ESSAYS ON THE THEORY OF ELLIPTIC HYPERGEOMETRIC FUNCTIONS

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Abstract. We give a brief review of the main results of the theory of elliptic hypergeometric functions — a new class of special functions of mathematical physics. We prove the most general univariate exact integration formula generalizing Euler's beta integral, which is called the elliptic beta integral. An elliptic analogue of the Gauss hypergeometric function is constructed together with the elliptic hypergeometric equation for it. Biorthogonality relations for this function and its particular subcases are described. We list known elliptic beta integrals on root systems and consider symmetry transformations for the corresponding elliptic hypergeometric functions of the higher order.

Contents

1. Introduction

Theory of special functions is widely used in theoretical and mathematical physics as a handbook collection of exact mathematical formulae together with the methods of their derivation. This concerns the series summation formulae, exactly computable integrals, symmetry transformations for functions, differential or other equations solvable in terms of “simple” functions, and so on. An impetuous buildup of such a database, which was taking place in the XIX century and which was on a top of priorities of the mathematics of that time, has changed in the XX century by an essential deterioration of interest to special functions, investigations of which started to be considered as a pursuit of a secondary importance. Such an attitude to investigations in this field was justified by an opinion that all principle types of interesting functions “with classical properties” (elliptic, hypergeometric, automorphic, and some other functions) have been found already, and it remains only to investigate them in more detail.

The theory of special functions of hypergeometric type was developing during several centuries, starting from the fundamental results obtained by Euler [1]. Gauss, Jacobi, Riemann, Kummer and other prominent mathematicians were contributing to its foundations. The Gauss hypergeomeric function ${}_2F_1(a, b; c; x)$ is a canonical example of functions of such a type. According to the approach by Pochhammer and Horn [2], the generalized plain hypergeometric series can be defined as the sums $\sum_n c_n$ for which the ratio c_{n+1}/c_n is a rational function of n. In 1847 Heine has introduced a q-analogue of the ${}_2F_1$-series ${}_2\varphi_1(a, b; c; q; x)$ [3]. The general series of such type $\sum_n c_n$ are characterized by the property that for them the ratio c_{n+1}/c_n is a rational function of q^n, where q is some complex parameter. Until recent time only these two classes of functions of hypergeometric type were known (including integral representations for them), and many papers were devoted to their investigations.

Unexpectedly, around ten years ago it became clear that there exist hypergeometric functions of the third type, which are related to elliptic curves. Such objects appeared for the first time within the quantum inverse scattering method developed for exactly solvable models of statistical mechanics [4, 5] as elliptic solutions of the Yang–Baxter equation [6, 7]. As shown by Frenkel and Turaev [8], these solutions (called elliptic $6j$-symbols) are expressed in terms of an elliptic generalization of the terminating very-well poised balanced q-hypergeometric series ${}_{10}\varphi_9$ with discrete values of the parameters. A generalized $(1+1)$-dimensional integrable chain similar to the discrete time Toda chain was constructed in the paper [9], and it was shown that the same terminating series with arbitrary parameters appears as a particular solution of the corresponding Lax pair equations.

The general formal definition of elliptic hypergeometric series, which was suggested and investigated in detail in [10], describes them as the series $\sum_n c_n$, for which the ratio c_{n+1}/c_n is equal to an elliptic function of n. Within this scheme, the case [8] is characterized by the presence of several interesting structural restrictions upon the coefficients c_n. There are certain difficulties in the description of infinite series connected with their convergency. Therefore the general elliptic hypergeometric functions are defined by the integral representations [11].

The Meijer function can be considered as the most general plain hypergeometric function [12]. It is defined by a contour integral of some ratio of the Euler gamma functions. For integral representations of more complicated functions of the hypergeometric type one needs the generalized gamma functions the theory of which was developed by Barnes [13] and Jackson [14] more than a century ago. The Jackson q-gamma function is necessary for the description of q-hypergeometric functions at $|q| < 1$. A more complicated function is needed when q lies on the unit circle, $|q| = 1$ [15, 16, 17, 18]. These functions, related to the Barnes gamma function of the second order, are actively used in the modern mathematical physics in the description of quantum integrable models and representations of quantum algebras [19, 20, 21, 22, 23, 24].

For the definition of elliptic hypergeometric integrals one needs the elliptic gamma function related to the Barnes gamma function of the third order. Importance of the elliptic gamma function was stressed by Ruijsenaars [18], who gave to it this name and considered some of its properties. Modular transformations of this function were described in [25]. In [11], a modified elliptic gamma function was built which remains also well defined in the case when one of the base parameters lies on the unit circle. Other aspects of this function were investigated in [26, 27, 28].

The first exact integration formula, which uses the elliptic gamma function, was constructed by the author in [29]. It represents the most general known univariate integral generalizing Euler's beta integral [1]. This elliptic beta integral serves as a basis for building general very-well poised elliptic hypergeometric functions. The first direction of generalizations consists in increasing the number of free parameters entering the integrand, which leads to the elliptic analogues of the functions ${}_{s+1}F_s$ [11, 30]. In particular, in this way one builds an elliptic analogue of the Gauss hypergeometric function and derives its properties [31]. The second direction of generalizations increases the number of integrations in such a way that the integrands acquire symmetries in the integration variables related to the root systems [11, 32, 33, 34, 35, 36]. One of such generalizations leads to an elliptic analogue of the Selberg integral. Over a short period of time, in the papers mentioned above and cited below in the list of references, a systematic theory of elliptic hypergeometric functions of one and many variables has been built. The present review is devoted to a brief description of this theory.

2. Generalized gamma functions

We use the symbols: $\mathbb{Z} = 0, \pm 1, \pm 2, \ldots$; $\mathbb{C}$ – the open complex plane; $\mathbb{C}^* = \mathbb{C}/\{0\}$; $\mathbb{R}$ – the real axis; $i = \sqrt{-1}$.

2.1. The Barnes multiple gamma function.

The Euler gamma function is a cornerstone of the theory of ordinary hypergeometric functions [1]. Different analogues of this function with many parameters have been considered in the mathematical literature of the beginning of the twentieth century. The most complete investigations of the generalized gamma functions belong to Barnes [13]. As a starting point in his work serves a function generalizing the Hurwitz zeta-function [1]

$$\zeta(s, u) = \sum_{n=0}^{\infty} \frac{1}{(u+n)^s}, \quad \mathrm{Re}(s) > 1,$$

which reduces to the Riemann zeta-function for $u = 1$. Because of the grown interest the Barnes theory was sufficiently widely discussed in the recent literature, for instance, in [19, 26, 27, 28, 37].

Let us take m quasiperiods $\omega_j \in \mathbb{C}$, which we assume to be linearly independent over $\mathbb{Z}$ for simplicity (the condition of incommensurability). For $s, u \in \mathbb{C}$ the Barnes zeta function is defined by the m-fold series

$$\zeta_m(s, u; \omega) = \sum_{n_1,\ldots,n_m=0}^{\infty} \frac{1}{(u+\Omega)^s}, \qquad \Omega = n_1\omega_1 + \ldots + n_m\omega_m,$$

converging for $\mathrm{Re}(s) > m$ and under the condition that all ω_j lie in one half-plane defined by some line passing through the zero point of the coordinates axes. Because of the latter requirement, the sequences $n_1\omega_1 + \ldots + n_m\omega_m$ do not have accumulation points on the finite plane for any $n_j \to +\infty$. It is convenient to assume for definiteness that $\mathrm{Re}(\omega_j) > 0$ or $\mathrm{Im}(\omega_j) > 0$.

The function $\zeta_m(s, u; \omega)$ satisfies the following set of finite difference equations

$$\zeta_m(s, u + \omega_j; \omega) - \zeta_m(s, u; \omega) = -\zeta_{m-1}(s, u; \omega(j)), \quad j = 1, \ldots, m, \tag{2.1}$$

where $\omega(j) = (\omega_1, \ldots, \omega_{j-1}, \omega_{j+1}, \ldots, \omega_m)$ and $\zeta_0(s, u; \omega) = u^{-s}$. It may be continued analytically (meromorphically) to the whole complex plane $\mathrm{Re}(s) \leq m$ with the simple poles at the points $s = 1, 2, \ldots, m$. The Barnes multiple gamma function is defined by the equality

$$\Gamma_m(u; \omega) = \exp(\partial\zeta_m(s, u; \omega)/\partial s)\big|_{s=0}.$$

It has an infinite product representation of the form

$$\frac{1}{\Gamma_m(u; \omega)} = e^{\sum_{k=0}^m \gamma_{mk}\frac{u^k}{k!}}\, u \prod_{n_1,\ldots,n_m=0}^{\infty}{}' \left(1 + \frac{u}{\Omega}\right) e^{\sum_{k=1}^m (-1)^k \frac{u^k}{k\Omega^k}}, \tag{2.2}$$

where γ_{mk} are some constants analogous to the Euler constant. The prime of the product means that the point $n_1 = \ldots = n_m = 0$ is skipped in it. In particular, the function $\Gamma_1(u; \omega)$ is directly related to the Euler gamma function $\Gamma(u)$,

$$\Gamma_1(u; \omega) = \frac{\omega^{u/\omega}}{\sqrt{2\pi\omega}} \Gamma\left(\frac{u}{\omega}\right),$$

which can be checked by straightforward manipulations with the Hurwitz zeta function. We note that Barnes used in [13] a different normalization of the Γ_m-function, in which one has $\gamma_{m0} = 0$.

The function $\Gamma_m(u; \omega)$ satisfies m finite difference equations of the first order, obtained by differentiation of equalities (2.1) at the point $s = 0$:

$$\Gamma_m(u + \omega_j; \omega) = \frac{1}{\Gamma_{m-1}(u; \omega(j))} \Gamma_m(u; \omega), \qquad j = 1, \ldots, m, \tag{2.3}$$

where $\Gamma_0(u; \omega) := u^{-1}$.

The following integral representations for the Euler gamma function are well known

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt = \frac{i}{2\sin(\pi s)} \int_{C_H} (-t)^{s-1} e^{-t} dt,$$

where in the first case $\mathrm{Re}(s) > 0$, and in the second expression $|\arg(-t)| < \pi$ and the Hankel contour C_H starts and finishes near the $+\infty$ point, turning around the half-axis $[0,\infty)$ counterclockwise. One can write with their help

$$\zeta_m(s,u;\omega)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{t^{s-1}e^{-ut}}{\prod_{k=1}^m(1-e^{-\omega_k t})}dt$$
$$=\frac{i\Gamma(1-s)}{2\pi}\int_{C_H}\frac{(-t)^{s-1}e^{-ut}}{\prod_{k=1}^m(1-e^{-\omega_k t})}dt$$

and analytically continue this function in s to the whole complex plane. Using the latter expression, Barnes has derived the following integral representation for the multiple gamma functions

$$\Gamma_m(u;\omega)=\exp\left(\frac{1}{2\pi i}\int_{C_H}\frac{e^{-ut}(\log(-t)+\gamma)}{t\prod_{k=1}^m(1-e^{-\omega_k t})}dt\right), \tag{2.4}$$

where γ is the Euler constant.

The values $\zeta_m(0,u;\omega)$ are expressed in terms of the multiple Bernoulli polynomials

$$\zeta_m(0,u;\omega)=\frac{(-1)^m}{m!}B_{m,m}(u|\omega),$$

which are defined by the generating function

$$\frac{x^m e^{xu}}{\prod_{k=1}^m(e^{\omega_k x}-1)}=\sum_{n=0}^\infty B_{m,n}(u|\omega_1,\ldots,\omega_m)\frac{x^n}{n!}. \tag{2.5}$$

We shall need in the following the first three diagonal polynomials

$$B_{1,1}(u|\omega_1)=\frac{u}{\omega_1}-\frac{1}{2},$$
$$B_{2,2}(u|\omega_1,\omega_2)=\frac{1}{\omega_1\omega_2}\left(u^2-(\omega_1+\omega_2)u+\frac{\omega_1^2+\omega_2^2}{6}+\frac{\omega_1\omega_2}{2}\right),$$
$$B_{3,3}(u|\omega_1,\omega_2,\omega_3)=\frac{1}{\omega_1\omega_2\omega_3}\left(u^3-\frac{3u^2}{2}\sum_{k=1}^3\omega_k+\frac{u}{2}\left(\sum_{k=1}^3\omega_k^2+3\sum_{j<k}\omega_j\omega_k\right)\right.$$
$$\left.-\frac{1}{4}\left(\sum_{k=1}^3\omega_k\right)\sum_{j<k}\omega_j\omega_k\right).$$

Theory of the plain hypergeometric functions is built with the help of the Euler gamma function or $\Gamma_1(u;\omega_1)$; the q-hypergeometric functions are tied to $\Gamma_2(u;\omega_1,\omega_2)$, and the elliptic hypergeometric functions "live" at the level of the Barnes multiple gamma function of the third order, respectively.

