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Summation formula involving harmonic numbers

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A b s t r a c t. Some identities of sums associated with harmonic numbers and binomial coefficients are developed. Integral representations and closed form identities of these sums are also given.

1. Preliminaries

Some usual notational terms are defined for the reader and will be utilized throughout this paper. The generalized hypergeometric function notation

$$ {}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}\middle| t\right] = \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{r!(b_1)_r \ldots (b_q)_r} t^r, $$

$$ \left(\begin{matrix} p, q \in \{0,1,2,3\ldots\};\ p \le q+1;\ p \le q \text{ and } |t| < \infty; \\ p = q+1 \text{ and } |t| < 1;\ p = q+1,\ |t| = 1 \text{ and} \\ \operatorname{Re}\left\{ \sum_{m=1}^{q} b_m - \sum_{m=1}^{p} a_m \right\} > 0,\ b_m, \gamma \notin \{0,-1,-2,-3,\ldots\} \end{matrix}\right) $$

and $(a)_r$ is Pochhammer's symbol defined by

$$ (a)_r = a(a+1)(a+2)\cdots(a+r-1), \qquad r > 0,\ (a)_0 = 1. $$

The Beta function

$$ B(s,t) = \int_0^1 z^{s-1}(1-z)^{t-1}\,dz = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \text{for} \quad \operatorname{Re}(s) > 0 \text{ and } \operatorname{Re}(t) > 0, $$

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and the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \text{for} \quad \mathrm{Re}(z) > 0.$$

The Riemann zeta function

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z}\,, \quad \mathrm{Re}(z) > 1$$

and the generalized Harmonic numbers in power α are defined by

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha}\,.$$

The nth Harmonic number, for $\alpha = 1$ is defined by

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t}\, dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler–Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \Big(\sum_{r=1}^{n} \frac{1}{r} - \log(n) \Big) = -\psi(1) \approx 0.5772156649\ldots\ .$$

The Polygamma functions $\psi^{(k)}(z)$, $k \in \mathbb{N}$, are defined by

$$\psi^{(k)}(z) := \frac{d^{k+1}}{dz^{k+1}} \log\Gamma(z) = \frac{d^k}{dz^k}\Big(\frac{\Gamma'(z)}{\Gamma(z)}\Big) =$$

$$= -\int_0^1 \frac{[\log(t)]^k\ t^{z-1}}{1-t}\, dt, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1,2,3,\ldots\}$$

and $\psi^{(0)}(z) = \psi(z)$, denotes the Psi, or Digamma function, defined by

$$\psi(z) = \frac{d}{dz} \log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

We also recall the well known series representation

$$\psi(z) = \sum_{r=0}^{\infty} \Big(\frac{1}{r+1} - \frac{1}{r+z} \Big) - \gamma.$$

In this paper it is intended to establish integral and closed form identities for sums of the reciprocal of binomial coefficients and harmonic numbers, so called "Euler sums".

There are many striking formulas discovered by Euler in relation to Euler sums, including

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+1)^2} = \zeta(3), \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^3} = \frac{5}{4}\zeta(4), \tag{1.1}$$

and the recurrence formula

$$(2n+1)\zeta(2n) = 2\sum_{r=1}^{n-1} \zeta(2r)\zeta(2n-2r),$$

which shows that in particular, for $n = 2$ we have $5\zeta(4) = 2(\zeta(2))^2$, and more generally, that $\zeta(2n)$ is a rational multiple of $(\zeta(2n))^2$. Another striking recursion known to EULER [6] is the following:

$$2\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^q} = (q+2)\zeta(q+1) - \sum_{r=1}^{q-2} \zeta(r+1)\zeta(q-r). \tag{1.2}$$

