RECIPROCAL POWER SUMS

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Abstract

In this paper we give an alternative proof and an integral representation for a reciprocal power sum given by Bhatnagar, namely $\sum_{n=1}^{p} \frac{1}{n\binom{n+z}{z}}$, and then generalize the result to

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z)+zQ^{(q)}(n,z)}{n}, \text{ where } Q^{(q)}(n,z)=\frac{d^q}{dz^q}\left(\binom{n+z}{z}^{-1}\right).$$

1. Introduction and Preliminaries

Bhatnagar [2] states the following telescoping theorem.

Theorem 1. ([2]) *Let u_k, v_k and w_k be three sequences, such that $u_k - v_k = w_k$. Then we have*

$$\sum_{k=0}^{n} \frac{w_k}{w_0} \cdot \frac{\prod_{r=0}^{k-1} u_r}{\prod_{r=1}^{k} v_r} = \frac{u_0}{w_0}\left(\frac{\prod_{r=1}^{n} u_r}{\prod_{r=1}^{n} v_r} - \frac{v_0}{u_0}\right) \tag{1}$$

provided none of the denominators in (1) are zero.

Bhatnagar then indicates that the identity

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\ldots(k+m)} = \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)\ldots(n+m)}\right) \tag{2}$$

can be proved by the telescoping theorem 1. In this paper we shall give an integral representation of (2) and then generalize the result to give identities for finite sums for products of generalized harmonic numbers and binomial coefficients. First we give some definitions which will be useful throughout this paper.

The harmonic numbers in power α are defined as

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha}.$$

and the n^{th} Harmonic number, for $\alpha = 1$,

$$H_n^{(1)} = H_n = \int\limits_0^1 \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi\left(n+1\right),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log n \right) = -\psi\left(1\right) \approx 0.5772156649.......$$

Let $\mathbb{C}$ be the set of complex numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3...\}$, then for $w \in \mathbb{C}\backslash\mathbb{Z}_0^-$, $(w)_n$ is Pochhammer's symbol defined by

$$(w)_n = \frac{\Gamma\left(w+n\right)}{\Gamma\left(n\right)} = \begin{cases} w(w+1)....(w+n-1), & \text{if } n \in \mathbb{N} \\ 1, & \text{if } n = 0, \end{cases} \tag{3}$$

here $\mathbb{Z}_0^-$ denotes the set of non positive integers and the Gamma and Beta functions are defined respectively as

$$\Gamma\left(z\right) = \int_0^\infty w^{z-1} e^{-w} dw, \text{ for } \operatorname{Re}\left(z\right) > 0,$$

and

$$B\left(s,z\right) = B\left(z,s\right) = \int_0^1 w^{s-1}\left(1-w\right)^{z-1} dw = \frac{\Gamma\left(s\right)\Gamma\left(z\right)}{\Gamma\left(s+z\right)}$$

for $\operatorname{Re}\left(s\right) > 0$ and $\operatorname{Re}\left(z\right) > 0$. The binomial coefficient is defined as

$$\left(\begin{array}{c} z \\ w \end{array} \right) = \frac{\Gamma\left(z+1\right)}{\Gamma\left(w+1\right)\Gamma\left(z-w+1\right)}$$

for z and w non-negative integers, where $\Gamma\left(x\right)$ is the Gamma function. The polygamma functions $\psi^{(k)}\left(z\right)$, $k \in \mathbb{N}$ are defined by

$$\psi^{(k)}\left(z\right) := \frac{d^{k+1}}{dz^{k+1}} \log\Gamma\left(z\right) = \frac{d^k}{dz^k}\left(\frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)}\right) = -\int\limits_0^1 \frac{\left[\log(t)\right]^k\ t^{z-1}}{1-t} dt,\ k \in \mathbb{N} \tag{4}$$

and $\psi^{(0)}\left(z\right) = \psi\left(z\right)$, denotes the Psi, or digamma function, defined by

$$\psi\left(z\right) = \frac{d}{dz}\log\Gamma\left(z\right) = \frac{\Gamma'\left(z\right)}{\Gamma\left(z\right)}.$$