2.2. **The elliptic gamma functions.**

Let $\omega_1,\omega_2,\omega_3$ denote complex parameters linearly independent over $\mathbb{Z}$ and lying in the right half-plane. We define with their help the base variables $p,q,r\in\mathbb{C}$:

$$q=e^{2\pi i\frac{\omega_1}{\omega_2}},\quad p=e^{2\pi i\frac{\omega_3}{\omega_2}},\quad r=e^{2\pi i\frac{\omega_3}{\omega_1}} \tag{2.6}$$

and their modular transformed ($\tau\to-1/\tau$) partners

$$\tilde{q}=e^{-2\pi i\frac{\omega_2}{\omega_1}},\quad \tilde{p}=e^{-2\pi i\frac{\omega_2}{\omega_3}},\quad \tilde{r}=e^{-2\pi i\frac{\omega_1}{\omega_3}}. \tag{2.7}$$

For $|p|,|q|<1$, the infinite products

$$(z;q)_\infty=\prod_{j=0}^\infty(1-zq^j),\quad (z;p,q)_\infty=\prod_{j,k=0}^\infty(1-zp^jq^k)$$

are well defined and satisfy q-difference equations

$$(qz;q)_\infty = \frac{(z;q)_\infty}{1-z},$$
$$(qz;q,p)_\infty = \frac{(z;q,p)_\infty}{(z;p)_\infty}, \qquad (pz;q,p)_\infty = \frac{(z;q,p)_\infty}{(z;q)_\infty}. \tag{2.8}$$

The shortened theta function $\theta(z;p)$

$$\theta(z;p) := (z;p)_\infty (pz^{-1};p)_\infty = \frac{1}{(p;p)_\infty}\sum_{k\in\mathbb{Z}}(-1)^k p^{k(k-1)/2} z^k \tag{2.9}$$

plays a key role in our considerations. It obeys the following simple symmetry transformations:

$$\theta(pz;p) = \theta(z^{-1};p) = -z^{-1}\theta(z;p), \tag{2.10}$$

and has zeros, $\theta(z;p)=0$, at $z=p^k,\ k\in\mathbb{Z}$. Evidently, $\theta(z;0)=1-z$. For $k>0$, we have

$$\theta(p^k z;p) = \frac{\theta(z;p)}{(-z)^k p^{\binom{k}{2}}}, \qquad \theta(p^{-k}z;p) = \frac{(-z)^k\theta(z;p)}{p^{\binom{k+1}{2}}}.$$

We shall need the $\tau\to -1/\tau$ modular transformation rule for $\theta(z;p)$, for the description of which it is necessary to use the exponential parameterization of variables:

$$\theta\left(e^{-2\pi i\frac{u}{\omega_1}}; e^{-2\pi i\frac{\omega_2}{\omega_1}}\right) = e^{\pi i B_{2,2}(u|\omega_1,\omega_2)}\theta\left(e^{2\pi i\frac{u}{\omega_2}}; e^{2\pi i\frac{\omega_1}{\omega_2}}\right), \tag{2.11}$$

where $B_{2,2}(u|\omega_1,\omega_2)$ is the second Bernoulli polynomial. In the following it is convenient to use the compact notations

$$\theta(a_1,\ldots,a_k;p) := \theta(a_1;p)\cdots\theta(a_k;p), \quad \theta(at^{\pm1};p) := \theta(at;p)\theta(at^{-1};p).$$

The simplest gamma function can be defined as a special meromorphic solution of the functional equation $f(u+\omega_1)=uf(u)$. Following Jackson's approach [14], we shall be connecting q-gamma functions with the meromorphic solutions of the equation

$$f(u+\omega_1) = (1-e^{2\pi iu/\omega_2})f(u), \tag{2.12}$$

where $q=e^{2\pi i\omega_1/\omega_2}$. Introducing the variable $z=e^{2\pi iu/\omega_2}$, this equation can be replaced by $f(qz)=(1-z)f(z)$. For $|q|<1$ its particular solution, analytical at the point $z=0$, is determined by a simple iteration, which yields the standard q-gamma function $\gamma_q(z)=1/(z;q)_\infty$ (which can be considered also as a q-exponential function [3]). This expression differs from the Jackson q-gamma function

$$\Gamma_q^{(J)}(u) = \frac{(q;q)_\infty}{(q^u;q)_\infty}(1-q)^{1-u}, \tag{2.13}$$

satisfying the equation $\Gamma_q^{(J)}(u+1)/\Gamma_q^{(J)}(u) = (1-q^u)/(1-q)$, by a simple change of the argument and by a simple multiplier. The limiting transition to the ordinary gamma function has the form $\lim_{q\to1}\Gamma_q^{(J)}(u)=\Gamma(u)$ [1, 3], but for a simplification of q-hypergeometric formulae it is more convenient to use the function $\gamma_q(z)$.

The modified q-gamma function, which remains well defined at $|q|=1$ as well, has the form

$$\gamma(u;\omega_1,\omega_2) = \exp\left(-\int_{\mathbb{R}+i0}\frac{e^{ux}}{(1-e^{\omega_1x})(1-e^{\omega_2x})}\frac{dx}{x}\right), \tag{2.14}$$

where the contour $\mathbb{R}+i0$ passes along the real axis turning over the point $x=0$ from above in an infinitesimal way. This function appeared in the number theory [15] and in the theory of completely integrable systems [17, 18, 19]. It figures in the literature under the different names: "the double sine" [16], "the non-compact quantum dilogarithm" [21], "the hyperbolic gamma function" [18].

Let $\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2) > 0$. Then integral (2.14) is convergent for $0 < \mathrm{Re}(u) < \mathrm{Re}(\omega_1+\omega_2)$. Under appropriate restrictions on u and $\omega_{1,2}$, integral (2.14) can be computed as a convergent sum of residues of the poles in the upper half-plane. For $\mathrm{Im}(\omega_1/\omega_2) > 0$ this leads to the expression

$$\gamma(u;\omega_1,\omega_2) = \frac{(e^{2\pi i u/\omega_1}\tilde{q};\tilde{q})_\infty}{(e^{2\pi i u/\omega_2};q)_\infty}, \tag{2.15}$$

which is continued analytically to the whole complex plane of u (this expression satisfies equation (2.12) in an evident way). This q-gamma function serves as a main "brick" in the construction of analytical q-hypergeometric functions at $|q|=1$, which were not considered in the literature until the recent time.

The modified q-gamma function is proportional to a ratio of two Barnes gamma functions of the second order. The general relation of such a type is derived with the help of integral representation (2.4) and has the form [27]

$$\exp\left(-\int_{\mathbb{R}+i0} \frac{e^{ux}}{\prod_{k=1}^m (e^{\omega_k x}-1)}\frac{dx}{x}\right) = e^{\frac{\pi i}{m!}B_{m,m}(u|\omega)} \frac{\Gamma_m(u;\omega)^{(-1)^m}}{\Gamma_m(\sum_{k=1}^m \omega_k - u;\omega)},$$

where $\mathrm{Re}(\omega_k) > 0$ and $0 < \mathrm{Re}(u) < \mathrm{Re}(\sum_{k=1}^m \omega_k)$. In [27] there were obtained also infinite product representations of these functions analogous to (2.15), which we are not describing here.

Already in the works of Barnes it was noticed that the Jacobi $\theta_1(u|\tau)$-function can be decomposed as a product of four multiple gamma functions of the second order with different arguments. The exact form of such a relation is (see, for instance, [26])

$$\theta(e^{2\pi i u};p) = \frac{e^{-\pi i B_{2,2}(u|1,\tau)}}{\Gamma_2(u;1,\tau)\Gamma_2(1+\tau-u;1,\tau)\Gamma_2(u-\tau;1,-\tau)\Gamma_2(1-u;1,-\tau)}, \tag{2.16}$$

where $p=e^{2\pi i\tau}$. The general relation between multiple gamma functions and infinite products of the Jackson type has the form [26]

$$\prod_{n_1,\ldots,n_m=0}^{\infty} (1-e^{2\pi i(u+\Omega)}) = \frac{e^{-\pi i \zeta_{m+1}(0,u;1,\alpha)}}{\Gamma_{m+1}(u;1,\alpha)\Gamma_{m+1}(1-u;1,-\alpha)},$$

where $\alpha=(\alpha_1,\ldots,\alpha_m)$, $\Omega = n_1\alpha_1+\ldots+n_m\alpha_m$, and $\mathrm{Im}(\alpha_j)>0$.

Following the logic of definitions of the q-gamma functions, we connect the elliptic gamma functions with meromorphic solutions of the finite difference equation

$$f(u+\omega_1) = \theta(e^{2\pi i u/\omega_2};p)f(u), \tag{2.17}$$

which passes to (2.12) for $p\to 0$. Using the factorization (2.9) and equalities (2.8), it is not difficult to see that the ratio

$$\Gamma(z;p,q) = \frac{(pqz^{-1};p,q)_\infty}{(z;p,q)_\infty} = \prod_{j,k=0}^{\infty} \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \tag{2.18}$$

where $|p|, |q| < 1$ and $z \in \mathbb{C}^*$, satisfies the equations

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q), \quad \Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q). \tag{2.19}$$

Thus, the function $f(u) = \Gamma(e^{2\pi i u/\omega_2}; p, q)$ defines a solution of equation (2.17) at $|q|, |p| < 1$, and it is called the (standard) elliptic gamma function [18]. Because non-trivial triply periodic functions do not exist, it can be defined uniquely as the meromorphic solution of the system of three equations:

$$f(u + \omega_2) = f(u), \qquad f(u + \omega_3) = \theta(e^{2\pi i u/\omega_2}; q)f(u)$$

and equation (2.17) with the normalization of the solution $f(\sum_{k=1}^3 \omega_k/2) = 1$. The reflection equation for this generalized gamma function has the form $\Gamma(z; p, q)\Gamma(pq/z; p, q) = 1$. For $p = 0$, we have $\Gamma(z; 0, q) = \gamma_q(z)$.

The modified elliptic gamma function, which is well defined for $|q| = 1$ as well, has the form [11]

$$G(u; \omega) = \Gamma(e^{2\pi i \frac{u}{\omega_2}}; p, q)\Gamma(re^{-2\pi i \frac{u}{\omega_1}}; \tilde{q}, r). \tag{2.20}$$

It defines the unique solution of three equations:

$$f(u + \omega_2) = \theta(e^{2\pi i u/\omega_1}; r)f(u), \qquad f(u + \omega_3) = e^{-\pi i B_{2,2}(u|\omega)}f(u)$$

and equation (2.17) with the normalization of the solution $f(\sum_{k=1}^3 \omega_k/2) = 1$.

It is easy to check [38], that the function

$$G(u; \omega) = e^{-\frac{\pi i}{3}B_{3,3}(u|\omega)}\Gamma(e^{-2\pi i \frac{u}{\omega_3}}; \tilde{r}, \tilde{p}), \tag{2.21}$$

where $|\tilde{p}|, |\tilde{r}| < 1$, satisfies the same three equations and the normalization as the function (2.20). Therefore these functions coincide, and their equality constitutes one of the laws of the modular transformations for the elliptic gamma function related to the $SL(3; \mathbb{Z})$-group [25]. From the expression (2.21) it follows, that $G(u; \omega)$ is a meromorphic function of u for $\omega_1/\omega_2 > 0$, i.e. $|q| = 1$.

Because of the relation $B_{3,3}(\sum_{k=1}^3 \omega_k - u|\omega) = -B_{3,3}(u|\omega)$, the reflection formula for the G-function has the form $G(a; \omega)G(b; \omega) = 1$, $a + b = \sum_{k=1}^3 \omega_k$. For $|q| < 1$ in the limit $p, r \to 0$ (i.e., $\mathrm{Im}(\omega_3/\omega_1)$, $\mathrm{Im}(\omega_3/\omega_2) \to +\infty$) expression (2.20) passes in an evident way to the modified q-gamma function $\gamma(u; \omega_1, \omega_2)$. The representation (2.21) provides an alternative way for the reduction to this function (such a limiting transition was rigorously justified in a different way in [18]). As follows from the results of [28], for a fixed domain of values of parameters the function $G(u; \omega)$ converges in this limit to $\gamma(u; \omega_1, \omega_2)$ exponentially fast and uniformly on compact subsets of this domain. This result is important for a rigorous justification of the corresponding degeneration of the elliptic hypergeometric integrals.

