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A q-ANALOGUE OF THE GENERALIZED FACTORIAL NUMBERS

Seok-Zun Song, Gi-Sang Cheon, Young-Bae Jun, and LeRoy B. Beasley

Abstract. In this paper, more generalized q-factorial coefficients are examined by a natural extension of the q-factorial on a sequence of any numbers. This immediately leads to the notions of the extended q-Stirling numbers of both kinds and the extended q-Lah numbers. All results described in this paper may be reduced to well-known results when we set $q = 1$ or use special sequences.

1. Introduction

During the last several decades, the Stirling numbers of the first and second kinds have been studied from many diverse viewpoints (see for example [2]-[17]). A viewpoint of Carlitz [2], motivated by the enumeration problem for Abelian groups, is to study the Stirling numbers as specializations of the q-Stirling numbers. Originally, he defined the *q-Stirling numbers of the second kind* as the numbers $S_q(n,k)$ in our notation such that

$$(1) \qquad [x]^n = \sum_{k=0}^{n} q^{\binom{k}{2}} S_q(n,k)[x]_k,$$

where $[x]$ and $[x]_k$ denote the *q-number* and the (*falling*) *q-factorial of order k*, respectively defined as

$$[x] = (1-q^x)/(1-q) = 1 + q + \cdots + q^{x-1},$$

and

$$(2) \qquad [x]_k = [x][x-1]\cdots[x-k+1] \ \ (k \geq 1), \quad [x]_0 = 1$$

for real numbers x and q.

Recently there has been interested in generalizing the q-factorial as well as q-Stirling numbers, see for example [3, 4, 14, 17]. This interest is largely motivated by Carlitz [2]. He found for some purposes it is convenient to generalize

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the q-Stirling numbers, and studied the coefficients $a_{n,k}(r)$ such that

$$[x+r]^n=\sum_{k=0}^{n} q^{\binom{k+r}{2}} a_{n,k}(r)[x]_k \tag{3}$$

for any real number r. The expression (3) may be written as

$$[x]^n=\sum_{k=0}^{n} q^{\binom{k}{2}+kr} S_q(n,k;r)[x-r]_k, \tag{4}$$

which reduces to (1) when $r=0$, where $S_q(n,k;r) := q^{\binom{r}{2}} a_{n,k}(r)$.

In 2004, Charalambides [4] (also see [12]) called $[x-r]_k$ and $S_q(n,k;r)$ the non-central q-factorial of order k with non-centrality parameter r and the non-central q-Stirling numbers of the second kind, respectively. Moreover, he (see [3, 4]) intensively investigated the generalized q-factorial coefficients with increment h, $R_q(n,k;h)$, and the non-central generalized q-factorial coefficients, $C_q(n,k;s,r)$, where

$$\prod_{k=0}^{n-1}[x-kh]=q^{-h\binom{n}{2}}\sum_{k=0}^{n} q^{\binom{k}{2}} R_q(n,k;h)[x]_k \tag{5}$$

and

$$\prod_{k=0}^{n-1}[sx+r-k]=q^{rn-\binom{n}{2}}\sum_{k=0}^{n} q^{s\binom{k}{2}} C_q(n,k;s,r)[x]_{k,q^s}. \tag{6}$$

In the present paper, more generalized q-factorial coefficients which involve $R_q(n,k;h)$ and $C_q(n,k;s,r)$ as well as central or non-central q-Stirling numbers of both kinds and q-Lah numbers are examined by a natural extension of the q-factorial. This immediately leads to the notion of the extended q-Stirling numbers of both kinds and the extended q-Lah numbers. Specifically, in Section 2 we first define the extended q-factorial for a sequence $\alpha=(\alpha_n)_{n\geq 0}$ of any numbers. Then we develop the extension of the generalized q-factorial coefficients and obtain the explicit formula by aid of Newton's general interpolation formula based on the divided differences. In Section 3, we give the definition of the extended q-Stirling numbers of both kinds and the extended q-Lah numbers, and we examine the connections with the generalized q-factorial coefficients. In Section 4, the non-central q-Stirling numbers of both kinds with an increment which generalize the non-central q-Stirling numbers examined in [4] are developed. In Section 5 we obtain some interesting matrix factorizations arising from the extended q-factorial coefficients, q-Stirling numbers of both kinds and q-Lah numbers. In particular, the LDU-factorization of the Vandermonde matrix is given. All results described in this paper may be reduced to well-known results when we set $q=1$ or use special sequences for α.

Finally we should mention that similar generalizations of the factorial coefficients can be found in the literature, see for example [6, 8, 14, 17]. However it should also be emphasized that we have focused our attention on q-analogies of the generalized factorial coefficients and related numbers for any sequences. In particular, our extended q-Stirling numbers allow a straightforward extension of the notions of the central and non-central q-Stirling numbers presented in [4, 12].