In a response to a letter from Goldbach, EULER [6] (see also [5]), considered sums of the form

$$E_{k,s} = \sum_{n=1}^{\infty} \frac{H_{n-1}^{(s)}}{n^k}$$

for k and s positive integers and extrapolated that $E_{k,s}$ $(k+s \le 13)$ can be evaluated in terms of the Riemann ζ functions. NIELSEN [10] gave an unambiguous version of Euler's proof for (1.2) and for $k+s = 2q+1$, $q \ge 1$, BORWEIN et al. [5] gave a complete proof for

$$\begin{aligned} E_{k,s} = \frac{A}{2}\left[\binom{k+s}{s} - A\right]\zeta(k+s) + \zeta(k)\zeta(s) - \\ - A\sum_{r=1}^{q}\left[\binom{2r-2}{s-1} + \binom{2r-2}{k-1}\right]\zeta(2r-1)\zeta(k+s-2r+1), \end{aligned} \tag{1.3}$$

where $A = 1$ for s odd and $A = -1$ for s even. Further work in the summation of harmonic numbers and binomial coefficients has also been done by BASU [4] and FLAJOLET and SALVY [7]. In this paper we intend to add, in a small way, some results related to (1.1) and (1.2). Specifically, we investigate integral representations and closed form representations for sums of the form

$$\sum_{n\ge 1} \frac{H_n^{(1)}}{\binom{bn+k}{k}\binom{cn+l}{l}}$$

for specific parameter values. The works of [1], [2], [3], [9], [11], [12], [13], [14] and [16] also investigate various representations of binomial sums and zeta functions in simpler form by the use of the Beta function and other techniques.

2. Harmonic numbers

The following result has been obtained in [11].

Lemma 1. *Let a be a positive real number with $j \geq 0$, $n > 0$, and let*

$$Q(a,j) = \binom{an+j}{j}^{-1}$$

be an analytic function in j. Then

$$Q^{(1)}(a,j) = \frac{dQ}{dj} = \begin{cases} -Q(a,j)P(a,j), \\ \qquad \text{where } P(a,j) = \sum_{r=1}^{an} \frac{1}{r+j} \text{ for } j > 0, \\ -Q(a,j)[\psi(j+1+an) - \psi(j+1)] - H_n^{(1)}, \\ \qquad \text{for } j = 0 \text{ and } a = 1, \end{cases}$$

and

$$Q^{(\lambda)}(a,j) = \frac{d^\lambda Q}{dj^\lambda} = -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(a,j) P^{(\lambda-1-\rho)}(a,j), \quad \text{for } \lambda \geq 2,$$

where

$$P^{(0)}(a,j) = \sum_{r=1}^{an} \frac{1}{r+j}, \quad \text{for } n = 1,2,3,\ldots, \quad \text{and} \quad Q^{(0)}(a,j) = Q(a,j).$$

For $i = 1,2,3,\ldots$

$$P^{(i)}(a,j) = \frac{d^i P}{dj^i} = \frac{d^i}{dj^i}\Big(\sum_{r=1}^{an} \frac{1}{r+j}\Big) = \tag{2.1}$$

$$= (-1)^i i! \sum_{r=1}^{an} \frac{1}{(r+j)^{i+1}} = (-1)^i i! \Big[\zeta(i+1,j+1) - \zeta(i+1,j+1+an)\Big],$$

$$\zeta(q,b) = \sum_{k=0}^{\infty} \frac{1}{(k+b)^q}$$

is the generalized Zeta function, where any term with $k+b=0$ is excluded.

Remark 1. Some particular and special cases of Lemma 1 are the following:

$$Q^{(1)}(a,j) = -\binom{an+j}{j}^{-1} \sum_{r=1}^{an} \frac{1}{r+j},$$

$$Q^{(2)}(a,j) = \binom{an+j}{j}^{-1} \Big[\Big(\sum_{r=1}^{an} \frac{1}{r+j}\Big)^2 + \sum_{r=1}^{an} \frac{1}{(r+j)^2}\Big] =$$

$$= \binom{an+j}{j}^{-1} \Big[\sum_{r=1}^{an} \sum_{s=1}^{r} \frac{2}{(r+j)(s+j)} \Big],$$

and

$$Q^{(3)}(a,j) = -\binom{an+j}{j}^{-1} \Big[\Big(\sum_{r=1}^{an} \frac{1}{r+j} \Big)^3 +$$

$$+ 2\sum_{r=1}^{an} \frac{1}{(r+j)^3} + 3\sum_{r=1}^{an} \frac{1}{(r+j)^2} \sum_{r=1}^{an} \frac{1}{r+j} \Big].$$