Using the theta function factorization (2.16), one can consider equation (2.17) as a composition of four equations for $\Gamma_3(u; \omega)$ with different arguments and quasiperiods. This permits to represent the elliptic gamma function as a ratio of four Barnes gamma functions of the third order [26]

$$\Gamma(e^{2\pi i u}; e^{2\pi i \tau}, e^{2\pi i \sigma}) = \frac{e^{-\frac{\pi i}{3}B_{3,3}(u|1,\sigma,\tau)}\Gamma_3(u; 1, \sigma, \tau)\Gamma_3(1 - u; 1, -\sigma, -\tau)}{\Gamma_3(1 + \sigma + \tau - u; 1, \sigma, \tau)\Gamma_3(u - \sigma - \tau; 1, -\sigma, -\tau)}.$$

For $0 < \mathrm{Im}(u) < \mathrm{Im}(\tau + \sigma)$, one has the representation

$$\Gamma(e^{2\pi i u}; e^{2\pi i \tau}, e^{2\pi i \sigma}) = \exp\left(-\frac{i}{2}\sum_{k=1}^{\infty} \frac{\sin(\pi k(2u - \tau - \sigma))}{k \sin(\pi k \tau)\sin(\pi k \sigma)}\right),$$

through which the elliptic gamma function appeared implicitly in the work of Baxter on the eight-vertex model [6] (see also [5, 25]).

3. The elliptic beta integral

As a first example of elliptic hypergeometric functions we describe the elliptic beta integral, which was discovered by the author in [29].

Theorem 1. *We consider six complex parameters t_j, $j = 1, \ldots, 6$, and two base variables p and q satisfying the constraints $|p|, |q|, |t_j| < 1$ and $\prod_{j=1}^6 t_j = pq$ (the balancing condition). Then the following equality is true*

$$\kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k; p, q), \tag{3.1}$$

where $\mathbb{T}$ denotes the unit circle with positive orientation and

$$\kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi i}.$$

Here and below we use the compact notation

$$\Gamma(a_1, \ldots, a_k; p, q) := \Gamma(a_1; p, q) \cdots \Gamma(a_k; p, q),$$
$$\Gamma(tz^{\pm 1}; p, q) := \Gamma(tz; p, q)\Gamma(tz^{-1}; p, q), \quad \Gamma(z^{\pm 2}; p, q) := \Gamma(z^2; p, q)\Gamma(z^{-2}; p, q).$$

Proof. We take variables $z, q, p \in \mathbb{C}$, $|q|, |p| < 1$, and five complex parameters $t_m, m = 1, \ldots, 5$, and compose the function

$$\rho(z, t_1, \ldots, t_5) = \frac{\prod_{m=1}^5 \Gamma(t_m z^{\pm 1}, A t_m^{-1}; p, q)}{\Gamma(z^{\pm 2}, A z^{\pm 1}; p, q) \prod_{1 \le m < s \le 5} \Gamma(t_m t_s; p, q)}, \tag{3.2}$$

where $A = \prod_{m=1}^5 t_m$. This function has sequences of poles converging to zero along the points

$$\mathcal{P} = \{t_m q^a p^b,\ A^{-1} q^{a+1} p^{b+1}\}_{m=1,\ldots,5,\ a,b=0,1,\ldots}$$

and diverging to infinity along their $z \to 1/z$ reciprocals $\mathcal{P}^{-1}$. Let C denote a contour on the complex plane with positive orientation, which separates the sets $\mathcal{P}$ and $\mathcal{P}^{-1}$ (the existence of such a contour is the only restriction on the parameters t_m). For instance, for $|t_m| < 1$, $|pq| < |A|$, the contour C can coincide with the unit circle $\mathbb{T}$. Let us prove now that

$$\int_C \rho(z, t_1, \ldots, t_5) \frac{dz}{z} = \frac{4\pi i}{(q;q)_\infty (p;p)_\infty}, \tag{3.3}$$

where from the needed formula will follow after the substitution $A = pq/t_6$.

The first step consists in the derivation of the following q-difference equation for the kernel:

$$\rho(z, qt_1, t_2, \ldots, t_5) - \rho(z, t_1, \ldots, t_5) = g(q^{-1}z, t_1, \ldots, t_5) - g(z, t_1, \ldots, t_5), \tag{3.4}$$

where

$$g(z, t_1, \ldots, t_5) = \rho(z, t_1, \ldots, t_5) \frac{\prod_{m=1}^5 \theta(t_m z; p)}{\prod_{m=2}^5 \theta(t_1 t_m; p)} \frac{\theta(t_1 A; p)}{\theta(z^2, Az; p)} \frac{t_1}{z}. \tag{3.5}$$

After the division of equation (3.4) by $\rho(z,t_1,\ldots,t_5)$, it takes the form

$$\frac{\theta(t_1z,t_1z^{-1};p)}{\theta(Az,Az^{-1};p)}\prod_{m=2}^5\frac{\theta(At_m^{-1};p)}{\theta(t_1t_m;p)}-1$$
$$=\frac{t_1\theta(t_1A;p)}{z\theta(z^2;p)\prod_{m=2}^5\theta(t_1t_m;p)}\left(\frac{z^4\prod_{m=1}^5\theta(t_mz^{-1};p)}{\theta(Az^{-1};p)}-\frac{\prod_{m=1}^5\theta(t_mz;p)}{\theta(Az;p)}\right)\tag{3.6}$$

Both sides of this equality define elliptic functions of $\log z$ (that is they are invariant under the transformation $z\to pz$) with equal sets of poles and their residues. For example,

$$\lim_{z\to A}\theta(Az^{-1};p)\binom{\text{left-hand}}{\text{side}}=\frac{\theta(t_1A,t_1A^{-1};p)}{\theta(A^2;p)}\prod_{m=2}^5\frac{\theta(At_m^{-1};p)}{\theta(t_1t_m;p)},$$

with the same result for the right-hand side. Therefore the difference of expressions in two sides of equality (3.6) defines an elliptic function without poles, that is a constant. This constant is equal to zero because equation (3.6) is checked in a trivial way for the choice $z=t_1$.

We integrate now (3.4) over the variable $z\in C$ and obtain

$$I(qt_1,t_2,\ldots,t_5)-I(t_1,\ldots,t_5)=\left(\int_{q^{-1}C}-\int_C\right)g(z,t_1,\ldots,t_5)\frac{dz}{z},\tag{3.7}$$

where $I(t_1,\ldots,t_5)=\int_C\rho(z,t_1,\ldots,t_5)dz/z$, and $q^{-1}C$ denotes the contour C dilated with respect to the point $z=0$ by the factor q^{-1}. Function (3.5) has sequences of poles converging to zero along the points $z=\{t_mq^ap^b,A^{-1}q^ap^{b+1}\}$ and diverging to infinity at $z=\{t_m^{-1}q^{-a-1}p^{-b},Aq^{-a-1}p^{-b-1}\}$ for $m=1,\ldots,5$ and $a,b=0,1,\ldots$. For the choice $C=\mathbb{T}$, it is seen that at $|t_m|<1$ and $|p|<|A|$ there are no poles in the annulus $1\le|z|\le|q|^{-1}$. Therefore we can deform $q^{-1}\mathbb{T}$ to $\mathbb{T}$ in (3.7) and obtain zero on the right-hand side. As a result, $I(qt_1,t_2,\ldots,t_5)=I(t_1,\ldots,t_5)$.

Requiring $|p|,|q|<|A|$, we have by symmetry in p and q that $I(pt_1,t_2,\ldots,t_5)=I(t_1,\ldots,t_5)$. Further transformations $t_1\to q^{\pm1}t_1$ and $t_1\to p^{\pm1}t_1$ can be performed only if they do not take parameters outside of the annulus of analyticity of the function $I(t_1,\ldots,t_5)$.

Let us suppose temporarily that p and q are real, $p<q$, and $p^n\neq q^k$ for any $n,k=0,1,\ldots$. Impose also the constraint that the arguments of $t_m^{\pm1}$, $m=1,\ldots,5$, and $A^{\pm1}$ differ from each other. Let C denotes now a contour encircling $\mathcal{P}$ and two cuts $c_1=[t_1,t_1p^2]$, $c_2=[(pq/A)p^{-2},pq/A]$ and excluding their $z\to1/z$ reciprocals. Then we can make transformations $t_1\to t_1q^k$, $k=1,2,\ldots$, until the moment when t_1q^k enters the interval $[t_1p,t_1p^2]$, after which we replace $t_1\to t_1p^{-1}$; this does not take out needed parameters outside of the intervals c_1 or c_2. In this way we obtain $I(q^jp^{-k}t_1,t_2,\ldots,t_5)=I(t_1,\ldots,t_5)$ for all $j,k=0,1,\ldots$ such that $q^jp^{-k}\in[1,p]$. Since the set of such points is dense, we come to the conclusion that I does not depend on t_1 and, by symmetry, on any t_m.

Alternatively, we can use the p-expansion $I(t_1,\ldots,t_5)=\sum_{n=0}^\infty I_n(t_1,\ldots,t_5)p^n$ and check validity of the equalities $I_n(qt_1,\ldots,t_5)=I_n(t_1,\ldots,t_5)$ termwise. The coefficients I_n are analytical in parameters near the points $t_m=0$ (the constraints on the absolute values of parameters from below appear from the requirement of convergency of the p-expansion). Therefore, we can simply iterate the dilations

$t_1 \to qt_1$ until reaching the limiting point. As a result, I_n and the integral I itself do not depend on t_1 and, consequently, on all t_m.

We conclude thus that the integral I is a constant depending only on p and q. In order to find its value, which is given by the right-hand side of (3.3), it is sufficient to consider the limit in parameters $t_1t_2 \to 1$. In this case two poles approach the contour of integration, and it is necessary to deform this contour so that it crosses over these poles. Then it appears that in the limit $t_1t_2 \to 1$ only the residues of these poles have finite values and the integral itself vanishes. (This procedure is described in more detail below.) After proving the integration formula in a restricted domain of parameter values, it can be continued analytically to the domain permissible by the contour of integration C. □

There are many ways to degenerate the elliptic beta integral. In the simplest case it is necessary to substitute $t_6 = pq/t_1 \ldots t_5$, use the reflection formula for $\Gamma(z; p, q)$, and take the limit $p \to 0$. After this, the elliptic beta integral degenerates to the "trigonometric" q-beta-integral of Rahman [39] (which is connected with the integral representation for a ${}_8\varphi_7$-series [40]):

$$\frac{(q;q)_\infty}{4\pi i}\int_{\mathbb{T}} \frac{(z^2;q)_\infty(z^{-2};q)_\infty(Az;q)_\infty(Az^{-1};q)_\infty}{\prod_{m=1}^5(t_mz;q)_\infty(t_mz^{-1};q)_\infty}\frac{dz}{z}$$
$$= \frac{\prod_{m=1}^5(At_m^{-1};q)_\infty}{\prod_{1\le m<s\le 5}(t_mt_s;q)_\infty},$$

where $A = \prod_{m=1}^5 t_m$, $|t_m| < 1$. Further simplification of this equality by taking the limit $t_5 \to 0$ leads to the famous Askey–Wilson integral

$$\frac{(q;q)_\infty}{4\pi i}\int_{\mathbb{T}} \frac{(z^2;q)_\infty(z^{-2};q)_\infty}{\prod_{m=1}^4(t_mz;q)_\infty(t_mz^{-1};q)_\infty}\frac{dz}{z} = \frac{(t_1t_2t_3t_4;q)_\infty}{\prod_{1\le m<s\le 4}(t_mt_s;q)_\infty},$$

which serves as a measure for the Askey–Wilson polynomials [41] – the most general orthogonal polynomials obeying the classical properties. The first proof of formula (3.1) was based on an elliptic extension of the Askey approach [42] to computation of the Rahman integral. Here we presented the proof obtained in the paper [43], which generalizes the method of computation of the Askey–Wilson integral from [44].

If we express in the given q-beta integrals infinite products $(a;q)_\infty$ in term of the Jackson q-gamma function and pass to the limit $q \to 1$, then we obtain "rational" beta integrals over non-compact contours containing the ordinary Euler gamma function [1]. Their further simplification by special choices of parameters leads to the classical Euler beta-integral

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

The elliptic beta integral (3.1) gives thus the most general (from known ones) exact univariate integration formula including into itself the Euler beta integral as a particular case.