2. Extended q-factorial coefficients

We begin this section defining the *extended q-factorial of order n associated with a sequence* $\alpha = (\alpha_n)_{n\geq 0}$ of any numbers, denoted by $f_n(x;\alpha)$, as

$$f_n(x;\alpha) = \begin{cases} [x-\alpha_0][x-\alpha_1]\cdots[x-\alpha_{n-1}] & \text{if } n \geq 1, \\ 1 & \text{if } n = 0. \end{cases} \tag{7}$$

For the sake of notational convenience we shall mean $\alpha-\alpha_0 = (\alpha_n-\alpha_0)_{n\geq 0}$, $\alpha - 1 = (\alpha_n - 1)_{n\geq 0}$ and $-\alpha = (-\alpha_n)_{n\geq 0}$. We note that $f_n(x;\alpha)$ may be considered as the central or non-central extended q-factorial along with $\alpha_0 = 0$ or $\alpha_0 \neq 0$, respectively. Also one may consider $f_n(x;\alpha-\alpha_0)$ as the central extended q-factorial by means of

$$f_n(x;\alpha-\alpha_0) = [x][x-(\alpha_1-\alpha_0)]\cdots[x-(\alpha_{n-1}-\alpha_0)], \quad n \geq 1.$$

We shall take $f_n(x;-\alpha)$ as the notation for the extended rising q-factorial of order n associated with $\alpha = (\alpha_n)_{n\geq 0}$ by means of

$$f_n(x;-\alpha) = [x+\alpha_0][x+\alpha_1]\cdots[x+\alpha_{n-1}], \; f_0(x;-\alpha) = 1.$$

By the q-Newton's formula (see [4]), the expansion of the extended q-factorial $f_n(x;\alpha)$ into a polynomial of q-factorial $[x]_k$ is given by

$$f_n(x;\alpha) = \sum_{k=0}^{n} \frac{1}{[k]!}\left[\Delta_q^k f_n(x;\alpha)\right]_{x=0}[x]_k, \tag{8}$$

where Δ_q^k is the q-difference operator of order k defined by $\Delta_q^k = (E-1)(E-q)\cdots(E-q^{k-1})$ together with the usual shift operator E.

We suppose that $\alpha = (\alpha_n)_{n\geq 0}$ and $\beta = (\beta_n)_{n\geq 0}$ are two distinct sequences. Applying Newton's general interpolation formula [11] based on the divided differences, we easily obtain the expansion of $f_n(x;\alpha)$ into a polynomial of $f_n(x;\beta)$ given by

$$f_n(x;\alpha) = \sum_{k=0}^{n} q^{\mathcal{C}_k(\beta)}\left(\mathcal{D}_q^k f_n(x;\alpha)\right)_{x=\beta_0} f_k(x;\beta), \tag{9}$$

where $\mathcal{C}_k(\beta) = \beta_1+\beta_2+\cdots+\beta_{k-1}$ and $\mathcal{D}_q^k$ is the q-divided difference operator of order k defined by

$$\mathcal{D}_q^k f_n(x;\alpha) = \frac{\mathcal{D}_q^{k-1} f_n(x;\alpha) - \mathcal{D}_q^{k-1} f_n(\beta_k;\alpha)}{[x]-[\beta_k]}. \tag{10}$$

The following theorem provides very useful another generalization for q-Stirling numbers of both kinds and q-Lah numbers in the next section.

Theorem 2.1. *Given the sequences $\alpha = (\alpha_n)_{n\geq 0}$ and $\beta = (\beta_n)_{n\geq 0}$ of any numbers, the followings are equivalent*:

(i) $f_n(x;\alpha) = q^{-\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} q^{\mathcal{C}_k(\beta)}\Omega_q(n,k;\alpha,\beta) f_k(x;\beta)$;

(ii) $\Omega_q(n,k;\alpha,\beta) = \Omega_q(n-1,k-1;\alpha,\beta) + ([\beta_k]-[\alpha_{n-1}])\Omega_q(n-1,k;\alpha,\beta)$, $\Omega_q(0,0;\alpha,\beta) = 1$;

(iii) $\Omega_q(n,k;\alpha,\beta) = \sum_{j=0}^{k} \frac{\prod_{i=0}^{n-1}([\beta_j]-[\alpha_i])}{\prod_{i=0,\ i\neq j}^{k}([\beta_j]-[\beta_i])}$ *if* $\beta_i \neq \beta_j$ *for each* $i,j = 0,\ldots,k$,

$\Omega_q(n,0;\alpha,\beta) = \prod_{i=0}^{n-1}([\beta_0]-[\alpha_i])$, $(n \geq 1)$,

where $\mathcal{C}_n(\alpha) := \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$ *and* $\mathcal{C}_0(\alpha) = 0$ *are assumed.*

Proof. Comparing (i) with (9), we get

$$\Omega_q(n,k;\alpha,\beta) = q^{\mathcal{C}_n(\alpha)}\left[\mathcal{D}_q^k f_n(x;\alpha)\right]_{x=\beta_0}.$$

