S(a,m,t) = \sum_{j=m+1}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{\binom{an+j}{j}} \tag{2.6}$$

$$= \sum_{j=m+1}^{\infty}\sum_{n=0}^{\infty} \frac{t^n (m)_n}{n!\,(an+1)_j} \tag{2.7}$$

$$= \int_{x=0}^{1} \frac{e^{1-x} - \sum\limits_{k=1}^{m} \frac{(1-x)^{k-1}}{(k-1)!}}{(1-tx^a)^m} dx. \tag{2.8}$$

When a *is a positive integer we can write*

$$S(a,m,t) = \sum_{j=m+1}^{\infty} \frac{1}{j!}\ {}_{a+1}F_a\left[\begin{matrix} m, \frac{1}{a}, \frac{2}{a}, \frac{3}{a}, \ldots, \frac{a}{a} \\ \frac{1+j}{a}, \frac{2+j}{a}, \frac{3+j}{a}, \ldots, \frac{a+j}{a} \end{matrix} \middle| t\right]. \tag{2.9}$$

Proof. Consider

$$S(a,m,t) = \sum_{j=m+1}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{t^n \binom{n+m-1}{n}}{\binom{an+j}{j}}$$
$$= \sum_{j=m+1}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \binom{n+m-1}{n} \frac{t^n \Gamma(j+1)\Gamma(an+1)}{\Gamma(an+j+1)}$$
$$= \sum_{j=m+1}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} \binom{n+m-1}{n} \frac{t^n \Gamma(j)\Gamma(an+1)}{\Gamma(an+j+1)},$$

and since

$$\frac{\Gamma(j)\Gamma(an+1)}{\Gamma(an+j+1)} = B(j,an+1)$$

then

$$S(a,m,t) = \sum_{j=m+1}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} \binom{n+m-1}{n} t^n B(j,an+1).$$

Now

$$S(a,m,t) = \sum_{j=m+1}^{\infty} \frac{1}{(j-1)!} \sum_{n=0}^{\infty} \binom{n+m-1}{n} t^n \int_{x=0}^{1} x^{an} (1-x)^{j-1} dx$$
$$= \sum_{j=m+1}^{\infty} \frac{(1-x)^{j-1}}{(j-1)!} \int_{x=0}^{1} \sum_{n=0}^{\infty} \binom{n+m-1}{n} (tx^a)^n dx$$
$$= \int_{x=0}^{1} \frac{1}{(1-tx^a)^m} \sum_{j=m+1}^{\infty} \frac{(1-x)^{j-1}}{(j-1)!} dx$$

by an allowable change of order of summation and integration, hence,

$$S(a,m,t) = \int_{x=0}^{1} \frac{e^{1-x} - \sum_{k=1}^{m} \frac{(1-x)^{k-1}}{(k-1)!}}{(1-tx^a)^m} dx$$

which is the result (2.8).

The hypergeometric function in (2.9) can be arrived at by considering the ratio of successive terms of (2.6). □

The following examples are highlighted.

1.

$$S\left(2,2,\frac{1}{4}\right) = \sum_{j=3}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{(n+1)\left(\frac{1}{4}\right)^n}{\binom{2n+j}{j}} = \int_{x=0}^{1} \frac{e^{1-x} - 2 + x}{\left(1-\left(\frac{x}{2}\right)^2\right)^2} dx$$
$$= \sum_{j=3}^{\infty} \frac{1}{j!}\, {}_3F_2\left[\begin{matrix} 2,1,\frac{1}{2} \\ \frac{1+j}{2}, \frac{2+j}{2} \end{matrix} \middle| \frac{1}{4}\right]$$
$$= \frac{e^3}{2}\left[E_i(-2) - E_i(-3)\right] - \frac{3}{2e}\left[E_i(1) - E_i(2)\right] - \frac{1}{2}\log(9)$$

2.