Corollary 2. *The following Frenkel and Turaev summation formula* [8] *is true:*

$$\sum_{n=0}^N \frac{\theta(t_5^2q^{2n};p)}{\theta(t_5^2;p)}\prod_{m=0}^5 \frac{\theta(t_mt_5)_n}{\theta(qt_m^{-1}t_5)_n}q^n = \frac{\theta(qt_5^2, \frac{q}{t_1t_2}, \frac{q}{t_1t_3}, \frac{q}{t_2t_3})_N}{\theta(\frac{q}{t_1t_2t_3t_5}, \frac{qt_5}{t_1}, \frac{qt_5}{t_2}, \frac{qt_5}{t_3})_N}, \tag{3.8}$$

where $t_4t_5 = q^{-N}$, $\prod_{m=0}^5 t_m = q$, *and the compact notation*

$$\theta(t_1,\ldots,t_k)_n := \prod_{j=1}^k \theta(t_j)_n$$

is used for products of the elliptic Pochhammer symbols

$$\theta(t)_n = \prod_{j=0}^{n-1} \theta(tq^j;p) = \frac{\Gamma(tq^n;p,q)}{\Gamma(t;p,q)}.$$

Proof. We replace in integral (3.1) $\mathbb{T}$ by a contour C separating sequences of the poles $z = t_jq^ap^b$, $j = 1,\ldots,6$, $a,b = 0,1,\ldots$ converging to zero from their $z \to 1/z$ reciprocals going to infinity. This permits us to remove the constraints $|t_j| < 1$ without changing the right-hand side of (3.1). Substitute now $t_6 = pq/A$, $A = \prod_{m=1}^5 t_m$, and suppose that $|t_m| < 1$, $m = 1,\ldots,4$, $|pt_5| < 1 < |t_5|$, $|pq| < |A|$, and that the arguments of t_m, $m = 1,\ldots,5$, and p,q are linearly independent over $\mathbb{Z}$. Then the following equality is true [32]:

$$\kappa \int_C \Delta_E(z,\underline{t})\frac{dz}{z} = \kappa \int_{\mathbb{T}} \Delta_E(z,\underline{t})\frac{dz}{z} + c_0(\underline{t}) \sum_{|t_5q^n|>1,\, n\geq 0} \nu_n(\underline{t}), \tag{3.9}$$

where $\Delta_E(z,\underline{t}) = \prod_{m=1}^5 \Gamma(t_mz^{\pm 1};p,q)/\Gamma(z^{\pm 2}, Az^{\pm 1};p,q)$ and

$$c_0(\underline{t}) = \frac{\prod_{m=1}^4 \Gamma(t_mt_5^{\pm 1};p,q)}{\Gamma(t_5^{-2}, At_5^{\pm 1};p,q)}, \qquad \nu_n(\underline{t}) = \frac{\theta(t_5^2q^{2n};p)}{\theta(t_5^2;p)} \prod_{m=0}^5 \frac{\theta(t_mt_5)_n}{\theta(qt_m^{-1}t_5)_n}\, q^n.$$

Here we introduced the new parameter t_0 with the help of the relation $\prod_{m=0}^5 t_m = q$. The multiplier κ is absent in the coefficient c_0 because of the relation $\lim_{z\to 1}(1 - z)\Gamma(z;p,q) = 1/(p;p)_\infty(q;q)_\infty$ and due to doubling of the number of residues (the latter follows from the symmetry of the kernel $z \to z^{-1}$).

In the limit $t_5t_4 \to q^{-N}$, $N = 0,1,\ldots$, the integral on the left-hand side of (3.9) (coinciding with (3.1)) and the multiplier $c_0(\underline{t})$ in front of the sum of residues in the right-hand side diverge. But the integral over the unit circle $\mathbb{T}$ on the right-hand side remains finite. After dividing all the terms by $c_0(\underline{t})$ and passing to the limiting equality, we obtain the summation formula (3.8), which was obtained for the first time in [8] by a completely different method. □

Other proofs of formula (3.8) are given in [3, 45, 46, 47, 48, 49]. For $p \to 0$ and fixed parameters, equality (3.8) is reduced to the Jackson sum for the terminating very-well-poised balanced q-hypergeometric series ${}_8\varphi_7$ [1]. The left-hand side of formula (3.8) represents thus an elliptic analogue of this q-series.

Using the modified elliptic gamma function, it is not difficult to construct the modified elliptic beta integral [38], one of the base variables for which can lie on the unit circle, say, $|q| = 1$.

Theorem 3. *Let* $Im(\omega_1/\omega_2) \geq 0$, $Im(\omega_3/\omega_1) > 0$, $Im(\omega_3/\omega_2) > 0$, *and six parameters* $g_j \in \mathbb{C}$, $j = 1,\ldots,6$, *satisfy the restrictions* $Im(g_j/\omega_3) < 0$ *and* $\sum_{j=1}^6 g_j = \sum_{k=1}^3 \omega_k$. *Then*

$$\tilde{\kappa} \int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{j=1}^6 G(g_j \pm u;\omega)}{G(\pm 2u;\omega)} \frac{du}{\omega_2} = \prod_{1\leq j<m\leq 6} G(g_j+g_m;\omega), \tag{3.10}$$

where

$$\tilde{\kappa} = -\frac{(q;q)_\infty (p;p)_\infty (r;r)_\infty}{2(\tilde{q};\tilde{q})_\infty}.$$

Here the integration goes along the cut with the end points $-\omega_3/2$ and $\omega_3/2$ and the convention $G(a \pm b;\omega) \equiv G(a+b;\omega)G(a-b;\omega)$ is used.

Proof. We substitute relation (2.21) into the left-hand side of (3.10) and obtain

$$\tilde{\kappa} e^{\pi i a/3} \int_{-\omega_3/2}^{\omega_3/2} \frac{\prod_{j=1}^6 \Gamma(e^{-2\pi i \frac{g_j \pm u}{\omega_3}}; \tilde{r}, \tilde{p})}{\Gamma(e^{\mp 4\pi i \frac{u}{\omega_3}}; \tilde{r}, \tilde{p})} \frac{du}{\omega_2}. \tag{3.11}$$

where $a = 2B_{3,3}(0|\omega) - 2\sum_{j=1}^6 B_{3,3}(g_j|\omega)$. Taken restrictions on the parameters permit us to use formula (3.1) with the substitutions

$$z \to e^{\frac{2\pi i}{\omega_3}u}, \quad t_j \to e^{-\frac{2\pi i}{\omega_3}g_j}, \quad p \to e^{-2\pi i \frac{\omega_1}{\omega_3}}, \quad q \to e^{-2\pi i \frac{\omega_2}{\omega_3}},$$

which yields for (3.11)

$$\frac{2\tilde{\kappa}\omega_3\omega_2^{-1} e^{\pi i a/3}}{(\tilde{r};\tilde{r})_\infty (\tilde{p};\tilde{p})_\infty} \prod_{1\le j<m\le 6} \Gamma(e^{-2\pi i \frac{g_j+g_m}{\omega_3}}; \tilde{r}, \tilde{p})$$
$$= \frac{2\tilde{\kappa}\omega_3\omega_2^{-1} e^{\pi i (a+b)/3}}{(\tilde{r};\tilde{r})_\infty (\tilde{p};\tilde{p})_\infty} \prod_{1\le j<m\le 6} G(g_j+g_m;\boldsymbol{\omega}),$$

where $b = \sum_{1\le j<m\le 6} B_{3,3}(g_j+g_m|\omega)$. A straightforward computation shows that

$$a+b = \frac{1}{4}\left(\sum_{k=1}^3 \omega_k\right)\left(\sum_{k=1}^3 \omega_k^{-1}\right).$$

Therefore for the choice

$$\tilde{\kappa}^{-1} = \frac{2\omega_3 e^{\frac{\pi i}{12}(\sum_{k=1}^3 \omega_k)(\sum_{k=1}^3 \omega_k^{-1})}}{\omega_2(\tilde{r};\tilde{r})_\infty (\tilde{p};\tilde{p})_\infty}$$

we obtain the needed result. After application of the modular transformation law for the Dedekind function

$$e^{-\frac{\pi i}{12\tau}} \left(e^{-2\pi i/\tau}; e^{-2\pi i/\tau}\right)_\infty = (-i\tau)^{1/2} e^{\frac{\pi i \tau}{12}} \left(e^{2\pi i \tau}; e^{2\pi i \tau}\right)_\infty \tag{3.12}$$

to infinite products entering definition of $\tilde{\kappa}$, we obtain

$$\tilde{\kappa}^{-1} = -2\sqrt{\frac{\omega_1}{i\omega_2}} \frac{e^{\frac{\pi i}{12}(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1})}}{(r;r)_\infty (p;p)_\infty}.$$

One more application of the relation (3.12) permits us to replace the exponential function by a ratio of infinite products, and this leads to the needed form of κ. □

If we take the limit $\mathrm{Im}(\omega_3) \to \infty$ in such a way that $p, r \to 0$, then the modified elliptic beta integral reduces to a q-beta integral of the Mellin–Barnes type. More precisely, for $\omega_{1,2}$ such that $\mathrm{Im}(\omega_1/\omega_2) \geq 0$ and $\mathrm{Re}(\omega_1/\omega_2) > 0$, we substitute $g_6 = \sum_{k=1}^3 \omega_k - \mathcal{A}$, where $\mathcal{A} = \sum_{j=1}^5 g_j$ and apply the inversion formula for $G(u;\omega)$. After that we set $\omega_3 = it\omega_2$, $t \to +\infty$, and obtain formally

$$\int_{\mathbb{L}} \frac{\prod_{j=1}^5 \gamma(g_j \pm u;\omega)}{\gamma(\pm 2u, \mathcal{A} \pm u;\omega)} \frac{du}{\omega_2} = -2\frac{(\tilde{q};\tilde{q})_\infty}{(q;q)_\infty} \frac{\prod_{1\le j<m\le 5}\gamma(g_j+g_m;\omega)}{\prod_{j=1}^5 \gamma(\mathcal{A}-g_j;\omega)}, \tag{3.13}$$

where $\gamma(u;\omega_1,\omega_2)$ denotes the modified q-gamma function, and the integration is taken along the line $\mathbb{L}\equiv i\omega_2\mathbb{R}$. This result is true provided the parameters satisfy the constraints $\mathrm{Re}(g_j/\omega_2)>0$ and $\mathrm{Re}((\mathcal{A}-\omega_1)/\omega_2)<1$. This integration formula represents a "hyperbolic" analogue of the Rahman integral; it was proved for the first time by Stokman in [50]. Because of the non-compactness of the integration contour, the described method of derivation of (3.13) is rigorous under the condition of uniform convergence of the function $G(u;\omega)$ to $\gamma(u;\omega_1,\omega_2)$, which follows from the results obtained by Rains in [28]. One can establish also formula (3.13) by the method, which was used above for proving the elliptic beta integral. The limit $q\to 1$ leads to the same rational beta integral as the "trigonometric" Rahman integral.

Summarizing the consideration of the present section, we see that the elliptic beta integral includes into itself the whole hierarchy of exactly computable integrals: two types of the q-beta integrals, the rational class of beta integrals, whose kernels are expressed in terms of the Euler gamma function, and the classical Euler beta integral. This scheme reflects the general picture of degenerations of the elliptic hypergeometric functions which was rigorously considered in [28].

4. General elliptic hypergeometric series and integrals

In the papers [10] and [11], the author has proposed definitions of general elliptic hypergeometric series and integrals, which will be considered in this section.

4.1. An elliptic analogue of the Meijer function.

Univariate contour integrals $\int_C \Delta(u)du$ are called the elliptic hypergeometric integrals, if the meromorphic function $\Delta(u)$ satisfies the following system of three equations

$$\Delta(u+\omega_k)=h_k(u)\Delta(u),\quad k=1,2,3, \tag{4.1}$$

where $\omega_{1,2,3}\in\mathbb{C}$ are linearly independent over $\mathbb{Z}$ parameters, and $h_k(u)$ are some elliptic functions with the periods ω_k, ω_{k+1} (we set $\omega_{k+3}=\omega_k$).