In the special case when $a = 1$ and $j = 0$ we may write

$$Q^{(1)}(1,0) = -H_n^{(1)}, \quad Q^{(2)}(1,0) = (H_n^{(1)})^2 + H_n^{(2)},$$

and

$$Q^{(3)}(1,0) = (H_n^{(1)})^3 + 3H_n^{(1)} H_n^{(2)} + 2H_n^{(3)}.$$

Our first result is the following

Theorem 1. *Let $a \geq 0$, $b \geq 0$, $c \geq 0$ be real positive numbers $j, k, l \geq 0$, and $|t| \leq 1$. Then we have*

$$\sum_{n=1}^{\infty} \frac{t^n \sum_{r=1}^{an} \frac{1}{r+j}}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} = \sum_{n=1}^{\infty} \frac{t^n [\psi(j+1+an) - \psi(j+1)]}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} \tag{2.2}$$

(2.3)

$$= -aklt \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^{k-1} (1-z)^{l-1} x^{a-1} y^b z^c \log(1-x)}{(1-tx^a y^b z^c)^2} \, dx \, dy \, dz,$$

where $|tx^a y^b z^c| < 1$.

Proof. To prove (2.3), consider the following expansion:

$$\sum_{n=1}^{\infty} \frac{t^n}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} =$$

$$= \sum_{n=1}^{\infty} \frac{t^n ankl \, \Gamma(an)\, \Gamma(j+1)\, \Gamma(bn+1)\, \Gamma(k)\, \Gamma(cn+1)\, \Gamma(l)}{\Gamma(an+j+1)\, \Gamma(bn+k+1)\, \Gamma(cn+l+1)} =$$

$$= akl \sum_{n=1}^{\infty} t^n n \, B(an, j+1)\, B(bn+1, k)\, B(cn+1, l) =$$

$$= akl \sum_{n=1}^{\infty} t^n n \, B(bn+1, k)\, B(cn+1, l) \int_0^1 x^{an-1} (1-x)^j \, dx.$$

Now we can differentiate both sides with respect to the parameter j, and by using Lemma 1, we have

$$\sum_{n=1}^{\infty} \frac{t^n \sum\limits_{r=1}^{an} \frac{1}{r+j}}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} =$$

$$= -akl \sum_{n=1}^{\infty} t^n n B(bn+1,k) B(cn+1,l) \int_0^1 x^{an-1}(1-x)^j \ln(1-x)\,dx =$$

$$= -akl \sum_{n=1}^{\infty} t^n n \int_0^1 x^{an-1}(1-x)^j \ln(1-x)\,dx \times$$

$$\times \int_0^1 y^{bn}(1-y)^{k-1}\,dy \int_0^1 z^{cn}(1-z)^{l-1}\,dz.$$

By the dominated convergence theorem, we have a justified change of order of sum and integral, thus we may write

$$\sum_{n=1}^{\infty} \frac{t^n \sum\limits_{r=1}^{an} \frac{1}{r+j}}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} =$$

$$= -akl \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^{k-1} (1-z)^{l-1} \ln(1-x)}{x} \times$$

$$\times \sum_{n=1}^{\infty} n (t x^a y^b z^c)^n \,dx\,dy\,dz =$$

$$= -aklt \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^{k-1} (1-z)^{l-1} x^{a-1} y^b z^c \log(1-x)}{(1-t x^a y^b z^c)^2}\,dx\,dy\,dz,$$

which is the result (2.3). □

Some consequences of Theorem 1 are the following.