$$S\left(1,4,\frac{1}{2}\right) = \sum_{j=5}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{\binom{n+3}{n}\left(\frac{1}{2}\right)^n}{\binom{n+j}{j}} = \sum_{j=5}^{\infty} \frac{1}{j!}\, {}_2F_1\left[\begin{matrix} 1,4 \\ j+1 \end{matrix} \middle| \frac{1}{2}\right]$$
$$= \frac{8}{3} \int_{x=0}^{1} \frac{6e^{1-x} - 16 + 15x - 6x^2 + x^3}{(2-x)^4} dx$$
$$= \frac{55}{9} - \frac{8e}{3} - \frac{8}{3e}\left[E_i(1) - E_i(2)\right] - \frac{8}{3}\log(2).$$

3.

$$S\left(1,8,-\frac{1}{4}\right) = \sum_{j=9}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{\binom{n+7}{n}\left(-\frac{1}{4}\right)^n}{\binom{n+j}{j}} = \sum_{j=9}^{\infty} \frac{1}{j!}\, {}_2F_1\left[\begin{matrix} 8,1 \\ 1+j \end{matrix} \middle| -\frac{1}{4}\right]$$

$$= \frac{4096}{315}\int_{x=0}^{1}\frac{1}{(4+x)^8}$$
$$\times\left(5040e^{1-x}-13700+13699x-6846x^2+2275x^3-560x^4+105x^5-14x^6+x^7\right)dx$$
$$= \frac{956e}{315}-\frac{401783}{66150}+\frac{4096e^5}{315}\left[E_i(-4)-E_i(-5)\right]+\frac{4096}{315}\log\left(\frac{5}{4}\right).$$

The following corollary can now be stated.

Corollary 1. *Let* $a\in\mathbb{R}^{+}\setminus\{0\}$, $m\geq 1,\ j\ \geq m+2$ *and* $|t|\leq 1,$ *then*

$$D(a,m,t):=\sum_{j=m+2}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{nt^n\binom{n+m-1}{n}}{\binom{an+j}{j}}$$
$$=\sum_{j=m+2}^{\infty}\sum_{n=0}^{\infty}\frac{nt^n(m)_n}{n!\,(an+1)_j}$$
$$= mt\int_{x=0}^{1}\frac{\left\{e^{1-x}-\sum\limits_{k=1}^{m+1}\frac{(1-x)^{k-1}}{(k-1)!}\right\}x^a}{(1-tx^a)^{m+1}}dx. \tag{2.10}$$

When a *is a positive integer we can write*

$$D(a,m,t)=mt\sum_{j=m+2}^{\infty}\frac{1}{j!\binom{a+j}{j}}\,{}_{a+1}F_a\left[\begin{array}{c}m+1,\frac{a+1}{a},\frac{a+2}{a},\frac{a+3}{a},\dots,\frac{a+a}{a}\\ \frac{a+1+j}{a},\frac{a+2+j}{a},\frac{a+3+j}{a},\dots,\frac{a+a+j}{a}\end{array}\middle|\,t\right].$$

Proof. Utilising the same ideas as Theorem 2 we can apply the operator $x\frac{d}{dx}\left\{\frac{1}{(1-tx^a)^m}\right\}$, hence we arrive at (2.10). □

Some interesting examples are:

1.

$$D(1,1,1)=\sum_{j=3}^{\infty}\frac{1}{j!}\sum_{n=1}^{\infty}\frac{n}{\binom{n+j}{j}}=\int_{x=0}^{1}\frac{x\left(e^{1-x}-2+x\right)}{(1-x)^2}dx$$
$$=\sum_{j=3}^{\infty}\frac{1}{(j+1)!}\,{}_2F_1\left[\begin{array}{c}2,2\\2+j\end{array}\middle|\,1\right]=\sum_{j=3}^{\infty}\frac{1}{(j-1)!\,(j-1)\,(j-2)}$$
$$=\frac{1}{4}\,{}_2F_2\left[\begin{array}{c}1,1\\3,3\end{array}\middle|\,1\right]=3-e$$