We also recall the relation, for $m = 1, 2, 3, ...$

$$H_{z-1}^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(z). \tag{5}$$

There are many results of the type (2) for finite, infinite and alternating sums. The result

$$\sum_{n=0}^{p} \frac{\binom{p}{n}}{\binom{q}{n}} = \frac{1}{1-\frac{p}{q+1}}, \text{ for } q \geq p \geq 0$$

appears in [5]. Other results appear in [1, 3, 4, 6, 7, 9, 11, 12, 13, 14, 15].

2. Generalizations

The next lemma deals with an alternative proof of (2) from which more generalized identities can be ascertained.

Lemma 2. *Let $z \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $t \in \mathbb{R}$ and $p \in \mathbb{N}$. Then*

$$\sum_{n=1}^{p} \frac{t^n}{n\binom{n+z}{z}} = t\int_0^1 \frac{(1-x)^z}{1-tx}\left(1-(tx)^p\right)dx \tag{6}$$

$$= \frac{t}{z+1}\left({}_2F_1\left[\begin{matrix}1,1\\2+z\end{matrix}\middle| t\right] - t^p\, {}_2F_1\left[\begin{matrix}1,1+p\\p+z+2\end{matrix}\middle| t\right]\right) \tag{7}$$

and when $t = 1$,

$$\sum_{n=1}^{p} \frac{1}{n\binom{n+z}{z}} = \int_0^1 (1-x)^{z-1}(1-x^p)\,dx = \frac{1}{z} - \frac{1}{z\binom{z+p}{p}} \tag{8}$$

where ${}_2F_1\left[\begin{matrix}\cdot,\cdot\\ \cdot\end{matrix}\middle| t\right]$ is the Gauss hypergeometric function.

Proof. First notice that, with t^n, (2) may be written as

$$\sum_{n=1}^{p} \frac{t^n}{n\binom{n+z}{z}} = \sum_{n=1}^{p} \frac{t^n\,\Gamma(n)\Gamma(z+1)}{\Gamma(n+z+1)} = \sum_{n=1}^{p} t^n\, B(n, z+1)$$

$$= \int_0^1 \frac{(1-x)^z}{x}\sum_{n=1}^{p}(tx)^n\, dx = t\int_0^1 \frac{(1-x)^z}{1-tx}\left(1-(tx)^p\right)\, dx,$$

which is the integral (6) and performing the integration, results in (7). When $t = 1$, we obtain (8). It may be of some interest to note that from (7), with $t = 1$, and (8) we have the hypergeometric identities

$$ {}_2F_1\left[\begin{array}{c} 1,1 \\ 2+z \end{array}\middle| 1\right] = 1 + \frac{1}{z} \text{ and } {}_2F_1\left[\begin{array}{c} 1,1+p \\ 2+z+p \end{array}\middle| 1\right] = 1 + \frac{1+p}{z}. $$

□

Remark 3. It is possible to massage (8) to produce identities of the form;

$$ \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} H_{r+p} = H_p + \frac{1}{n\binom{n+p}{p}} + \frac{H_n}{n} - \frac{1}{n} $$

for p a positive integer.

The next lemma deals with the derivatives of binomial coefficients.