Corollary 1. *For $a=b=1$, $t=1$ and $j=l=0$, we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+k}{k}} = -k \int_0^1 \int_0^1 \frac{(1-y)^{k-1} y \log(1-x)}{(1-xy)^2}\,dx\,dy = \frac{k}{(k-1)^2} \quad \textit{for} \;\; k>1. \tag{2.4}$$

This is a result that is also claimed by Cloitre and reported in [15], and later proved by Sofo [14].

R e m a r k 2. For $a=b=1$, $t=\frac{1}{2}$ and $j=l=0$, SOFO [14] obtained the following new result:

$$\sum_{n=1}^{\infty}\frac{(\frac{1}{2})^n H_n^{(1)}}{\binom{n+k}{k}}=(-1)^{k+1}k\Bigg(\ln^2(2)+$$

$$+2\sum_{r=1}^{k-1}\binom{k-1}{r}\frac{(-1)^r}{r}\ln(2)+2\sum_{r=1}^{k-1}\binom{k-1}{r}\frac{(-1)^r(1-2^r)}{r^2}\Bigg).$$

The following new results are also consequences of Theorem 1.

C o r o l l a r y 2. *Let $a=b=c=1$, $j=0$, $l=1$, $k\geq 2$, and $t=1$. Then we have*

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)\binom{n+k}{k}}=-k\int_0^1\int_0^1\int_0^1\frac{(1-y)^{k-1}yz\ln(1-x)}{(1-xyz)^2}\,dx\,dy\,dz= \tag{2.5}$$

$$=k\zeta(3)-kH_{k-1}^{(3)}. \tag{2.6}$$

P r o o f. To prove (2.6), consider (2.5) and expand as follows:

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)\binom{n+k}{k}}=\sum_{n=1}^{\infty}\frac{k!H_n^{(1)}}{(n+1)^2(n+2)_{k+1}}=\sum_{n=1}^{\infty}\frac{k!H_n^{(1)}}{(n+1)^2}\sum_{r=2}^{k}\frac{A_r}{(n+r)},$$

where

$$A_r=\lim_{n\to(-r)}\left\{\frac{n+r}{\prod\limits_{r=2}^{k}(n+r)}\right\}=\frac{(-1)^r}{(k-r)!\,(r-2)!}=\frac{2(-1)^r}{k!}\binom{k}{r}\binom{r}{2}.$$

Now, by rearrangement we get

$$\sum_{n=1}^{\infty}\frac{k!\,H_n^{(1)}}{(n+1)^2}\Bigg[\sum_{r=2}^{k}\frac{2(-1)^r}{(n+r)k!}\binom{k}{r}\binom{r}{2}\Bigg]=$$

$$=\sum_{r=2}^{k}\frac{2(-1)^r}{k!}(\binom{k}{r}\binom{r}{2}\sum_{n=1}^{\infty}\frac{k!\,H_n^{(1)}}{(n+1)^2(n+r)};$$

and by partial fraction decomposition we get

$$(2.7)\qquad = \sum_{r=2}^{k} 2(-1)^r \binom{k}{r}\binom{r}{1} \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(r-1)} \times$$

$$\times \Big[\frac{1}{(n+1)^2} - \frac{1}{(r-1)(n+1)} + \frac{1}{(r-1)(n+r)}\Big] =$$

$$= \sum_{r=2}^{k} (-1)^r \binom{k}{r}\binom{r}{1} \sum_{n=1}^{\infty} H_n^{(1)} \Big[\frac{1}{(n+1)^2} - \frac{1}{(n+r)(n+1)}\Big].$$

Since we have

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+1)^2} = \zeta(3) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+1)(n+r)} = \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{2(r-1)},$$