Using the identity $f_n(x;\alpha) = q^{-\alpha_{n-1}}([x]-[\alpha_{n-1}])f_{n-1}(x;\alpha)$, we get

$$\begin{aligned}
\Omega_q(n,k;\alpha,\beta) &= q^{\mathcal{C}_n(\alpha)}\mathcal{D}_q^k f_n(x;\alpha)|_{x=\beta_0} \\
&= q^{\mathcal{C}_{n-1}(\alpha)}\mathcal{D}_q^k([x]-[\beta_k])f_{n-1}(x;\alpha)|_{x=\beta_0} \\
&\quad + ([\beta_k]-[\alpha_{n-1}])q^{\mathcal{C}_{n-1}(\alpha)}\mathcal{D}_q^k f_{n-1}(x;\alpha)|_{x=\beta_0} \\
&= q^{\mathcal{C}_{n-1}(\alpha)}\mathcal{D}_q^{k-1}\{\mathcal{D}_q([x]-[\beta_k])f_{n-1}(x;\alpha)\}|_{x=\beta_0} \\
&\quad + ([\beta_k]-[\alpha_{n-1}])\Omega_q(n-1,k;\alpha,\beta) \\
&= \Omega_q(n-1,k-1;\alpha,\beta) + ([\beta_k]-[\alpha_{n-1}])\Omega_q(n-1,k;\alpha,\beta),
\end{aligned}$$

which proves (i) $\Leftrightarrow$ (ii).

The relation (i) $\Leftrightarrow$ (iii) is easily seen by

$$\mathcal{D}_q^k f_n(\beta_0;\alpha) = \frac{f_n(\beta_0;\alpha)}{\omega_k(\beta_0,\beta)} + \frac{f_n(\beta_1;\alpha)}{\omega_k(\beta_1,\beta)} + \cdots + \frac{f_n(\beta_k;\alpha)}{\omega_k(\beta_k,\beta)},\ (k = 0,1,\ldots,n),$$

where $\omega_k(\beta_j,\beta) = \frac{q^{\mathcal{C}_{k+1}(\beta)}f_{k+1}(x;\beta)}{[x]-[\beta_j]}|_{x=\beta_j}$ with $\omega_0(\beta_j,\beta) = 1$.

Hence the proof is complete. □

Theorem 2.1 suggests that many well-known results can be derived when we set $q = 1$ or use special sequences α and β. We call the coefficients $\Omega_q(n,k;\alpha,\beta)$ the *extended q-factorial coefficients associated with the sequences* α *and* β.

By aid of the formula

$$\Delta_q^k f_n(x;\alpha) = \sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}}\begin{bmatrix}k\\ j\end{bmatrix} f_n(x+j;\alpha), \tag{11}$$

we can easily derive the following corollary from Theorem 2.1.

Corollary 2.2. *Given the sequence $\alpha = (\alpha_n)_{n\geq 0}$ of any numbers, the followings are equivalent*:

(i) $f_n(x;\alpha) = q^{-\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} q^{\binom{k}{2}} \Omega_q(n,k;\alpha)[x]_k$;

(ii) $\Omega_q(n,k;\alpha) = \frac{q^{\mathcal{C}_n(\alpha)-\binom{k}{2}}}{[k]!} \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} f_n(j;\alpha)$;

(iii) $\Omega_q(n,k;\alpha) = \Omega_q(n-1,k-1;\alpha) + ([k]-[\alpha_{n-1}])\Omega_q(n-1,k;\alpha)$, $\Omega_q(0,0;\alpha) = 1$.

We observe that the formula (i) above corollary may be regarded as an extension of both (5) and (6). If each α_n is replaced by nh, then we obtain immediately $R_q(n,k;h) = \Omega_q(n,k;\alpha)$. If each α_n is replaced by $\frac{n-r}{s}$, then we obtain

$$\begin{aligned}\prod_{k=0}^{n-1}[sx+r-k] &= [s]^n \prod_{k=0}^{n-1}\left[x - \frac{k-r}{s}\right]_{q^s} \\ &= q^{rn-\binom{n}{2}} \sum_{k=0}^{n} q^{s\binom{k}{2}} \left\{[s]^n q^{(s-1)(nr-\binom{n}{2})}\Omega_{q^s}(n,k;\alpha)\right\}[x]_{k,q^s}.\end{aligned}$$

It implies that $C_q(n,k;s,r) = [s]^n q^{(s-1)(nr-\binom{n}{2})}\Omega_{q^s}(n,k;\alpha)$.