Lemma 4. *Let $z \geq 0$, $n > 0$ and let $Q(n,z) = \binom{n+z}{z}^{-1}$ be an analytic function of $z \in \mathbb{C} \setminus \{-1,-2,-3,...\}$. Then,*

$$ Q^{(1)}(n,z) = \frac{dQ}{dz} = \begin{cases} -Q(n,z)P(n,z), \quad \textit{where} \\ P(n,z) = \sum\limits_{r=1}^{n} \frac{1}{r+z} \quad \textit{for } z > 0 \\ -Q(n,z)\left[\psi(z+1+n) - \psi(z+1)\right] \\ -H_n, \quad \textit{for } z = 0, \end{cases} $$

and

$$ Q^{(\lambda)}(n,z) = \frac{d^\lambda Q}{dz^\lambda} = -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(n,z) P^{(\lambda-1-\rho)}(n,z), \quad \textit{for } \lambda \geq 2 \tag{9} $$

where $P^{(0)}(n,z) = \sum\limits_{r=1}^{n} \frac{1}{r+z}$, for $n \in \mathbb{N}$ and $Q^{(0)}(n,z) = Q(n,z)$. For $i \in \mathbb{N}$,

$$ P^{(i)}(n,z) = \frac{d^i P}{dz^i} = \frac{d^i}{dz^i}\left(\sum_{r=1}^{n} \frac{1}{r+z}\right) = (-1)^i i! \sum_{r=1}^{n} \frac{1}{(r+z)^{i+1}}. $$

A proof of Lemma 4 is given in [8].

Now we list some particular cases of Lemma 4.

$$ Q^{(1)}(n,z) = -\binom{n+z}{z}^{-1} \sum_{r=1}^{n} \frac{1}{r+z}, $$

$$\begin{aligned} Q^{(2)}(n,z) &= \binom{n+z}{z}^{-1}\left[\left(\sum_{r=1}^{n}\frac{1}{r+z}\right)^2+\sum_{r=1}^{n}\frac{1}{(r+z)^2}\right] \\ &= \binom{n+z}{z}^{-1}\left[\sum_{r=1}^{n}\sum_{s=1}^{r}\frac{2}{(r+z)(s+z)}\right], \end{aligned}$$

$$Q^{(3)}(n,z) = -\binom{n+z}{z}^{-1}\left[\begin{array}{c} \left(\sum\limits_{r=1}^{n}\frac{1}{r+z}\right)^3+2\sum\limits_{r=1}^{n}\frac{1}{(r+z)^3} \\ +3\sum\limits_{r=1}^{n}\frac{1}{(r+z)^2}\sum\limits_{r=1}^{n}\frac{1}{r+z} \end{array}\right]$$

and

$$Q^{(4)}(n,z) = \binom{n+z}{z}^{-1}\left[\begin{array}{c} 6\sum\limits_{r=1}^{n}\frac{1}{(r+z)^2}\left(\sum\limits_{r=1}^{n}\frac{1}{r+z}\right)^2 \\ +8\sum\limits_{r=1}^{n}\frac{1}{(r+z)^3}\sum\limits_{r=1}^{n}\frac{1}{r+z}+3\left(\sum\limits_{r=1}^{n}\frac{1}{(r+z)^2}\right)^2 \\ +\left(\sum\limits_{r=1}^{n}\frac{1}{r+z}\right)^4+6\sum\limits_{r=1}^{n}\frac{1}{(r+z)^4} \end{array}\right].$$

In the special case when $z=0$ we may write

$$Q^{(1)}(n,0) = -H_n, \tag{10}$$

$$Q^{(2)}(n,0) = (H_n)^2 + H_n^{(2)}, \tag{11}$$

$$Q^{(3)}(n,0) = -(H_n)^3 - 3H_nH_n^{(2)} - 2H_n^{(3)} \tag{12}$$

and

$$Q^{(4)}(n,0) = (H_n)^4 + 6(H_n)^2H_n^{(2)} + 8H_nH_n^{(3)} + 3\left(H_n^{(2)}\right)^2 + 6H_n^{(4)}. \tag{13}$$

We can generalize (8) as follows.

Lemma 5. *Let q be a positive integer, $Q(n,z) = \binom{n+z}{z}^{-1}$, and $Q^{(q)}(n,z)$ be defined by (9). Then*

$$\sum_{n=1}^{p}\frac{Q^{(q)}(n,z)}{n} = \int_0^1 (1-x)^{z-1}\ (1-x^p)\ \log^q(1-x)\ dx$$

for $z \neq \mathbb{Z}_0^-$.