The general elliptic function of the order s with the periods ω_2 and ω_3 can be represented in the form (see the Appendix)

$$h_1(u)=y_1\prod_{j=1}^{s}\frac{\theta(t_je^{2\pi iu/\omega_2};p)}{\theta(w_je^{2\pi iu/\omega_2};p)},$$

where y_1 is an arbitrary constant and t_j,w_j are some parameters satisfying the balancing condition $\prod_{j=1}^s t_j=\prod_{j=1}^s w_j$ (we remind that $p=e^{2\pi i\omega_3/\omega_2}$). Using properties of the function $\Gamma(z;p,q)$, it is not difficult to build the general solution of the $k=1$ equation in (4.1) for $|q|<1$:

$$\Delta(u)=y_1^{u/\omega_1}\varphi(u)\prod_{j=1}^{s}\frac{\Gamma(t_jz;p,q)}{\Gamma(w_jz;p,q)},\qquad z=e^{2\pi iu/\omega_2},$$

where $\varphi(u+\omega_1)=\varphi(u)$ is an arbitrary periodic function. So, if we would restrict ourselves to a single equation for $\Delta(u)$, then our definition of the integrals would be highly non-unique.

An arbitrary elliptic function of the order ℓ with the periods ω_1 and ω_3 has the form

$$h_2(u) = y_2 \prod_{j=1}^{\ell} \frac{\theta(\tilde{t}_j e^{-2\pi i u/\omega_1}; r)}{\theta(\tilde{w}_j e^{-2\pi i u/\omega_1}; r)},$$

where $|r| < 1$, y_2 is an arbitrary constant, and the parameters satisfy the constraint $\prod_{j=1}^{\ell} \tilde{t}_j = \prod_{j=1}^{\ell} \tilde{w}_j$. The $k = 2$ equation from (4.1) serves now as a constraint for the function $\varphi(u)$. It is easy to prove that the common solution of these two equations for $|q| < 1$ has the form

$$\Delta(u) = \phi(u) \prod_{j=1}^{s} \frac{\Gamma(t_j e^{2\pi i u/\omega_2}; p, q)}{\Gamma(w_j e^{2\pi i u/\omega_2}; p, q)} \prod_{j=1}^{\ell} \frac{\Gamma(\tilde{t}_j e^{-2\pi i u/\omega_1}; \tilde{q}, r)}{\Gamma(\tilde{w}_j e^{-2\pi i u/\omega_1}; \tilde{q}, r)}, \tag{4.2}$$

where $\phi(u)$ is an arbitrary function satisfying the equations $\phi(u+\omega_1) = y_1\phi(u)$ and $\phi(u + \omega_2) = y_2\phi(u)$. $\phi(u)$ is thus a meromorphic theta function with the special quasiperiodicity multipliers the general form of which is easily established (see the Appendix):

$$\begin{aligned}\phi(u) &= e^{cu+d} \prod_{k=1}^{m} \frac{\theta(a_k e^{2\pi i u/\omega_2}; q)}{\theta(b_k e^{2\pi i u/\omega_2}; q)} \\ &= e^{cu+d} \prod_{k=1}^{m} \frac{\Gamma(pa_k e^{2\pi i u/\omega_2}, b_k e^{2\pi i u/\omega_2}; p, q)}{\Gamma(a_k e^{2\pi i u/\omega_2}, pb_k e^{2\pi i u/\omega_2}; p, q)},\end{aligned}$$

where m is an arbitrary integer, the parameter d is arbitrary, and the parameters a_k, b_k, c are connected with y_1 and y_2 by the relations $y_2 = e^{c\omega_2}$ and $y_1 = e^{c\omega_1} \prod_{k=1}^{m} b_k a_k^{-1}$.

Due to the representation in terms of the elliptic gamma functions, the function $\phi(u)$ can be reduced to the pure exponential factor by the replacement of s in $\Delta(u)$ by $s + 2m$ and the choice of parameters $t_{s+k} = pa_k, t_{s+m+k} = b_k, w_{s+k} = a_k, w_{s+m+k} = pb_k$, $k = 1, \ldots, m$, which does not violate the balancing condition $\prod_{j=1}^{s+2m} t_j = \prod_{j=1}^{s+2m} w_j$. Since s was arbitrary from the very beginning, we can set $\phi(u) = e^{cu+d}$ without loss of generality.

As a result, two equations determine already the kernel $\Delta(u)$, i.e. for the elliptic function $h_3(u)$ we automatically obtain

$$h_3(u) = e^{c\omega_3} \prod_{j=1}^{s} \frac{\theta(t_j e^{2\pi i u/\omega_2}; q)}{\theta(w_j e^{2\pi i u/\omega_2}; q)} \prod_{j=1}^{\ell} \frac{\theta(r^{-1}\tilde{w}_j e^{-2\pi i u/\omega_1}; \tilde{q})}{\theta(r^{-1}\tilde{t}_j e^{-2\pi i u/\omega_1}; \tilde{q})}.$$

To summarize, for $|q| < 1$ the most general elliptic hypergeometric integral has the form [11, 31]

$$\int_C e^{cu+d} \prod_{j=1}^{s} \frac{\Gamma(t_j e^{2\pi i u/\omega_2}; p, q)}{\Gamma(w_j e^{2\pi i u/\omega_2}; p, q)} \prod_{j=1}^{\ell} \frac{\Gamma(\tilde{t}_j e^{-2\pi i u/\omega_1}; \tilde{q}, r)}{\Gamma(\tilde{w}_j e^{-2\pi i u/\omega_1}; \tilde{q}, r)}\, du \tag{4.3}$$

with two balancing conditions for the parameters indicated above and some integration contour C.

Consider the definition of integrals for $|q| = 1$. It appears that now even the function $h_2(u)$ cannot be arbitrary. In this case it is necessary to take $\ell = s$ and choose the parameters $t_j, \tilde{t}_j$, and $w_j, \tilde{w}_j$ in such a way that all Γ-functions are combined to the modified elliptic gamma functions $G(u;\omega)$ (it is in this way that this function was built in [11]). This leads to the integrals of the form

$$\int e^{cu+d}\prod_{j=1}^{s}\frac{G(u+g_j;\omega)}{G(u+v_j;\omega)}\,du, \tag{4.4}$$

where the parameters g_j and v_j satisfy the balancing condition $\sum_{j=1}^{s}(g_j-v_j)=0$ together with the relations $t_j=e^{2\pi i g_j/\omega_2}$, $w_j=e^{2\pi i v_j/\omega_2}$, $\tilde{t}_j=re^{-2\pi i g_j/\omega_1}$, $\tilde{w}_j=re^{-2\pi i v_j/\omega_1}$, and $y_{1,2}=e^{c\omega_{1,2}}$.

The case $|q|>1$ appears to be equivalent to the case $|q|<1$ after a change of parameters and leads to the integrals

$$\int_C e^{cu+d}\prod_{j=1}^{s}\frac{\Gamma(q^{-1}w_je^{2\pi iu/\omega_2};p,q^{-1})}{\Gamma(q^{-1}t_je^{2\pi iu/\omega_2};p,q^{-1})}\prod_{j=1}^{\ell}\frac{\Gamma(\tilde{q}^{-1}\tilde{w}_je^{-2\pi iu/\omega_1};\tilde{q}^{-1},r)}{\Gamma(\tilde{q}^{-1}\tilde{t}_je^{-2\pi iu/\omega_1};\tilde{q}^{-1},r)}\,du. \tag{4.5}$$

Functions (4.3), (4.4), and (4.5) can be called as elliptic analogues of the Meijer function, because for some particular choice of parameters and of the integration contour C they degenerate to that function [12]. During this degeneration procedure, at the intermediate steps there appear various q-analogues of the Meijer function, including the cases considered in [51]. The more general theta hypergeometric analogues of the Meijer function, for which the kernels $\Delta(u)$ satisfy the system of equations (4.1) with $h_k(u)$ given by arbitrary meromorphic theta functions, are built in [11]; we do not consider them here.

4.2. **Well poised and very-well poised integrals.**

We consider integrals (4.3) with $\ell=c=d=0$ and replace the integration variable by $z=e^{2\pi iu/\omega_2}$. Until now we did not fix the integration contour C. Let us choose it as the unit circle $\mathbb{T}$ oriented counterclockwise. As a result, we obtain integrals of the form

$$\int_{\mathbb{T}}\Delta(z)\frac{dz}{z}, \qquad \Delta(z)=\prod_{k=1}^{s}\frac{\Gamma(t_kz;p,q)}{\Gamma(w_kz;p,q)}$$

with $\prod_{k=1}^{s}t_k=\prod_{k=1}^{s}w_k$. In the case when the conditions $w_kt_k=pq$, $k=1,\dots,s$ are satisfied, the integrals take the form

$$\int_{\mathbb{T}}\Delta^{(s)}(z)\frac{dz}{z}, \qquad \Delta^{(s)}(z)=\prod_{k=1}^{s}\Gamma(t_kz,t_k/z;p,q),$$

and are called well poised integrals. The balancing condition for them takes the form $\prod_{k=1}^{s}t_k^2=(pq)^s$ or $\prod_{k=1}^{s}t_k=\mu(pq)^{s/2}$ with the ambiguity in the sign choice $\mu=\pm1$. The reflection formula $\Gamma(a,b;p,q)=1$, $ab=pq$, shows that the choice of parameters $t_jt_k=pq$ plays an essential role, since it reduces the number of parameters in $\Delta^{(s)}(z)$. In particular, for $t_k^2=pq$ the variable t_k drops out of formulae completely. The function

$$h^{(p)}(z):=\frac{\Delta^{(s)}(qz)}{\Delta^{(s)}(z)}=\prod_{k=1}^{s}\frac{\theta(t_kz;p)}{\theta(pqz/t_k;p)}$$

is evidently p-elliptic, that is $h^{(p)}(pz) = h^{(p)}(z)$. Denoting $u_k := t_k z$, $v_k := pqz/t_k$, $\lambda := pqz^2$, we can rewrite $h^{(p)}(z)$ as

$$h^{(p)}(u_1, \ldots, u_s; \lambda) = \prod_{k=1}^{s} \frac{\theta(u_k; p)}{\theta(v_k; p)} \tag{4.6}$$

with the conditions of well poisedness $u_k v_k = \lambda$, $k = 1, \ldots, s$, and balancing $\prod_{k=1}^{s} u_k = \mu \lambda^{s/2}$.

Let us consider all possible p-shifts of the parameters $u_1, \ldots, u_s$ and λ:

$$u_k \to p^{n_k} u_k, \qquad \lambda \to p^N \lambda, \quad n_k, N \in \mathbb{Z},$$

and require that $h^{(p)}$ is invariant under the maximally possible group of these transformations. The balancing condition leads to the constraint $\sum_{k=1}^{s} n_k = sN/2$. For $N = 0$ it is easy to check that $h^{(p)}(\ldots, pu_a, \ldots, p^{-1}u_b, \ldots; \lambda) = h^{(p)}(u_1, \ldots, u_s; \lambda)$, i.e. $h^{(p)}$ is an elliptic function of all its parameters. The transformations with $N \neq 0$ are more complicated and depend on the parity of the variable s. For odd s the integer N must be even. The full symmetry group is generated then by the transformations with $N = 0$ and, say, $n_1 = s$, $n_k = 0, k \neq 1$, and $N = 2$ yielding

$$\frac{h^{(p)}(p^s u_1, u_2, \ldots, u_s; p^2\lambda)}{h^{(p)}(u_1, \ldots, u_s; \lambda)} = \frac{\lambda^s}{\prod_{k=1}^{s} u_k^2} = 1,$$

that is the value of μ is not fixed.

As to the even values $s = 2m$, in this case there are no constraints on N, and it is sufficient to consider the transformation corresponding to the choice $n_1 = m$, $N = 1$, with all other $n_k = 0$. Then we have

$$\frac{h^{(p)}(p^m u_1, u_2, \ldots, u_{2m}; p\lambda)}{h^{(p)}(u_1, \ldots, u_{2m}; \lambda)} = \frac{\lambda^m}{\prod_{k=1}^{2m} u_k} = \mu.$$

Requiring this transformation to be a symmetry, we fix uniquely the balancing condition, $\mu = 1$.