2.

$$D\left(2,2,\frac{1}{4}\right)=\sum_{j=4}^{\infty}\frac{1}{j!}\sum_{n=1}^{\infty}\frac{n(n+1)\left(\frac{1}{4}\right)^n}{\binom{2n+j}{j}}=\sum_{j=4}^{\infty}\frac{1}{(j+2)!}\,{}_3F_2\left[\left.\begin{array}{c}2,3,\frac{3}{2}\\ \frac{j+3}{2},\frac{j+2}{2}\end{array}\right|\frac{1}{4}\right]$$
$$=\frac{1}{4}\int_{x=0}^{1}\frac{\left(e^{1-x}-5+4x-x^2\right)x^2}{\left(1-x^2\right)^3}dx$$
$$=\frac{5}{2}-e+\frac{5e^3}{4}\left[E_i(-3)-E_i(-2)\right]+\frac{1}{4e}\left[E_i(2)-E_i(1)\right]-\frac{7}{8}\log\left(3\right).$$

3.

$$D\left(\frac{1}{2},1,1\right)=\sum_{j=2}^{\infty}\frac{1}{j!}\sum_{n=1}^{\infty}\frac{n}{\binom{\frac{n}{2}+j}{j}}=\int_{x=0}^{1}\frac{\left(e^{1-x}-1\right)\sqrt{x}}{1-\sqrt{x}}dx$$

$$=\Gamma\left(\frac{3}{2}\right)\sum_{j=2}^{\infty}\frac{1}{\Gamma\left(\frac{3}{2}+j\right)}\,{}_{j+1}F_j\left[\left.\begin{array}{c}2,3,5,7,...,2j+1\\ 4,6,8,...,2j\end{array}\right|1\right]$$
$$=\frac{32}{3}-e\left\{2+\sqrt{\pi}Erf(1)\right\}.$$

where $Erf(z)=\frac{2}{\sqrt{\pi}}\int\limits_{t=0}^{z}e^{-t^2}dt$ is the integral of the Gaussian distribution.

The above technique may be generalised further by taking the consecutive derivative operator $x\frac{d}{dx}\left\{\bullet\right\}$

The series (2.1) for $S\left(a,j\right)$ can be generalised in the following manner.

Theorem 3. *Let* $a\geq b\in\mathbb{R}^{+}\backslash\left\{0\right\}$, $m\geq1,\ j\ \geq m+1$ *and* $|t|\leq1,$ *then*

$$S\left(a,b,m,t\right)=\sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{t^n\binom{n+m-1}{n}}{\binom{an+j}{bn}}\tag{2.11}$$
$$=\sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{t^n\left(m\right)_n}{n!\binom{an+j}{bn}}$$
$$=mbt\int_{x=0}^{1}\frac{\left\{e^{1-x}-\sum\limits_{k=0}^{m}\frac{(1-x)^k}{k!}\right\}x^b\left(1-x\right)^{a-b}}{x\left(1-tx^b\left(1-x\right)^{a-b}\right)^{m+1}}dx\tag{2.12}$$

Proof. Consider

$$\begin{aligned}
S\left(a,b,m,t\right) &= \sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\frac{t^{n}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)}{\left(\begin{array}{c} an+j\\ bn\end{array}\right)}\\
&= \sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)\frac{t^{n}\Gamma\left(bn+1\right)\Gamma\left(\left(a-b\right)n+j+1\right)}{\Gamma\left(an+j+1\right)}\\
&= b\sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)\frac{nt^{n}\Gamma\left(bn\right)\Gamma\left(\left(a-b\right)n+j+1\right)}{\Gamma\left(an+j+1\right)}\\
&= b\sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)nt^{n}B\left(bn,\left(a-b\right)n+j+1\right)\\
&= b\sum_{j=m+1}^{\infty}\frac{1}{j!}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)nt^{n}\int\limits_{x=0}^{1}x^{bn-1}(1-x)^{(a-b)n+j}dx
\end{aligned}$$