3. Extended q-Stirling and q-Lah numbers

Since the extended q-factorial $f_n(x;\alpha)$ is a polynomial of the q-number $[x]$ of degree $n \geq 1$, by following Carlitz [2] it would be natural to define more generalized q-Stirling numbers of the first and second kind related to $f_n(x;\alpha)$ denoted by $s_q(n,k;\alpha)$ and $S_q(n,k;\alpha)$, respectively as follows:

$$f_n(x;\alpha) = q^{-\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} s_q(n,k;\alpha)[x]^k, \tag{12}$$

and

$$[x]^n = \sum_{k=0}^{n} q^{\mathcal{C}_k(\alpha)} S_q(n,k;\alpha) f_k(x;\alpha). \tag{13}$$

In a similar way, we may express the extended rising q-factorial $f_n(x;-\alpha)$ associated with the sequence $\alpha = (\alpha_n)_{n\geq 0}$ by

$$f_n(x;-\alpha) = q^{\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} q^{\mathcal{C}_k(\alpha)} L_q(n,k;\alpha) f_k(x;\alpha). \tag{14}$$

We should also mention that (12) and (13) yield the following orthogonality relation:

$$\sum_{k=0}^{n} s_q(i,k;\alpha) S_q(k,j;\alpha) = \sum_{k=0}^{n} S_q(i,k;\alpha) s_q(k,j;\alpha) = \delta_{ij}, \tag{15}$$

where δ_{ij} is the Kronecker's delta.

Further, given sequences $\alpha = (\alpha_n)_{n\geq 0}$ and $\beta = (\beta_n)_{n\geq 0}$ of any numbers the extended q-factorial $f_n(x;\alpha)$ may be expressed by

$$\begin{aligned} f_n(x;\alpha) &= q^{-\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} s_q(n,k;\alpha) \left(\sum_{j=0}^{k} q^{\mathcal{C}_j(\beta)} S_q(k,j;\beta) f_j(x;\beta) \right) \\ &= q^{-\mathcal{C}_n(\alpha)} \sum_{k=0}^{n} q^{\mathcal{C}_k(\beta)} \left(\sum_{j=k}^{n} s_q(n,j;\alpha) S_q(j,k;\beta) \right) f_k(x;\beta). \end{aligned}$$

Comparing above expression with (i) of Theorem 2.1, we obtain the following theorem.

Theorem 3.1. *Let $\alpha = (\alpha_n)_{n\geq 0}$ and $\beta = (\beta_n)_{n\geq 0}$ be sequences of any numbers. Then we have*

$$\Omega_q(n,k;\alpha,\beta) = \sum_{j=k}^{n} s_q(n,j;\alpha) S_q(j,k;\beta). \tag{16}$$

If $\alpha = (n)_{n\geq 0}$, then $f_k(x;\alpha)$ reduces to q-factorial $[x]_k$. It means $s_q(n,k;\alpha)$, $S_q(n,k;\alpha)$ and $L_q(n,k;\alpha)$ reduce to the ordinary q-Stirling numbers of the first and second kind and q-Lah numbers, respectively. Further, by setting $q = 1$ these three numbers reduce to Comtet's generalized Stirling numbers of both kinds [6] and the generalized Lah numbers which are related to the generalized factorial $(x;\alpha)_n := (x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{n-1})$ for any sequence $\alpha = (\alpha_n)_{n\geq 0}$, respectively. For the reason, we will call the coefficients $s_q(n,k;\alpha)$, $S_q(n,k;\alpha)$ and $L_q(n,k;\alpha)$ appearing in (12), (13) and (14), respectively, the *extended q-Stirling numbers of first and second kind* and the *extended q-Lah numbers associated with the sequence $\alpha = (\alpha_n)_{n\geq 0}$*.

We note that these special numbers may be obtained in some connections with the extended q-factorial coefficients $\Omega_q(n,k;\alpha,\beta)$:

(i) $s_q(n,k;\alpha) = \Omega_q(n,k;\alpha,0)$;
(ii) $S_q(n,k;\alpha) = \Omega_q(n,k;0,\alpha)$;
(iii) $L_q(n,k;\alpha) = \Omega_q(n,k;-\alpha,\alpha)$.

The following result can be easily derived from Theorem 2.1.

Theorem 3.2. *Given a sequence $\alpha = (\alpha_n)_{n\geq 0}$ of any numbers, we have*:

(i) $s_q(n,k;\alpha) = s_q(n-1,k-1;\alpha) - [\alpha_{n-1}]s_q(n-1,k;\alpha),\ s_q(0,0;\alpha) = 1$;
(ii) $S_q(n,k;\alpha) = S_q(n-1,k-1;\alpha) + [\alpha_k]S_q(n-1,k;\alpha),\ S_q(0,0;\alpha) = 1$;
(iii) $L_q(n,k;\alpha) = L_q(n-1,k-1;\alpha) + ([\alpha_k]-[-\alpha_{n-1}])L_q(n-1,k;\alpha)$, $L_q(0,0;\alpha) = 1$;
(iv) $s_q(n,k;\alpha) = (-1)^{n-k} \sum_{\substack{c_0+c_1+\cdots+c_{n-1}=n-k \\ c_i\in\{0,1\}}} [\alpha_0]^{c_0}[\alpha_1]^{c_1}\cdots[\alpha_{n-1}]^{c_{n-1}}$;
(v) $S_q(n,k;\alpha) = \sum_{\substack{c_0+c_1+\cdots+c_k=n-k \\ c_i\in\{0,1,\ldots,n\}}} [\alpha_0]^{c_0}[\alpha_1]^{c_1}\cdots[\alpha_k]^{c_k}$;
(vi) $L_q(n,k;\alpha) = \sum_{j=k}^{n} s_q(n,j;-\alpha)S_q(j,k;\alpha)$;