It is not difficult to check that for any $\mu = \pm 1$ the following equality is true

$$\Delta^{(s)}(p^i q^j z)\Delta^{(s)}(z) = \Delta^{(s)}(p^i z)\Delta^{(s)}(q^j z)$$

for all $i, j \in \mathbb{Z}$. Vice versa, from this condition one can derive the balancing condition with $\mu = \pm 1$. Passing to the limits $z \to \pm p^{-i/2} q^{-j/2}$ and using the symmetry $\Delta^{(s)}(z) = \Delta^{(s)}(z^{-1})$, we obtain $\Delta^{(s)}(\pm p^{i/2} q^{j/2})^2 = \Delta^{(s)}(\pm p^{i/2} q^{-j/2})^2$. The straightforward computation yields

$$\Delta^{(s)}(\pm p^{i/2} q^{j/2}) = ((\mp 1)^s \mu)^{ij}\, \Delta^{(s)}(\pm p^{i/2} q^{-j/2}).$$

For even values $s = 2m$ and $\mu = 1$, we obtain

$$\Delta^{(2m)}(\pm p^{i/2} q^{j/2}) = \Delta^{(2m)}(\pm p^{i/2} q^{-j/2}), \tag{4.7}$$

provided the functions on both sides are well defined. The latter requirement is satisfied provided we do not hit the poles, that is $t_r \neq \pm p^{a/2} q^{b/2}$, $a, b \in \mathbb{Z}$, for all r.

We call the integrals $\int_{\mathbb{T}} \Delta(z) dz/z$ very-well poised, if their integration kernels have the form

$$\Delta_{vwp}^{(2m+6)}(z) = \frac{\prod_{k=1}^{2m+6} \Gamma(t_k z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} = \theta(z^2; p)\theta(z^{-2}; q) \prod_{k=1}^{2m+6} \Gamma(t_k z^{\pm 1}; p, q).$$

This kernel can be obtained from $\Delta^{(2m+14)}(z)$ by restriction of the parameters

$$t_{2m+7},\ldots,t_{2m+14}=(\pm(pq)^{1/2},\pm p^{1/2}q,\pm pq^{1/2},\pm pq);$$

this follows from the reflection formula and the argument duplication formula for the function $\Gamma(z;p,q)$:

$$\Gamma(z^2;p,q)=\Gamma(\pm z,\pm q^{1/2}z,\pm p^{1/2}z,\pm(pq)^{1/2}z;p,q).$$

The balancing condition takes now the form

$$\prod_{k=1}^{2m+6}t_k=\mu(pq)^{m+1}, \tag{4.8}$$

where we count the sign choice $\mu=1$ as canonical, since it leads to additional symmetries. The kernel $\Delta_{vwp}^{(2m+6)}(z)$ can be obtained also from $\Delta^{(2m+12)}(z)$ by imposing the constraints $t_{2m+7},\ldots,t_{2m+12}=(\pm p^{1/2}q,\pm pq^{1/2},\pm pq)$, since the choice $t_k^2=pq$ simply removes the corresponding Γ-factors. However, we shall not use such a reduction, because it changes the sign in the balancing condition: $\prod_{k=1}^{2m+6}t_k=-\mu(pq)^{m+1}$.

In the following we shall be studying the very-well poised elliptic hypergeometric integrals of the form

$$I^{(m)}(t_1,\ldots,t_{2m+6})=\kappa\int_{\mathbb{T}}\frac{\prod_{k=1}^{2m+6}\Gamma(t_kz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z}$$

with the "correct" balancing condition $\prod_{k=1}^{2m+6}t_k=(pq)^{m+1}$. These integrals represent elliptic analogues of the plain hypergeometric functions ${}_{m+1}F_m$. In particular, for $m=0$ we obtain the elliptic beta integral. The constraints $t_{2m+5}=(pq)^{1/2}$, $t_{2m+6}=-(pq)^{1/2}$ reduce these integrals to $I^{(m-1)}(t_1,\ldots,t_{2m+4})$ with the balancing condition $\prod_{k=1}^{2m+4}t_k=-(pq)^m$. Appearance of the "–" sign on the right-hand side simply indicates that these integrals should be considered as some generalizations of ${}_{m+1}F_m$-functions, and not of ${}_mF_{m-1}$. For example, such a choice in the elliptic beta integral yields after taking into account of the relation $\Gamma(-pq;p,q)=2(-p;p)_\infty(-q;q)_\infty$ the following:

$$\frac{1}{2\pi i}\int_C\frac{\prod_{k=1}^4\Gamma(t_kz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z}$$
$$=4(p^2;p^2)_\infty(q^2;q^2)_\infty\prod_{1\le j<k\le4}\Gamma(t_jt_k;p,q)\prod_{k=1}^4\Gamma(pqt_k^2;p^2,q^2),$$

where $\prod_{k=1}^4t_k=-1$ and the contour C separates sequences of poles converging to zero from those going to infinity. For $p\to0$ we obtain a special case of the Askey–Wilson integral.

The multiplier appearing in integrals' kernel from the very-well poisedness constraints for parameters leads to an interesting property [34]

$$\Delta_{vwp}^{(2m+6)}(p^{i/2}q^{j/2})=-\mu^{ij}\Delta_{vwp}^{(2m+6)}(p^{i/2}q^{-j/2}), \tag{4.9}$$

that is to the "–" sign in the right-hand side for the canonical choice $\mu=1$, which is sharply distinct from (4.7). Indeed, for $i=0$ or $j=0$ we have zeros in both parts of the equality, and in other cases we obtain

$$\frac{\lim_{z\to p^{i/2}q^{j/2}}\Gamma_{p,q}^{-1}(z^2,z^{-2})}{\lim_{z\to p^{i/2}q^{-j/2}}\Gamma_{p,q}^{-1}(z^2,z^{-2})} = \frac{\theta(p^{-i}q^{-j};q)\theta(p^iq^j;p)}{\theta(p^{-i}q^j;q)\theta(p^iq^{-j};p)} = -(pq)^{-2ij},$$

which together with the described properties of $\Delta^{(2m+6)}(z)$ yields the presented formula. For $i,j\neq 0$ there appears a non-commutativity of two limits:

$$\frac{\lim_{z\to p^{i/2}q^{j/2}}\Delta_{vwp}^{(2m+6)}(z)}{\lim_{z\to p^{i/2}q^{-j/2}}\Delta_{vwp}^{(2m+6)}(z)} = -\mu^{ij}$$
$$\neq \lim_{t_{2m+7},\ldots,t_{2m+14}\to(\pm(pq)^{1/2},\pm p^{1/2}q,\pm pq^{1/2},\pm pq)} \frac{\Delta^{(2m+14)}(p^{i/2}q^{j/2})}{\Delta^{(2m+14)}(p^{i/2}q^{-j/2})} = \mu^{ij},$$

although both of them are well defined. The reason for such a contradiction consists in the use in the very-well poisedness condition of the "forbidden" values of parameters leading to poles and zeros of $\Delta^{(2m+14)}(z)$ for $z=p^{i/2}q^{j/2}$.

As shown in [34], the product $\prod_{1\leq j\leq k\leq m}(t_jt_k;p,q)_\infty I^{(m)}(\underline{t})$ is a holomorphic function of parameters $t_j\in\mathbb{C}^*$. Additionally, the "–" sign on the right-hand side of (4.9) guarantees that the integral $I^{(m)}(\underline{t})$ is holomorphic in the points $t_k^2=p^{-a}q^{-b}$, $a,b=0,1,\ldots$. If some of the parameters take the forbidden values $\pm p^{a/2}q^{b/2}$, $a,b\in\mathbb{Z}$, then the latter property disappears.

4.3. **Series.**

According to the general definition [10], formal series $\sum_{n\in\mathbb{Z}}c_n$ are called the elliptic hypergeometric series, if the ratio of neighbouring coefficients c_{n+1}/c_n is an elliptic function of $n\in\mathbb{C}$. This definition lies in the stream of ideas of Pochhammer and Horn which are used for building the plain and q-hypergeometric series [2].

As we saw already on the example of the Frenkel–Turaev sum, the elliptic hypergeometric series appear as sums of residues of certain sequences of poles of the elliptic hypergeometric integrals' kernels. Indeed, let us consider the poles of the integrand in (4.3) located at the points $u=a+\omega_1 n$, $n=0,1,\ldots$, for some parameter a, and denote residues of these poles as c_n. For $u\to a+\omega_1 n$, we have $\Delta(u)\to c_n/(u-a-\omega_1 n)+O(1)$. Now it is not difficult to notice that the ratio

$$\lim_{u\to a+\omega_1 n}\frac{\Delta(u+\omega_1)}{\Delta(u)} = \frac{c_{n+1}}{c_n} = \lim_{u\to a+\omega_1 n} h_1(u) = h_1(a+\omega_1 n)$$

is an elliptic function of n with the periods ω_2/ω_1 and ω_3/ω_1, which demonstrates the general connection between the integrals and series.

An arbitrary elliptic function $h(n)$ of the order $s+1$ with the periods ω_2/ω_1 and ω_3/ω_1 has the form

$$h(n) = y\prod_{k=1}^{s+1}\frac{\theta(t_kq^n;p)}{\theta(w_kq^u;p)}, \tag{4.10}$$

where $\prod_{k=1}^{s+1}t_k=\prod_{k=1}^{s+1}w_k$. We define the elliptic Pochhammer symbol $\theta(t)_n$ as the solution of the recurrence relation $c_{n+1}=\theta(tq^n;p)\,c_n$ with the initial condition $c_0=1$:

$$\theta(t)_n = \begin{cases} \prod_{j=0}^{n-1}\theta(tq^j;p), & n>0 \\ \displaystyle\prod_{j=1}^{-n}\frac{1}{\theta(tq^{-j};p)}, & n<0. \end{cases}$$

Then it is easy to deduce the explicit form of the formal bilateral elliptic hypergeometric series

$$ {}_{s+1}G_{s+1}\left(\begin{matrix} t_1,\dots,t_{s+1} \\ w_1,\dots,w_{s+1}\end{matrix};q,p;y\right) = \sum_{n\in\mathbb{Z}}\prod_{k=1}^{s+1}\frac{\theta(t_k)_n}{\theta(w_k)_n}\,y^n $$

with the normalization of the zeroth coefficient $c_0 = 1$. Choosing $w_{s+1} = q$ and replacing $t_{s+1} \to t_0$, we obtain the unilateral series

$$ {}_{s+1}E_s\left(\begin{matrix} t_0,t_1,\dots,t_s \\ w_1,\dots,w_s\end{matrix};q,p;y\right) = \sum_{n=0}^{\infty}\frac{\theta(t_0,t_1,\dots,t_s)_n}{\theta(q,w_1,\dots,w_s)_n}\,y^n. \qquad (4.11) $$

For fixed t_j and w_j, in the limit $p \to 0$ we have $\theta(t)_n \to (t;q)_n = (1-t)(1-qt)\dots(1-q^{n-1}t)$, and the function ${}_{s+1}E_s$ reduces to the q-hypergeometric series [3]

$$ {}_{s+1}\varphi_s\left(\begin{matrix} t_0,t_1,\dots,t_s \\ w_1,\dots,w_s\end{matrix};q;y\right) = \sum_{n=0}^{\infty}\frac{(t_0;q)_n\dots(t_s;q)_n}{(q;q)_n(w_1;q)_n\dots(w_s;q)_n}\,y^n $$

with the condition $\prod_{j=0}^{s} t_j = q\prod_{j=1}^{s} w_j$. Parameterizing $t_j = q^{u_j}$ and $w_j = q^{v_j}$, in the limit $q \to 1$ we obtain the series

$$ {}_{s+1}F_s\left(\begin{matrix} u_0,u_1,\dots,u_s \\ v_1,\dots,v_s\end{matrix};y\right) = \sum_{n=0}^{\infty}\frac{(u_0)_n\dots(u_s)_n}{n!(v_1)_n\dots(v_s)_n}\,y^n, $$

where $(a)_n = a(a+1)\dots(a+n-1)$ denotes the standard Pochhammer symbol and $u_0+\ldots+u_s = 1+v_1+\ldots+v_s$. The latter constraint for the parameters is not essential, since it disappears already for the ${}_sF_{s-1}$-function obtained after taking the limit $u_s \to \infty$.

Investigation of the conditions of convergence of the infinite series (4.11) represents a serious problem and it was not solved completely to the present moment. Therefore in the applications of the E-series it is usually assumed that they terminate because of the condition $t_k = q^{-N}p^M$ for some k, where $N = 0,1,\dots,$ $M \in \mathbb{Z}$. It is worth of noting that the formal ${}_2E_1$-series does not represent a natural elliptic generalization of the Gauss hypergeometric function, because it does not obey natural analogues of many important properties of the ${}_2F_1$-function.