interchanging sum and integral, we have

$$\begin{aligned}
S(a,b,m,t) &= b\sum_{j=m+1}^{\infty}\frac{1}{j!}\int\limits_{x=0}^{1}\frac{(1-x)^{j}}{x}\sum_{n=0}^{\infty}\left(\begin{array}{c} n+m-1\\ n\end{array}\right)n\left(tx^{b}(1-x)^{(a-b)}\right)^{n}\ dx\\
&= b\sum_{j=m+1}^{\infty}\frac{1}{j!}\int\limits_{x=0}^{1}\frac{(1-x)^{j}mtx^{b}(1-x)^{(a-b)}}{x\left(1-tx^{b}(1-x)^{(a-b)}\right)^{m+1}}\ dx\\
&= bmt\int\limits_{x=0}^{1}\frac{x^{b}(1-x)^{(a-b)}}{x\left(1-tx^{b}(1-x)^{(a-b)}\right)^{m+1}}\sum_{j=m+1}^{\infty}\frac{(1-x)^{j}}{j!}\ dx\\
&= mbt\int_{x=0}^{1}\frac{\left\{e^{1-x}-\sum\limits_{k=0}^{m}\frac{(1-x)^{k}}{k!}\right\}x^{b}\left(1-x\right)^{a-b}}{x\left(1-tx^{b}\left(1-x\right)^{a-b}\right)^{m+1}}dx
\end{aligned}$$

which is (2.12). □

Some examples now follow, with the minimum of detail.

1.

$$S(2,1,2,-1) = \sum_{j=3}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{\binom{2n+j}{n}} = \sum_{j=3}^{\infty} \frac{1}{j!}\, {}_3F_2 \left[\begin{array}{c} 2,1,j+1 \\ \frac{j+1}{2}, \frac{j+2}{2} \end{array} \middle| -\frac{1}{4} \right]$$

$$= \int_{x=0}^{1} \frac{\left\{2e^{1-x} - 5 + 4x - x^2\right\}(1-x)}{\left(x^2-x-1\right)^3} dx$$

$$= \frac{5-7\sqrt{5}}{250} e^{\phi} \left[E_i(-\phi) - E_i(\alpha)\right] + \frac{5+7\sqrt{5}}{250} e^{\alpha} \left[E_i(-\alpha) - E_i(\phi)\right]$$

$$+ \frac{2\sqrt{5}}{125} \left[\ln(\phi) - \ln(-\alpha)\right] + \frac{1}{125} \left[190 - 65e\right]$$

where $\phi = \frac{1+\sqrt{5}}{2}$, is the Golden ratio and $\alpha = \frac{1-\sqrt{5}}{2}$.

2.

$$S\left(2,1,5,-\frac{1}{2}\right) = \sum_{j=6}^{\infty} \frac{1}{j!} \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)^n \binom{n+4}{n}}{\binom{2n+j}{n}} = \sum_{j=6}^{\infty} \frac{1}{j!}\, {}_3F_2 \left[\begin{array}{c} 5,1,j+1 \\ \frac{j+1}{2}, \frac{j+2}{2} \end{array} \middle| -\frac{1}{8} \right]$$

$$= -\frac{4}{3} \int_{x=0}^{1} \frac{\left(120e^{1-x} - 326 + 325x - 160x^2 + 50x^3 - 10x^4 + x^5\right)(1-x)}{\left(2+x-x^2\right)^6} dx$$

$$= \frac{76e^2}{6581} \left[E_i(-1) - E_i(-2)\right] + \frac{1528}{6561e} \left[E_i(1) - E_i(2)\right] + \frac{58385}{34992}$$

$$- \frac{1216e}{2187} + \frac{376}{2187} \ln(2).$$

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