Proof. The proof follows upon differentiating (8) q times. □

The following is an example for $q=2$.

Example 6. For $q = 2$,

$$\sum_{n=1}^{p} \frac{\left(\sum\limits_{r=1}^{n} \frac{1}{r+z}\right)^2 + \sum\limits_{r=1}^{n} \frac{1}{(r+z)^2}}{n \begin{pmatrix} n+z \\ z \end{pmatrix}} = \frac{2}{z^3} - \frac{1}{z \begin{pmatrix} z+p \\ p \end{pmatrix}} \left[\begin{array}{c} (\psi(z) - \psi(z+p+1))^2 \\ +\psi'(z) - \psi'(z+p+1) \end{array} \right]$$

$$= \frac{2}{z^3} - \frac{1}{z \begin{pmatrix} z+p \\ p \end{pmatrix}} \left[\begin{array}{c} H_{z-1}^2 - H_{z-1}^{(2)} + H_{z+p}^2 + H_{z+p}^{(2)} \\ -2H_{z-1}H_{z+p} \end{array} \right].$$

Theorem 7. *Let $z \in \mathbb{R} \backslash \mathbb{Z}_0^-$ and let p, $q \in \mathbb{N}$. Then*

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} \quad = \quad -Q^{(q)}(p,z), \tag{14}$$

and when $z = 0$,

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,0)}{n} = -Q^{(q)}(p,0), \tag{15}$$

where $Q^{(0)}(n,z) = \frac{1}{\begin{pmatrix} n+z \\ z \end{pmatrix}}$ and $Q^{(0)}(n,0) = 1$.

Proof. From the left hand side of (8), let $F(n,z) = \sum\limits_{n=1}^{p} \frac{Q(n,z)}{n}$ so that $zF(n,z) = 1 - Q(p,z)$. Differentiating q times with respect to z results in

$$qF^{(q-1)}(n,z) + zF^{(q)}(n,z) = -Q^{(q)}(p,z)$$

or

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z)}{n} + \sum_{n=1}^{p} \frac{zQ^{(q)}(n,z)}{n} = -Q^{(q)}(p,z),$$

arriving at (14). From Lemma 5, we may write in integral form

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} \quad = \quad -Q^{(q)}(p,z)$$

$$= \int_0^1 (1-x)^{z-1} \ (1-x^p) \ \log^{q-1}(1-x) \left(q + z\log(1-x)\right) \, dx.$$

For $z = 0$
$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,0)}{n} = -Q^{(q)}(p,0),$$
and hence (15). $\square$

The following is an example.

Example 8. For $q = 3$
$$\sum_{n=1}^{p} \frac{3Q^{(2)}(n,z)}{n} + \sum_{n=1}^{p} \frac{zQ^{(3)}(n,z)}{n} = -Q^{(3)}(p,z),$$
in explicit form
$$\sum_{n=1}^{p} \frac{3\left[(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}\right]}{n\begin{pmatrix} n+z \\ z \end{pmatrix}}$$
$$+\sum_{n=1}^{p} \frac{z\left[\begin{array}{c} (H_{n+z} - H_z)^3 \\ +2\left(H_{n+z}^{(3)} - H_z^{(3)}\right) + 3\left(H_{n+z}^{(2)} - H_z^{(2)}\right)(H_{n+z} - H_z) \end{array}\right]}{n\begin{pmatrix} n+z \\ z \end{pmatrix}}$$
$$= \frac{\left[(H_{p+z} - H_z)^3 + 2\left(H_{p+z}^{(3)} - H_z^{(3)}\right) + 3\left(H_{p+z}^{(2)} - H_z^{(2)}\right)(H_{p+z} - H_z)\right]}{\begin{pmatrix} p+z \\ z \end{pmatrix}}$$
and when $z = 0$
$$\sum_{n=1}^{p} \frac{3\left((H_n)^2 + H_n^{(2)}\right)}{n} = (H_p)^3 + 2H_p^{(3)} + 3H_p^{(2)}H_p\ .$$

The next Lemma as stated in [10], gives an alternate representation for $Q^{(q)}(p,z)$.