(2.7) can be written as follows:

$$\sum_{r=2}^{k} (-1)^r \binom{k}{r}\binom{r}{1} \Big[\zeta(3) - \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{2(r-1)}\Big] =$$

$$= \sum_{r=2}^{k} (-1)^r \binom{k}{r}\binom{r}{1} \zeta(3) - \sum_{r=2}^{k} (-1)^r \binom{k}{r}\binom{r}{1} \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{2(r-1)} =$$

$$= k\zeta(3) - kH_{k-1}^{(3)}, \quad \text{for } k \ge 2,$$

which is the result (2.6). □

Another result is embodied in the following

Corollary 3. *Let $a = b = c = 1$, $j = 0$, $l = 2$, $k \ge 2$, and $t = 1$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+2}{2}\binom{n+k}{k}} = -2 \int_0^1 \int_0^1 \int_0^1 \frac{(1-y)^{k-1}(1-z)yz\ln(1-x)}{(1-xyz)^2}\, dx\, dy\, dz =$$

$$= 2k^2\zeta(3) + 2k(k-1)\zeta(2) - 2k^2 H_k^{(3)} - 2k(k-1)H_k^{(2)} - 2(k-1).$$

Proof. Expand as follows:

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+2}{2}\binom{n+k}{k}} = \sum_{n=1}^{\infty} \frac{2H_n^{(1)}}{(n+2)(n+1)\binom{n+k}{k}} =$$

$$= \sum_{n=1}^{\infty} \frac{2k!H_n^{(1)}}{(n+2)^2(n+1)^2(n+3)_{k+1}} = \sum_{n=1}^{\infty} \frac{2k!H_n^{(1)}}{(n+2)^2(n+1)^2} \sum_{r=3}^{k} \frac{B_r}{(n+r)},$$

where

$$B_r = \lim_{n\to(-r)} \left\{ \frac{n+r}{\prod\limits_{r=3}^{k}(n+r)} \right\} = \frac{(-1)^{r+1}}{(k-r)!\,(r-3)!} =$$

$$= \frac{3!\,(-1)^{r+1}}{k!} \binom{k}{r} \binom{r}{3}, \qquad k \geq 3.$$

Now, by rearrangement we get

$$\sum_{n=1}^{\infty} \frac{2H_n^{(1)}}{(n+2)^2(n+1)^2} \Big[\sum_{r=3}^{k} \frac{3!\,(-1)^{r+1}}{(n+r)} \binom{k}{r} \binom{r}{3} \Big] =$$

$$= 2\sum_{r=3}^{k} 3!\,(-1)^{r+1} \binom{k}{r} \binom{r}{3} \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+2)^2(n+1)^2(n+r)}\,;$$

by partial fraction decomposition we get

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+2}{2}\binom{n+k}{k}} = 2\sum_{r=3}^{k} 3!\,(-1)^{r+1} \binom{k}{r} \binom{r}{3} \sum_{n=1}^{\infty} H_n^{(1)} \Big[\frac{1}{(r-1)^2(n+1)^2} +$$

$$+ \frac{1}{(r-2)^2(n+2)^2} - \frac{2r-1}{(r-1)^2(n+1)} +$$

$$+ \frac{2r-5}{(r-2)^2(n+2)} + \frac{1}{(r-1)^2(r-2)^2(n+r)} \Big] =$$

$$= 2\sum_{r=3}^{k} 3!\,(-1)^{r+1} \binom{k}{r} \binom{r}{3} \sum_{n=1}^{\infty} H_n^{(1)} \Big[\frac{1}{(r-1)^2(n+1)^2} -$$

$$- \frac{n(2r-3)}{(r-1)(r-2)(n+1)(n+2)(n+r)} + \frac{1}{(r-2)^2(n+2)^2} -$$

$$- \frac{2r^2-3r-1}{(r-1)(r-2)(n+1)(n+2)(n+r)} \Big] =$$

$$= 2\sum_{r=3}^{k} \frac{3!\,(-1)^{r+1}}{(r-1)} \binom{k}{r} \binom{r}{3} \zeta(3) +$$

$$+ 2\sum_{r=3}^{k} \frac{3!\,(-1)^{r+1}}{(r-2)} \binom{k}{r} \binom{r}{3} (\zeta(3)+\zeta(2)-2) -$$

$$- \sum_{r=3}^{k} \frac{3!\,(-1)^{r+1}(2r-3)}{(r-1)^2(r-2)^2} \binom{k}{r} \binom{r}{3} (-4(r-1) + r(H_{r-1}^{(1)})^2 + rH_{r-1}^{(2)}) -$$

$$- \sum_{r=3}^{k} \frac{3!\,(-1)^{r+1}(2r^2-3r-1)}{(r-1)^2(r-2)^2} \binom{k}{r} \binom{r}{3} (2(r-1) - (H_{r-1}^{(1)})^2 - H_{r-1}^{(2)}).$$