Series (4.11) are called well poised, if the following constraints on the parameters are satisfied $t_0q = t_1w_1 = \ldots = t_sw_s$. The balancing condition for them takes the form $t_1\cdots t_s = \pm q^{(s+1)/2}t_0^{(s-1)/2}$, and the functions $h(n)$ and ${}_{s+1}E_s$ become invariant with respect to the transformations $t_j \to pt_j$, $j = 1,\dots,s-1$, and $t_0 \to p^2t_0$ (for this it is necessary to count t_s as a dependent parameter). In the same way as in the case of integrals, for odd s and the “+” sign choice in the balancing condition there appears an additional symmetry — the functions $h(n)$ and ${}_{s+1}E_s$ become invariant with respect to the transformation $t_0 \to pt_0$ (with the compensating transformation $t_s \to p^{(s-1)/2}t_s$). The function $h(n)$ becomes thus an elliptic function of all free parameters $\log t_j$, $j = 0,\dots,s-1$, with equal periods, that is there appears some kind of “total ellipticity” [10, 31].

The next structural restriction, which is needed for building the most interesting examples of the series, looks as follows

$$ t_{s-3} = q\sqrt{t_0}, \quad t_{s-2} = -q\sqrt{t_0}, \quad t_{s-1} = q\sqrt{t_0/p}, \quad t_s = -q\sqrt{pt_0} $$

and is called the very-well poisedness condition (it is related to doubling of the argument for theta functions). In view of the importance of the very-well poised elliptic hypergeometric series, there is a special notation for them [52]:

$$\begin{aligned} {}_{s+1}V_s(t_0;t_1,\ldots,t_{s-4};q,p;y) &= \sum_{n=0}^{\infty}\frac{\theta(t_0q^{2n};p)}{\theta(t_0;p)}\prod_{m=0}^{s-4}\frac{\theta(t_m)_n}{\theta(qt_0t_m^{-1})_n}(qy)^n \\ &= {}_{s+1}E_s\left(\begin{matrix} t_0,t_1,\ldots,t_{s-4},q\sqrt{t_0},-q\sqrt{t_0},q\sqrt{t_0/p},-q\sqrt{pt_0} \\ qt_0/t_1,\ldots,qt_0/t_{s-4},\sqrt{t_0},-\sqrt{t_0},\sqrt{pt_0},-\sqrt{t_0/p}\end{matrix};q,p;-y\right), \end{aligned}$$

where $\prod_{k=1}^{s-4}t_k=\pm t_0^{(s-5)/2}q^{(s-7)/2}$ (in this balancing condition for odd s the "+" sign choice is considered as canonical). The choice $t_j=\pm\sqrt{qt_0}$ removes completely this parameter and reduces the indices of the series by 1 (this choice can change the sign in the balancing condition). For sufficiently large values of s, one can choose four parameter values in such a way that the very-well poised part in the series coefficients cancels out, and it becomes again only the well poised series. For $y=1$, this argument is dropped in the notation of ${}_{s+1}V_s$-series. In this scheme, the Frenkel–Turaev formula (3.8) yields a closed form expression for the terminating ${}_{10}V_9(t_0;t_1,\ldots,t_5;q,p)$-series.

In order to consider modular transformations it is necessary to use the parameterization $t_k=q^{u_k}$, $w_k=q^{v_k}$, $q=e^{2\pi i\sigma}$, $p=e^{2\pi i\tau}$ and replace $\theta(q^a;p)$-functions by the "elliptic numbers" $[a]=\theta_1(\sigma a|\tau)$. Suppose that the parameter y does not depend on τ. Then, it is not difficult to verify with the help of formula (A.3) that the functions $h(n)$ and, respectively, ${}_{s+1}E_s$ will be modular invariant under the restriction $\sum_{k=0}^{s}u_k^2=1+\sum_{k=1}^{s}v_k^2$ defined modulo $2\tau/\sigma^2$. For completeness of the description, we present an explicit expression for the most interesting ${}_{s+1}V_s$-series with odd s in this notation:

$${}_{2m}V_{2m-1}(q^{u_0};q^{u_1},\ldots,q^{u_{2m-5}};q,p;y)=\sum_{n=0}^{\infty}\frac{[2n+u_0]}{[u_0]}\prod_{k=0}^{2m-5}\frac{[u_k]_n}{[1+u_0-u_k]_n}y^n,$$

where $[a]_n=[a][a+1]\ldots[a+n-1]$ and the balancing condition has the form $\sum_{k=1}^{2m-5}u_k=m-4+(m-3)u_0$. This series is automatically modular invariant. In [3], the symbols ${}_{s+1}e_s$ and ${}_{s+1}v_s$ were suggested for the additive system of notation for series, but we do not use them here.

Importance of the balancing condition for the plain and q-hypergeometric functions was known for a long time, because in its presence there appear some additional identities [3]. The same is true for the notions of well poisedness and very-well poisedness. However, the corresponding constraints on the parameters were appearing in a rather ad hoc manner, and their deep meaning was missing. The elliptic hypergeometric functions clarify the origin of these old concepts. Namely, the balancing condition is connected with the condition of double periodicity of the main elliptic function used in the construction of series or integrals. The condition of well poisedness is connected with the condition of ellipticity in all parameters determining the divisor of this elliptic function. The condition of very-well poisedness is tied to the rule of doubling of the argument of theta functions. Strictly speaking all these notions are defined in fact in a self-contained manner only at the elliptic level. Indeed, there are limiting transitions from the elliptic hypergeometric identities involving ${}_{s+1}V_s$-series to the q-hypergeometric relations such that there appear basic ${}_{s+1}\varphi_s$-functions which do not obey one of the mentioned properties [35, 52].

The degeneration of ${}_{s+1}E_s$-series described above at $p \to 0$ leads to ${}_{s+1}\varphi_s$-series with the constraint for parameters resembling the old q-balancing condition [1, 3], but not coinciding with it. The limit $p \to 0$ for ${}_{s+1}V_s$-series with fixed parameters leads to the very-well poised balanced ${}_{s-1}\varphi_{s-2}$-series having their own notation [3]:

$$\lim_{p\to 0} {}_{s+1}V_s(t_0; t_1, \ldots, t_{s-4}; q, p; z) = \sum_{n=0}^{\infty} \frac{1 - t_0 q^{2n}}{1 - t_0} \prod_{k=0}^{s-4} \frac{(t_k; q)_n}{(qt_0/t_k; q)_n} (qz)^n$$
$$=: {}_{s-1}W_{s-2}(t_0; t_1, \ldots, t_{s-4}; q, qz).$$

In a remarkable way, the balancing condition for the ${}_{s+1}V_s$-series coincides in this case with the usual balancing condition for the ${}_{s-1}W_{s-2}$-series [3, 10]. The possibility of fixing the sign in the balancing condition for odd s from the requirement of existence of an additional symmetry strengthens the "elliptic" point of view on the functions of hypergeometric type and indicates on an indispensable connection of these two classes of functions.

In conclusion of this section, let us mention that the quadratic transformations for the ${}_{s+1}V_s$-series and related summation formulae were considered in [49, 52, 53, 54]. Other specific elliptic hypergeometric series were investigated in the papers [55, 56]. An interesting application of a ${}_{s+1}E_s$-series with the nontrivial power variable y appeared recently in [57].

5. An elliptic analogue of the Gauss hypergeometric function

5.1. **Definition of the V-function and a connection with the root system E_7.**

The Euler integral representation for the ${}_2F_1(a, b; c; x)$-function differs from the beta integral by the presence in the integrand of an additional term depending on two new parameters [1]. An elliptic analogue of the Gauss hypergeometric function, which we shall be denoting by the symbol $V(t_1, \ldots, t_8; p, q)$, is also given by a two-parameter extension of the elliptic beta integral [31]:

$$V(t_1, \ldots, t_8; p, q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^{8} \Gamma(t_j z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \tag{5.1}$$

where eight parameters $t_1, \ldots, t_8 \in \mathbb{C}$ and two basic variables $p, q \in \mathbb{C}$ satisfy the constraints $|t_j|, |p|, |q| < 1$ and the balancing condition $\prod_{j=1}^{8} t_j = p^2 q^2$. For other values of the parameters t_j the V-function is defined by analytical continuation of the integral (5.1). This continuation is build by the replacement of the contour of integration $\mathbb{T}$ by a contour C separating the sequences of poles of the integrand at $z = t_j p^a p^b$, $a, b = 0, 1, \ldots$, converging to zero from the poles at $z = t_j^{-1} p^{-a} p^{-b}$, diverging to infinity; this does not assume now the restrictions $|t_j| < 1$. Shrinking the contour C, one can represent the resulting function as an integral over $\mathbb{T}$ and the sum of residues crossed by the contour during this deformation.

Let a pair of parameters satisfies the condition $t_j t_k = pq$. Then the V-function is reduced to the elliptic beta integral (3.1), which follows from the reflection formula for the elliptic gamma function $\Gamma(z; p, q)$.

Let us consider symmetries of the V-function. Evidently, it is invariant with respect to the permutation of p and q and S_8-group of permutations of t_j related to the root system A_7. It appears that there exists a transformation extending S_8

to the Weyl group for the exceptional root system E_7. It is derived with the help of the double integral

$$\kappa \int_{\mathbb{T}^2} \frac{\prod_{j=1}^4 \Gamma(a_j z^{\pm 1}, b_j w^{\pm 1}; p, q)\, \Gamma(c z^{\pm 1} w^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}, w^{\pm 2}; p, q)} \frac{dz}{z} \frac{dw}{w},$$

where $a_j, b_j, c \in \mathbb{C}$, $|a_j|, |b_j|, |c| < 1$, and $c^2 \prod_{j=1}^4 a_j = c^2 \prod_{j=1}^4 b_j = pq$. Computation of the integrals over z or over w in different orders with the help of formula (3.1) yields the fundamentally important relation [11]

$$V(\underline{t}) = \prod_{1 \le j < k \le 4} \Gamma(t_j t_k, t_{j+4} t_{k+4}; p, q)\, V(\underline{s}), \tag{5.2}$$

where $V(\underline{t}) = V(t_1, \ldots, t_8; p, q)$ and

$$\left\{ \begin{array}{ll} s_j = \varepsilon t_j, & j = 1,2,3,4 \\ s_j = \varepsilon^{-1} t_j, & j = 5,6,7,8 \end{array} \right. ; \quad \varepsilon = \sqrt{\frac{pq}{t_1 t_2 t_3 t_4}} = \sqrt{\frac{t_5 t_6 t_7 t_8}{pq}}$$

and $|t_j|, |s_j| < 1$.

Repetition of transformation (5.2) with the parameters $s_{3,4,5,6}$ playing the same role as $t_{1,2,3,4}$ and subsequent permutation of the parameters t_3, t_4 with t_5, t_6 result in the equality

$$V(\underline{t}) = \prod_{j,k=1}^{4} \Gamma(t_j t_{k+4}; p, q)\; V(T^{\frac{1}{2}}/t_1, \ldots, T^{\frac{1}{2}}/t_4, U^{\frac{1}{2}}/t_5, \ldots, U^{\frac{1}{2}}/t_8), \tag{5.3}$$

where $T = t_1 t_2 t_3 t_4$, $U = t_5 t_6 t_7 t_8$ and $|T|^{1/2} < |t_j| < 1$, $|U|^{1/2} < |t_{j+4}| < 1$, $j = 1,2,3,4$. Equating the right-hand sides of equalities (5.2) and (5.3) and expressing parameters t_j in terms of s_j, we obtain the third transformation

$$V(\underline{s}) = \prod_{1 \le j < k \le 8} \Gamma(s_j s_k; p, q)\, V(\sqrt{pq}/s_1, \ldots, \sqrt{pq}/s_8), \tag{5.4}$$

where $|pq|^{1/2} < |s_j| < 1$ for all j.

Let us connect parameters of the function $V(\underline{t})$ with the coordinates of an Euclidean space $x_j \in \mathbb{R}^8$ by the relations $t_j = e^{2\pi i x_j}(pq)^{1/4}$. Denote as $\langle x, y \rangle$ the scalar product in $\mathbb{R}^8$ and as e_i – an orthonormal basis satisfying the condition $\langle e_i, e_j \rangle = \delta_{ij}$. The root system A_7 consists of the vectors $v = \{e_i - e_j,\ i \neq j\}$, and its Weyl group S_8 – from the reflections $x \to S_v(x) = x - 2v\langle v, x\rangle / \langle v, v\rangle$, acting in the hyperplane orthogonal to the vector $\sum_{i=1}^8 e_i$. In this case the coordinates of the vectors $x = \sum_{i=1}^8 x_i e_i$ satisfy the relation $\sum_{i=1}^8 x_i = 0$, which is guaranteed by the balancing condition.