(vii) $\sum_{n=0}^{\infty} S_q(n,k;\alpha)x^n = \frac{x^k}{(1-[\alpha_0]x)(1-[\alpha_1]x)\cdots(1-[\alpha_k]x)}$;

(viii) $\sum_{k=0}^{n} s_q(n,k;\alpha)x^{n-k} = (1-[\alpha_0]x)(1-[\alpha_1]x)\cdots(1-[\alpha_{n-1}]x)$.

We point out that the above theorem may be also derived from some known results for Comtet's generalized Stirling numbers [6].

Corollary 3.3. *Let* $\alpha = (\alpha_n)_{n\geq 0}$ *be a sequence of any numbers. Then we have*:

(i) $S_q(n,k;\alpha) = \sum_{j=k}^{n} q^{(j-k)\alpha_0}[\alpha_0]^{n-j}\binom{n}{j}S_q(j,k;\alpha-\alpha_0)$;

(ii) $s_q(n,k;\alpha) = \sum_{j=k}^{n} q^{(n-k)\alpha_0}[-\alpha_0]^{j-k}\binom{j}{k}s_q(n,j;\alpha-\alpha_0)$.

Proof. We only give the proof of (i) since (ii) may be proved analogously. First, let $\alpha_0 \neq 0$ and let $k = 0$. Since $S_q(0,0;\alpha-\alpha_0) = 1$ and $S_q(n,0;\alpha-\alpha_0) = 0$ for $n \geq 1$, (i) gives $S_q(n,0;\alpha) = [\alpha_0]^n$. Hence (i) holds for $k = 0$. Now the proof is by induction on n. Clearly (i) holds when $n = 1$. Assume that $n \geq 2$, $k \geq 1$. From (ii) of Theorem 3.2 we have

$$\begin{aligned} S_q(n,k;\alpha) = &\sum_{j=k-1}^{n-1} q^{(j-k+1)\alpha_0}[\alpha_0]^{n-1-j}\binom{n-1}{j}S_q(j,k-1;\alpha-\alpha_0) \\ &+ [\alpha_k]\sum_{j=k}^{n-1} q^{(j-k)\alpha_0}[\alpha_0]^{n-1-j}\binom{n-1}{j}S_q(j,k;\alpha-\alpha_0). \end{aligned}$$

Apply $[\alpha_k] = [\alpha_0] + q^{\alpha_0}[\alpha_k - \alpha_0]$ to the last equation and combine the first and second summation using the Pascal identity and $S_q(j,k-1;\alpha-\alpha_0) + [\alpha_k - \alpha_0]S_q(j,k;\alpha-\alpha_0) = S_q(j+1,k;\alpha-a_0)$. Then it is easily seen that

$$\begin{aligned} S_q(n,k;\alpha) &= \sum_{j=k}^{n-1} q^{(j-k)\alpha_0}[\alpha_0]^{n-j}\binom{n}{j}S_q(j,k;\alpha-\alpha_0) \\ &\quad + q^{(n-k)\alpha_0}S_q(n,k;\alpha-\alpha_0) \\ &= \sum_{j=k}^{n} q^{(j-k)\alpha_0}[\alpha_0]^{n-j}\binom{n}{j}S_q(j,k;\alpha-\alpha_0), \end{aligned}$$

which completes the proof. $\square$

We conclude this section describing the familiar formulae for the q-Stirling numbers $s_q(n,k)$ and $S_q(n,k)$, and q-Lah numbers $L_q(n,k)$. By a q-binomial coefficient we shall mean

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]!}{[r]![n-r]!},$$

where n and r are nonnegative integers and $[n]! = [n][n-1]\cdots[1]$.

The following may be easily deduced from Theorem 3.2 and Corollary 3.3:

(i) $S_q(n,k) = \frac{1}{[k]!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}-\binom{k}{2}}\begin{bmatrix}k\\j\end{bmatrix}_q[j]^n$;

(ii) $s_q(n,k) = \sum_{j=k}^{n}(-1)^{j-k}q^{n-j}\binom{j-1}{k-1}s_q(n-1,j-1)$;

(iii) $L_q(n,k) = q^{\binom{k}{2}-\binom{n}{2}}\begin{bmatrix}n-1\\k-1\end{bmatrix}_q\frac{[n]!}{[k]!}$.

We also note that the expression (i), (iii) were obtained in [17] and (ii) was obtained in [16].