Collecting harmonic terms and isolating the terms $\zeta(3)$ and $\zeta(2)$, we have

$$= \sum_{r=3}^{k} 2(-1)^{r+1}(2r-3)\binom{k}{r}\binom{r}{1}\zeta(3) + \sum_{r=3}^{k} 2(-1)^{r+1}(r-1)\binom{k}{r}\binom{r}{1}\zeta(2) -$$

$$- \sum_{r=3}^{k} 2(-1)^{r+1}\frac{(4r-9)(r-1)}{(r-2)}\binom{k}{r}\binom{r}{1} -$$

$$- \sum_{r=3}^{k}(-1)^{r+1}\binom{k}{r}\binom{r}{1}\Big(\frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{(r-1)(r-2)}\Big) =$$

$$= 2k^2\zeta(3) - 2k(k-1)\zeta(2) -$$

$$- \sum_{r=3}^{k}(-1)^{r+1}\binom{k}{r}\frac{r}{(r-2)}\Big(2(r-1)(4r-9) + \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{(r-1)}\Big) =$$

$$= 2k^2\zeta(3) + 2k(k-1)\zeta(2) - 2k^2 H_k^{(3)} - 2k(k-1)H_k^{(2)} - 2(k-1) =$$

$$= \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+2}{2}\binom{n+k}{k}}\,. \qquad \square$$

Remark 3. An observation from Corollary 3 is the fact that the following sum

$$\sum_{r=3}^{k}(-1)^{r+1}\binom{k}{r}\frac{r}{(r-2)}\Big(2(r-1)(4r-9) + \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{(r-1)}\Big) =$$

$$= 2k^2 H_k^{(3)} + 2k(k-1)H_k^{(2)} + 2(k-1)$$

involves rational numbers. Hence we deduce

$$\sum_{r=3}^{k}(-1)^{r}\binom{k}{r}\frac{r}{(r-1)(r-2)}\Big((H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}\Big) = \tag{2.8}$$

$$= 2k^2 H_k^{(3)} + 2k(k-1)H_k^{(2)} + 2k(k-1)H_{k-1}^{(1)} - 2(2k-1)^2,$$

since it can be shown that

$$\sum_{r=3}^{k}(-1)^{r+1}\binom{k}{r}\frac{2r(r-1)(4r-9)}{(r-2)} = 2k\Big[4k-3-(k-1)(\gamma+\psi(k))\Big];$$

and by the identity

$$\psi(k) = H_{k-1}^{(1)} - \gamma,$$

we have

$$\sum_{r=3}^{k}(-1)^{r+1}\binom{k}{r}\frac{2r(r-1)(4r-9)}{(r-2)} = 2k\Big[4k-3-(k-1)H_{k-1}^{(1)}\Big].$$

Now, we consider another extension of Theorem 1.

T h e o r e m 2. *Let $a = b = c = 1$, $j = 0$, $l = k$, $k \geq 2$, and $t = 1$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+k}{k}^2} = -k^2 \int_0^1 \int_0^1 \int_0^1 \frac{((1-y)(1-z))^{k-1} yz \ln(1-x)}{(1-xyz)^2} \, dx \, dy \, dz =$$

$$= -\frac{1}{(k+1)^2} \int_0^1 \log(1-x) \; {}_3F_2\Big[\begin{matrix} 2,\, 2,\, 2 \\ 2+k,\, 2+k \end{matrix} \Big| \, x \Big] \, dx =$$