Lemma 9. *Let q and p be positive integers, $Q(p,z) = \begin{pmatrix} p+z \\ z \end{pmatrix}^{-1}$, and $Q^{(q)}(p,z)$ be defined by (9). It was proved in [10] that an alternate representation for $Q^{(q)}(p,z)$ is:*
$$(-1)^{q-1}\, q! \sum_{r=0}^{p} (-1)^r \binom{p}{r} \frac{r}{(r+z)^{q+1}} \quad = \quad Q^{(q)}(p,z) \tag{16}$$

$$= \frac{(-1)^q \, q! \; p}{(z+1)^{q+1}} \quad {}_{q+2}F_{q+1}\left[\begin{array}{c} \overbrace{z+1, ..., z+1}^{(q+1)-terms}, 1-p \\ \underbrace{z+2, ..., z+2}_{(q+1)-terms} \end{array} \middle| 1 \right]. \tag{17}$$

From (14) and (16) we see that

$$\sum_{n=1}^{p} \frac{qQ^{(q-1)}(n,z) + zQ^{(q)}(n,z)}{n} \quad = \quad (-1)^q \;\; q! \sum_{r=0}^{p} \binom{p}{r} \frac{(-1)^r \;\, r}{(r+z)^{q+1}} = -Q^{(q)}(p,z).$$

From Example 2 with $q = 3$, we have

$$6\sum_{r=0}^{p} (-1)^{r+1} \binom{p}{r} \frac{r}{(r+z)^4} = \sum_{n=1}^{p} \frac{3\left[(H_{n+z} - H_z)^2 + H_{n+z}^{(2)} - H_z^{(2)}\right]}{n \begin{pmatrix} n+z \\ z \end{pmatrix}}$$

$$+ \sum_{n=1}^{p} \frac{z \left[\begin{array}{c} (H_{n+z} - H_z)^3 \\ \\ +2\left(H_{n+z}^{(3)} - H_z^{(3)}\right) + 3\left(H_{n+z}^{(2)} - H_z^{(2)}\right)(H_{n+z} - H_z) \end{array} \right]}{n \begin{pmatrix} n+z \\ z \end{pmatrix}}.$$

The inversion formula, see [5], states

$$g(n) = \sum_k (-1)^k \binom{n}{k} f(k) \Leftrightarrow f(n) = \sum_k (-1)^k \binom{n}{k} g(k),$$

and hence

$$\sum_{n=0}^{p} (-1)^n \binom{p}{n} Q^{(q)}(n,z) = (-1)^{q-1} \;\; q! \frac{p}{(p+z)^{q+1}},$$

and when $z = 0$,

$$\sum_{n=0}^{p} (-1)^n \binom{p}{n} Q^{(q)}(n,0) = \frac{(-1)^{q-1} \;\; q!}{p^q}.$$

It can also be noted that

$$\begin{aligned} \sum_{p=1}^{\infty} \sum_{n=0}^{p} (-1)^n \binom{p}{n} Q^{(q)}(n,z) &= (-1)^{q-1} \;\; q! \sum_{p=1}^{\infty} \frac{p}{(p+z)^{q+1}} \\ &= (-1)^q \left(q\psi^{(q-1)}(z+1) + z\psi^{(q)}(z+1) \right) \\ &= (-1)^q \, q! \left(\zeta(q) - H_z^{(q)} - z\zeta(q+1) + zH_z^{(q+1)} \right), \end{aligned}$$

by (5); and when $z = 0$,

$$\sum_{p=1}^{\infty}\sum_{n=0}^{p}(-1)^n\binom{p}{n}Q^{(q)}(n,0)=(-1)^q\,q!\zeta\left(q\right).$$

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