4. Examples

Let us define the *non-central generalized q-factorial with both increment h and non-centrality parameter r of order n* denoted by $[x;r,h]_n$ as

$$[x;r,h]_n := [x-r][x-r-h]\cdots[x-r-(n-1)h]. \tag{17}$$

By setting $h=1$ or $r=0$, $[x;r,h]_n$ is reduced to the noncentral q-factorial with non-centrality parameter r or the central generalized q-factorial with increment h given in (4) and (5), respectively.

In this section, we investigate the q-Stirling numbers of both kinds and the q-Lah numbers corresponding to the generalized q-factorial $[x;r,h]_n$ as the special examples of our extended q-factorial coefficients. First note that $[x;r,h]_n$ may be considered as $f_n(x;\alpha)$ with $\alpha = (r+nh)_{n\geq 0}$ by means of $f_n(x;r+nh) := [x-r][x-(r+h)]\cdots[x-(r+(n-1)h)]$. Thus (12), (13) and (14) respectively can be expressed by

(i) $[x;r,h]_n = q^{-nr-\binom{n}{2}h}\sum_{k=0}^{n}s_q(n,k;r,h)[x]^k$;

(ii) $[x]^n = \sum_{k=0}^{n}q^{kr+\binom{k}{2}h}S_q(n,k;r,h)[x;r,h]_k$;

(iii) $[x;-r,-h]_k = q^{nr+\binom{n}{2}h}\sum_{k=0}^{n}q^{kr+\binom{k}{2}h}L_q(n,k;r,h)[x;r,h]_k$.

We call $s_q(n,k;r,h)$, $S_q(n,k;r,h)$ and $L_q(n,k;r,h)$ the *non-central q-Stirling numbers of the first and second kind with both increment h and non-centrality parameter r* and *non-central q-Lah numbers with both increment h and non-centrality parameter r*, respectively.

Theorem 4.1. *The $S_q(n,k;r,h)$ satisfy the following explicit formula*:

$$S_q(n,k;r,h) = \frac{q^{-\binom{k}{2}h-kr}}{[h]^k[k]_{q^h}!}\sum_{j=0}^{k}(-1)^{k-j}q^{\binom{k-j}{2}h}\begin{bmatrix}k\\j\end{bmatrix}_{q^h}[r+jh]^n. \tag{18}$$

Proof. From the expression (iii) for $\Omega_q(n,k;\alpha,\beta)$ with $\alpha\equiv 0$ in Theorem 2.1, we have

$$S_q(n,k;r,h) = \sum_{j=0}^{k}\frac{[\beta_j]^n}{\prod_{i=0,\ i\neq j}^{k}([\beta_j]-[\beta_i])}. \tag{19}$$

By aid of $[x]-[y]=q^y[x-y]$, $[xh]=[x]_{q^h}[h]$ and $[-x]=-q^{-x}[x]$ with the q^h-number $[x]_{q^h}$ and $\beta=(r+nh)_{n\geq 0}$, we get

$$\begin{aligned}\prod_{i=0,\ i\neq j}^{k}([\beta_j]-[\beta_i]) &= q^{\mathcal{C}_{k+1}(\beta)-\beta_j}[jh][(j-1)h]\cdots[h][-h][-2h]\cdots[-(k-j)h]\\ &= (-1)^{k-j}q^{kr-(j-\binom{k+1}{2}+\binom{k-j+1}{2})h}[h]^k[j]_{q^h}![k-j]_{q^h}!\\ &= (-1)^{k-j}q^{kr-(\binom{k-j}{2}-\binom{k}{2})h}[h]^k[k]_{q^h}!\left(\frac{[j]_{q^h}![k-j]_{q^h}!}{[k]_{q^h}!}\right).\end{aligned}$$

Hence (18) follows from (19). $\square$

The following is an immediate consequence of Corollary 3.3.

Corollary 4.2. *The $s_q(n,k;r,h)$ and $S_q(n,k;r,h)$ satisfy the following relations*:

(i) $s_q(n,k;r,h)=\sum_{j=k}^{n} q^{(n-k)r}[-r]^{j-k}\binom{j}{k}s_q(n,j;h)$;

(ii) $S_q(n,k;r,h)=\sum_{j=k}^{n} q^{(j-k)r}[r]^{n-j}\binom{n}{j}S_q(j,k;h)$,

where $s_q(n,k;h)$ and $S_q(n,k;h)$ are the central generalized q-Stirling numbers of the first and second kind with an increment h, i.e., $r=0$, investigated in [3].