$$= \sum_{r=2}^{k} (\lambda_r \zeta(3) + \mu_r \zeta(2) + \nu_r),$$

where

$$\lambda_r = r^2 \binom{k}{r}^2 \Big(2 + 2(r-1)(H_{r-2}^{(1)} - H_{k-r}^{(1)})\Big), \qquad \mu_r = r^2 \binom{k}{r}^2 H_{r-1}^{(1)}$$

and

$$\nu_r = r^2 \binom{k}{r}^2 \Big(- H_{r-1}^{(1)} H_{r-1}^{(2)} -$$

$$- H_{r-1}^{(3)} - \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{r-1} + \Big((H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}\Big)(H_{k-r}^{(1)} - H_{r-2}^{(1)})\Big).$$

P r o o f. Consider the following expansion:

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{n+k}{k}^2} = \sum_{n=1}^{\infty} \frac{(k!)^2 \; H_n^{(1)}}{(n+1)^2((n+2)_{k+1})^2} = \tag{2.9}$$

$$= \sum_{n=1}^{\infty} \frac{(k!)^2 H_n^{(1)}}{(n+1)^2} \sum_{r=2}^{k} \Big[\frac{A_r}{(n+r)} + \frac{B_r}{(n+r)^2} \Big].$$

Since

$$B_r = \lim_{n(-r)} \left\{ \frac{(n+r)^2}{\prod_{r=2}^{k} (n+r)^2} \right\} = \frac{1}{[(k-r)!\,(r-2)!]^2} =$$

$$= \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2, \qquad k \geq 2$$

and

$$A_r = \lim_{n \to (-r)} \frac{d}{dn} \left\{ \frac{(n+r)^2}{\prod_{r=2}^{k} (n+r)^2} \right\} = -2 \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2 (H_{k-r}^{(1)} - H_{r-2}^{(1)});$$

it follows from (2.9) that

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2}\sum_{r=2}^{k}\left(\frac{2}{k!}\binom{k}{r}\binom{r}{2}\right)^2\Big[\frac{1}{(n+r)^2}-\frac{2(H_{k-r}^{(1)}-H_{r-2}^{(1)})}{n+r}\Big]= \tag{2.10}$$

$$=\sum_{r=2}^{k}\left(\frac{2}{k!}\binom{k}{r}\binom{r}{2}\right)^2\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2(n+r)^2}+$$

$$+2\sum_{r=2}^{k}\left(\frac{2}{k!}\binom{k}{r}\binom{r}{2}\right)^2(H_{r-2}^{(1)}-H_{k-r}^{(1)})\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2(n+r)}\,.$$

First, we note that

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2(n+r)}=\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(r-1)(n+1)^2}-\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(r-1)(n+1)(n+r)}$$

and using (1.1) gives

$$=\frac{1}{(r-1)}\zeta(3)-\frac{1}{2(r-1)^2}\Big(H_{r-1}^{(2)}+(H_{r-1}^{(1)})^2\Big).$$

Similarly, we have

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{(n+1)^2(n+r)^2}=$$

$$=\sum_{n=1}^{\infty}H_n^{(1)}\Big[\frac{1}{(r-1)^2(n+1)^2}+\frac{1}{(r-1)^2(n+r)^2}-\frac{2}{(r-1)^2(n+1)(n+r)}\Big]=$$

$$=\frac{2\zeta(3)}{(r-1)^2}+\frac{\zeta(2)H_{r-1}^{(1)}}{(r-1)^2}-\frac{H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}}{(r-1)^2}-\frac{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}}{(r-1)^3}\,;$$

and after some lengthy algebraic manipulations (and with the aid of Mathematica [17]), by (2.10) we can write

$$\sum_{r=2}^{k}\left(\binom{k}{r}\binom{r}{1}\right)^2\Big[2\zeta(3)+\zeta(2)H_{r-1}^{(1)}-$$

$$-H_{r-1}^{(1)}H_{r-1}^{(2)}-H_{r-1}^{(3)}-\frac{(H_{r-1}^{(1)})^2+H_{r-1}^{(2)}}{r-1}\Big]+$$