5. Matrix factorizations

In this section, we develop special matrices arising from the extended q-factorial coefficients $\Omega_q(n,k;\alpha,\beta)$. Define the $n\times n$ matrix $\Omega_q(n;\alpha,\beta)$ by

$$[\Omega_q(n;\alpha,\beta)]_{i,j}=\begin{cases}\Omega_q(i,j;\alpha,\beta) & \text{if} \quad i\geq j\geq 0,\\ 0 & \text{otherwise,}\end{cases}$$

where the rows and columns are indexed by $0,1,\dots,n-1$. Similarly the matrices $s_q(n;\alpha)$, $S_q(n;\alpha)$ and $L_q(n;\alpha)$ corresponding to $s_q(n,k;\alpha)$, $S_q(n,k;\alpha)$ and $L_q(n,k;\alpha)$ respectively can be defined as $\Omega_q(n;\alpha,\beta)$. We will see how these matrices are connected with each other and are related to the q-Vandermonde matrix $V([\alpha_0],[\alpha_1],\dots,[\alpha_{n-1}])$.

We first define the $n\times n$ ($n\geq 2$) matrix $F_q^{(\ell)}(\alpha)$, $\ell=0,\dots,n-2$, for any numbers $\alpha_0,\alpha_1,\dots$ by

$$F_q^{(\ell)}(\alpha)=I_{n-\ell-2}\oplus\begin{bmatrix}1 & & & \\ [\alpha_0] & 1 & & \\ & \ddots & \ddots & \\ & & [\alpha_\ell] & 1\end{bmatrix},$$

where $I_{n-\ell-2}$ is the identity matrix of order $n-\ell-2$, and $\oplus$ denotes the direct sum of matrices and unspecified entries are all zeros.

Lemma 5.1. *The $n \times n$ matrix $\Omega_q(n;\alpha,\beta)$, $n \geq 2$, may be factorized by*

$$\Omega_q(n;\alpha,\beta) = (F_q^{(n-2)}(\alpha))^{-1}(I_1 \oplus \Omega_q(n-1;\alpha,\beta))F_q^{(n-2)}(\beta). \tag{20}$$

Proof. From the recurrence relation for $\Omega_q(n,k;\alpha,\beta)$ in Theorem 2.1, we have

$$\begin{aligned}&\Omega_q(n,k;\alpha,\beta) + [\alpha_{n-1}]\Omega_q(n-1,k;\alpha,\beta)\\ &= \Omega_q(n-1,k-1;\alpha,\beta) + [\beta_k]\Omega_q(n-1,k;\alpha,\beta),\end{aligned}$$

which implies that

$$F_q^{(n-2)}(\alpha)\Omega_q(n;\alpha,\beta) = (I_1 \oplus \Omega_q(n-1;\alpha,\beta))F_q^{(n-2)}(\beta).$$

Hence the proof is complete. □

The following is an immediate consequence of (20) and (16).

Corollary 5.2. *The $n \times n$ matrix $\Omega_q(n;\alpha,\beta)$, $n \geq 2$, may be factorized by*

(i) $\Omega_q(n;\alpha,\beta) = (F_q^{(0)}(\alpha)\cdots F_q^{(n-2)}(\alpha))^{-1}(F_q^{(0)}(\beta)\cdots F_q^{(n-2)}(\beta))$;
(ii) $\Omega_q(n;\alpha,\beta) = s_q(n;\alpha)S_q(n;\beta)$.

For example, if $n = 4$, then we have:

$$\begin{aligned}\Omega_q(n;\alpha,\beta) &= \left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&[\alpha_0]&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&[\alpha_0]&1&0\\0&0&[\alpha_1]&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\ [\alpha_0]&1&0&0\\0&[\alpha_1]&1&0\\0&0&[\alpha_2]&1\end{bmatrix}\right)^{-1}\\ &\quad\times\left(\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&[\beta_0]&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&[\beta_0]&1&0\\0&0&[\beta_1]&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\ [\beta_0]&1&0&0\\0&[\beta_1]&1&0\\0&0&[\beta_2]&1\end{bmatrix}\right)\\ &= \begin{bmatrix}1&0&0&0\\ -[\alpha_0]&1&0&0\\ [\alpha_0][\alpha_1]&-([\alpha_0]+[\alpha_1])&1&0\\ -[\alpha_0][\alpha_1][\alpha_2]&[\alpha_0][\alpha_1]+[\alpha_0][\alpha_2]+[\alpha_1][\alpha_2]&-([\alpha_0]+[\alpha_1]+[\alpha_2])&1\end{bmatrix}\\ &\quad\times\begin{bmatrix}1&0&0&0\\ [\beta_0]&1&0&0\\ [\beta_0]^2&[\beta_0]+[\beta_1]&1&0\\ [\beta_0]^3&[\beta_0]^2+[\beta_0][\beta_1]+[\beta_1]^2&[\beta_0]+[\beta_1]+[\beta_2]&1\end{bmatrix}\\ &= s_q(n;\alpha)S_q(n;\beta).\end{aligned}$$

The following is an immediate consequence of (15), Theorem 3.1, Theorem 3.2, Corollary 3.3, and Corollary 5.2.