$$+\sum_{r=2}^{k}\left(\binom{k}{r}\binom{r}{1}\right)^2\Big(H_{r-2}^{(1)}-H_{k-r}^{(1)}\Big)\Big[2(r-1)\zeta(3)-H_{r-1}^{(2)}-(H_{r-1}^{(1)})^2\Big]=$$

$$= \sum_{r=2}^{k} \left(\binom{k}{r} \binom{r}{1} \right)^2 \Big[2 + 2(r-1)(H_{r-2}^{(1)} - H_{k-r}^{(1)}) \Big] \zeta(3) +$$

$$+ \sum_{r=2}^{k} \left(\binom{k}{r} \binom{r}{1} \right)^2 H_{r-1}^{(1)} \, \zeta(2) + \sum_{r=2}^{k} \left(\binom{k}{r} \binom{r}{1} \right)^2 \Big[- H_{r-1}^{(1)} H_{r-1}^{(2)} -$$

$$- H_{r-1}^{(3)} - \frac{(H_{r-1}^{(1)})^2 + H_{r-1}^{(2)}}{r-1} + \Big(H_{r-2}^{(1)} - H_{k-r}^{(1)} \Big) \Big(- H_{r-1}^{(2)} - (H_{r-1}^{(1)})^2 \Big) \Big].$$

On the other hand, we can compute the integral as follows:

$$-k^2 \int_0^1 \int_0^1 \int_0^1 \frac{((1-y)(1-z))^{k-1} yz \ln(1-x)}{(1-xyz)^2} \, dx \, dy \, dz =$$

$$= -\frac{k}{k+1} \int_0^1 \int_0^1 (1-y)^{k-1} y \ln(1-x) \ {}_2F_1 \Big[\begin{matrix} 2, \, 2 \\ 2+k \end{matrix} \Big| \, xy \Big] \, dx \, dy =$$

$$= -\frac{1}{(k+1)^2} \int_0^1 \log(1-x) \ {}_3F_2 \Big[\begin{matrix} 2, \, 2, \, 2 \\ 2+k, \, 2+k \end{matrix} \Big| \, x \Big] \, dx. \qquad \square$$

Another result for the nonunitary coefficients of n, while following the method used in the proof of Corollary 3, is the following

Corollary 4. *Let $a = b = 1$, $c = 2$, $j = 0$, $l = 2$, $k \geq 2$, and $t = 1$. Then we have*

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{\binom{2n+2}{2} \binom{n+k}{k}} =$$

$$= -2k \int_0^1 \int_0^1 \int_0^1 \frac{(1-y)^{k-1}(1-z) y z^2 \ln(1-x)}{(1-xyz^2)^2} \, dx \, dy \, dz =$$

$$= k \, B\Big(\frac{1}{2}, k\Big) \zeta(2) - k\zeta(3) - 2k \, B\Big(\frac{1}{2}, k\Big) \ln^2(2) +$$

$$+ \sum_{r=2}^{k} (-1)^r \binom{k}{r} \binom{r}{1} \Big[\frac{H_{r-1}^{(2)} + (H_{r-1}^{(1)})^2}{2(r-1)(2r-1)} \Big].$$

Remark 4. Results can be easily obtained from Theorem 1 which are similar to those obtained by Janous [8]. For $a = 2$, $b = c = 2$, $j = 0$, $l = 1$, $k = 1$, and $t = 1$, we have

$$\sum_{n=1}^{\infty} \frac{H_{2n}^{(1)}}{(2n+1)^2} = -2 \int_0^1 \int_0^1 \int_0^1 \frac{xy^2 z^2 \log(1-x)}{(1-x^2y^2z^2)^2} \, dx \, dy \, dz = \frac{7}{16} \zeta(3).$$

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Формула суммирования с гармоническими числами

АНТОНИ СОФО

Получены некоторые тождества для сумм, связанных с гармоническими числами и биномиальными коэффициентами. Получены также интегральные представления и тождества в замкнутой форме для таких сумм.