Theorem 5.3. *The $n \times n$ matrices $s_q(n;\alpha)$, $S_q(n;\alpha)$, $L_q(n;\alpha)$ have the following matrix factorizations*:

(i) $s_q(n;\alpha)S_q(n;\alpha) = I_n$;
(ii) $S_q(n;\alpha) = F_q^{(0)}(\alpha)F_q^{(1)}(\alpha)\cdots F_q^{(n-2)}(\alpha)$;
(iii) $s_q(n;\alpha) = \begin{cases} (I_1 \oplus \hat{s}_q(n-1;\alpha-\alpha_0))P_n(-[\alpha_0]) & \text{if } \alpha_0 \neq 0, \\ I_1 \oplus (\hat{s}_q(n-1;\alpha-1)P_{n-1}^{-1}) & \text{if } \alpha_0 = 0; \end{cases}$
(iv) $S_q(n;\alpha) = \begin{cases} P_n([\alpha_0])(I_1 \oplus \hat{S}_q(n-1;\alpha-\alpha_0)) & \text{if } \alpha_0 \neq 0, \\ I_1 \oplus (P_{n-1}\hat{S}_q(n-1;\alpha-1)) & \text{if } \alpha_0 = 0; \end{cases}$
(v) $L_q(n;\alpha) = s_q(n;-\alpha)S_q(n;\alpha)$;
(vi) $(L_q(n;\alpha))^{-1} = L_q(n;-\alpha)$,

where $\hat{s}_q(n-1;\alpha-\alpha_0)$ *and* $\hat{S}_q(n-1;\alpha-\alpha_0)$ *are* $(n-1)\times(n-1)$ *matrices defined by*

$$[\hat{s}_q(n-1;\alpha-\alpha_0)]_{i,j} = \begin{cases} q^{(i-j)\alpha_0}s_q(n-1;i,j;\alpha-\alpha_0) & \text{if } n-1 \geq i \geq j \geq 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$[\hat{S}_q(n-1;\alpha-\alpha_0)]_{i,j} = \begin{cases} q^{(i-j)\alpha_0}S_q(n-1;i,j;\alpha-\alpha_0) & \text{if } n-1 \geq i \geq j \geq 1, \\ 0 & \text{otherwise}. \end{cases}$$

Further, the extended q-factorial $f_n(x;\alpha)$ can be used to obtain the LDU-decomposition of the $n \times n$ *q-Vandermonde matrix*

$$V_q(n;\alpha) := V([\alpha_0],[a_1],\ldots,[\alpha_{n-1}]) = ([\alpha_j]^i)_{ij\geq 0}.$$

Let $U_q(n;\alpha)$ denote the $n\times n$ matrix defined by $[U_q(n;\alpha)]_{ij} = f_i(\alpha_j;\alpha)$, and let $D_q(n;\alpha) = \operatorname{diag}(q^{\mathcal{C}_0(\alpha)}, q^{\mathcal{C}_1(\alpha)},\ldots,q^{\mathcal{C}_{n-1}(\alpha)})$. Note that $U_q(n;\alpha)$ is an upper triangular matrix since $f_i(\alpha_j;\alpha) = 0$ if $i > j$.

The following theorem may be easily obtained by setting $x = \alpha_j$ and $n = i$ for each $i,j = 0,1,\ldots,n-1$ in (13) (also see [16]).

Theorem 5.4. *Given a sequence* $\alpha = (\alpha_n)_{n\geq 0}$ *of any different numbers, the q-Vandermonde matrix* $V_q(n;\alpha)$ *may be factorized by*

$$V_q(n;\alpha) = S_q(n;\alpha)D_q(n;\alpha)U_q(n;\alpha),$$

where $S_q(n;\alpha)$ *is the generalized q-Stirling matrix of the second kind.*

Thus given a sequence $\alpha = (\alpha_n)_{n\geq 0}$ of any different numbers, $V_q(n;\alpha)$ is nonsingular and by using $q^{\alpha_i}[\alpha_j - \alpha_i] = [\alpha_j] - [\alpha_i]$ we obtain

$$\det V_q(n;\alpha) = \prod_{0\leq i<j\leq n}([\alpha_j] - [\alpha_i]).$$

Remark. If we take $\alpha_k \equiv x$ for all $k = 0,1,\ldots,n-1$, then $S_q(n;\alpha)$ is exactly the same as $P_n([x]) = [\binom{i}{j}[x]^{i-j}]$. Thus our results examined in the present paper may be used to obtain a q-analogue of the Pascal matrix. We omit the details here.

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Seok-Zun Song
Department of Mathematics
Jeju National University
Cheju 690-756, Korea
E-mail address: `szsong@cheju.ac.kr`

Gi-Sang Cheon
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail address: `gscheon@skku.edu`

Young-Bae Jun
Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: `ybjun@gsnu.ac.kr`

LeRoy B. Beasley
Department of Mathematics and Statistics
Utah State University
Logan, UT 84322-3900, U.S.A
E-mail address: lbeasley@math.usu.edu