Transformation of the coordinates in (5.2) corresponds to the reflection $S_v(x)$ with respect to the vector $v = (\sum_{i=5}^8 e_i - \sum_{i=1}^4 e_i)/2$ of the length $\langle v, v \rangle = 2$. It extends the group S_8 to the Weyl group for the exceptional root system E_7. A different proof of equalities (5.2)–(5.4) is given by Rains in [34], where a relation to the E_7-group is indicated for the first time.

In [11] it was shown that the V-function is reduced to the product of two ${}_{12}V_{11}$-series for special discrete values of one of the parameters. Let us denote $A = t_0 \ldots t_4$;

then

$$
\begin{aligned}
&V(t_0,\ldots,t_5,t_5^{-1}p^{n+1}q^{m+1},A^{-1}p^{1-n}q^{1-m};p,q)\\
&\quad=\frac{\prod_{0\le j<k\le 4}\Gamma(t_jt_k;p,q)}{\prod_{j=0}^4\Gamma(t_j^{-1}A;p,q)}\frac{\Gamma(p^{n+1}q^{m+1}t_5^{-1}t_0^{\pm1},t_5t_0^{\pm1},At_0^{\pm1};p,q)}{\Gamma(p^nq^mAt_0^{\pm1};p,q)}\\
&\quad\times {}_{12}V_{11}\left(\frac{At_0}{q};\frac{At_5}{q},t_0t_1,t_0t_2,t_0t_3,t_0t_4,q^{-m},\frac{Aq^m}{t_5};q,p\right)\\
&\quad\times {}_{12}V_{11}\left(\frac{At_0}{p};\frac{At_5}{p},t_0t_1,t_0t_2,t_0t_3,t_0t_4,p^{-n},\frac{Ap^n}{t_5};p,q\right), \qquad (5.5)
\end{aligned}
$$

where the contour of integration in the definition of V-function is chosen in such a way that it separates sequences of the integrand poles converging to zero and diverging to infinity. For $m=0$, we obtain an integral representation of a separate terminating ${}_{12}V_{11}$-series. Since the left-hand side of (5.5) is symmetric in $t_0,\ldots,t_4$, the same should hold for the right-hand side as well, which follows from the symmetry transformation for series to be considered below.

5.2. **The elliptic hypergeometric equation.**

The addition formula for theta functions (A.5), being written in the form

$$t_8\theta(t_7t_8^{\pm1},t_6z^{\pm1};p)+t_6\theta(t_8t_6^{\pm1},t_7z^{\pm1};p)+t_7\theta(t_6t_7^{\pm1},t_8z^{\pm1};p)=0,$$

leads to the connection formula

$$\frac{t_6V(qt_6)}{\theta(t_6t_7^{\pm1},t_6t_8^{\pm1};p)}+\frac{t_7V(qt_7)}{\theta(t_7t_6^{\pm1},t_7t_8^{\pm1};p)}+\frac{t_8V(qt_8)}{\theta(t_8t_6^{\pm1},t_8t_7^{\pm1};p)}=0, \qquad (5.6)$$

where $V(qt_j)$ denotes the V-function with the parameter t_j replaced by qt_j (this leads to the balancing condition $\prod_{j=1}^8 t_j=p^2q$). Indeed, it is easy to check that the same equality holds for the V-function kernel, after integration of which over z one obtains (5.6).

A substitution of transformation (5.4) in (5.6) yields

$$
\begin{aligned}
&\frac{\prod_{j=1}^5\theta\left(t_6t_j/q;p\right)}{t_6\theta(t_7/t_6,t_8/t_6;p)}V(q^{-1}t_6)+\frac{\prod_{j=1}^5\theta\left(t_7t_j/q;p\right)}{t_7\theta(t_6/t_7,t_8/t_7;p)}V(q^{-1}t_7)\\
&\qquad+\frac{\prod_{j=1}^5\theta\left(t_8t_j/q;p\right)}{t_8\theta(t_6/t_8,t_7/t_8;p)}V(q^{-1}t_8)=0, \qquad (5.7)
\end{aligned}
$$

where $\prod_{j=1}^8 t_j=p^2q^3$.

Let us consider three equations appearing after the replacements $t_8\to qt_8$ in (5.7) and $t_6\to q^{-1}t_6$ or $t_7\to q^{-1}t_7$ in (5.6). Excluding from them the functions $V(q^{-1}t_6,qt_8)$ and $V(q^{-1}t_7,qt_8)$, we obtain the elliptic hypergeometric equation [31, 58]:

$$
\begin{aligned}
&\mathcal{A}(t_1,\ldots,t_6,t_7,t_8,q;p)\Big(U(qt_6,q^{-1}t_7)-U(\underline{t})\Big) \qquad (5.8)\\
&\qquad+\mathcal{A}(t_1,\ldots,t_7,t_6,t_8,q;p)\Big(U(q^{-1}t_6,qt_7)-U(\underline{t})\Big)+U(\underline{t})=0,
\end{aligned}
$$

where

$$\mathcal{A}(t_1,\ldots,t_8,q;p):=\frac{\theta(t_6/qt_8,t_8t_6,t_8/t_6;p)}{\theta(t_6/t_7,t_7/qt_6,t_6t_7/q;p)}\prod_{k=1}^5\frac{\theta(t_7t_k/q;p)}{\theta(t_8t_k;p)} \qquad (5.9)$$

and

$$U(\underline{t}) := U(\underline{t}; p, q) = \frac{V(\underline{t}; p, q)}{\prod_{k=6}^{7} \Gamma(t_k t_8^{\pm 1}; p, q)}.$$

The potential $\mathcal{A}(t_1, \ldots, t_8, q; p)$ is an elliptic function of all its parameters, i.e. it does not change under the transformations $t_j \to pt_j$, $j = 1, \ldots, 7$, provided we count t_8 as a parameter depending on others through the balancing condition. Because of the symmetry of the U-function in p and q, it satisfies another elliptic hypergeometric equation, which is obtained from (5.8) by permutation of p and q. A change of variables brings the function $\mathcal{A}(t_1, \ldots, t_8, q; p)$ to the completely S_8-symmetric form (4.6) with arbitrary $u_1, \ldots, u_8$ and $\mu = 1$ (a remark from D. Zagier to the author). However, the elliptic hypergeometric equation itself is maximally S_6-symmetric [31, 58].

Let us denote $t_6 = cx, t_7 = c/x$ and pass to a new set of parameters

$$\varepsilon_k = \frac{q}{ct_k},\ k = 1, \ldots, 5, \quad \varepsilon_8 = \frac{c}{t_8}, \quad \varepsilon_7 = \frac{\varepsilon_8}{q}.$$

We fix the parameter ε_6 from the requirement that the balancing condition takes the form $\prod_{k=1}^{8} \varepsilon_k = p^2 q^2$, which yields $c = \sqrt{\varepsilon_6 \varepsilon_8}/p^2$. Now, after the replacement of $U(\underline{t})$ by some unknown function $f(x)$ in (5.8), we can rewrite this elliptic hypergeometric equation in the form of a q-difference equation of the second order:

$$A(x)\left(f(qx) - f(x)\right) + A(x^{-1})\left(f(q^{-1}x) - f(x)\right) + \nu f(x) = 0, \tag{5.10}$$

$$A(x) = \frac{\prod_{k=1}^{8} \theta(\varepsilon_k x; p)}{\theta(x^2, qx^2; p)}, \qquad \nu = \prod_{k=1}^{6} \theta\left(\frac{\varepsilon_k \varepsilon_8}{q}; p\right), \tag{5.11}$$

where one can explicitly see the S_6-group of symmetries in parameters. We have already one functional solution of this equation:

$$f_1(x; \underline{\varepsilon}; p, q) = \frac{V(q/c\varepsilon_1, \ldots, q/c\varepsilon_5, cx, c/x, c/\varepsilon_8; p, q)}{\Gamma(c^2 x^{\pm 1}/\varepsilon_8, x^{\pm 1}\varepsilon_8; p, q)}. \tag{5.12}$$

In order to build other linearly independent solutions, one can use symmetries of the equation (5.10) which do not represent symmetries of the function (5.12). For instance, the second solution can be obtained by multiplication of one of the parameters $\varepsilon_1, \ldots, \varepsilon_5$ or x by the powers of p or by permutations of $\varepsilon_1, \ldots, \varepsilon_5$ with ε_6.

Since the function $\mathcal{A}(\underline{t}, q; p)$ in equation (5.8) does not change after the replacements $t_6 \to p^{-1}t_6, t_7 \to pt_7$, the function $U(p^{-1}t_6, pt_7)$ also represents a solution of this equation. The Casoratian (a discrete Wronskian) of these two solutions was computed in the paper [59]. Let us multiply equation (5.8) by $U(p^{-1}t_6, pt_7)$, and the equation for $U(p^{-1}t_6, pt_7)$ by $U(\underline{t})$ and subtract one of the resulting relations from another. As a result, we obtain the equality

$$\mathcal{A}(t_1, \ldots, t_6, t_7, t_8, q; p) D(p^{-1}t_6, q^{-1}t_7) = \mathcal{A}(t_1, \ldots, t_7, t_6, t_8, q; p) D(p^{-1}q^{-1}t_6, t_7), \tag{5.13}$$

where D denotes the Casoratian

$$D(t_6, t_7) = U(pqt_6, t_7)U(t_6, pqt_7) - U(qt_6, pt_7)U(pt_6, qt_7).$$

After substitutions $t_6 \to pt_6$, $t_7 \to qt_7$ the balancing condition takes the form $t_6 t_7 = pq/t_1 \ldots t_5 t_8$, and (5.13) can be considered as a q-difference equation of the first order in t_7. It is easily solved, and the solution is defined up to the multiplication by an arbitrary q-periodic function $\varphi(qt_7) = \varphi(t_7)$.

Repeating the same procedure with the equations appearing after the permutation of the parameters p and q, we find $\varphi(pt_7) = \varphi(t_7)$, so that $\varphi(t_7)$ does not depend on t_7. Further investigation of the structure of pole residues of the V-functions in $D(t_6, t_7)$, which are crossed by the integration contour in the limit $t_7 \to t_8^{-1}$, fixes completely the form of φ and leads to the identity [59]

$$V(pqt_6, t_7)V(t_6, pqt_7) - t_6^{-2}t_7^{-2}V(qt_6, pt_7)V(pt_6, qt_7) = \frac{\prod_{1\leq j<k\leq 8}\Gamma(t_jt_k; p, q)}{\Gamma(t_6^{\pm 1}t_7^{\pm 1}; p, q)}, \tag{5.14}$$

where t_6 and t_7 can be replaced by any other pair of parameters.

The described solutions of the elliptic hypergeometric equation exist for $|q| < 1$, whereas the equation itself (5.8) does not demand such a restriction. Because of the symmetry

$$\mathcal{A}\left(\frac{p^{1/2}}{t_1}, \ldots, \frac{p^{1/2}}{t_8}, q; p\right) = \mathcal{A}\left(t_1, \ldots, t_8, q^{-1}; p\right),$$

the transformation $t_j \to p^{1/2}/t_j$, $j = 1, \ldots, 8$, leads to the change of the base variable $q \to q^{-1}$ in (5.8). This yields the following solution of the elliptic hypergeometric equation for $|q| > 1$ [59]

$$U(\underline{t}; p, q) = \frac{V(p^{1/2}/t_1, \ldots, p^{1/2}/t_8; p, q^{-1})}{\prod_{k=6}^{7}\Gamma(p/t_kt_8, t_8/t_k; p, q^{-1})}. \tag{5.15}$$

In order to build solutions of equation (5.8) (or (5.10)) for $|q| = 1$, one can use the parameterization of the base variables (2.6), (2.7), make substitutions $t_k = e^{2\pi i g_k/\omega_2}$ and repeat the whole chain of arguments given above with the replacement of $\Gamma(z; p, q)$ by the modified elliptic gamma function $G(u; \omega)$. In the same way as in the case of modified elliptic beta integral, we obtain again the V-function, but with a different parameterization. This procedure appears to be equivalent to the application of the modular transformation $(\omega_2, \omega_3) \to (-\omega_3, \omega_2)$ to solutions described above. Indeed, the potential $\mathcal{A}(e^{2\pi i g_1/\omega_2}, \ldots, e^{2\pi i g_8/\omega_2}, q; p)$ is invariant under this transformation, but $U(t_1, \ldots, t_8; p, q)$ gets transformed to $U(e^{-2\pi i g_1/\omega_3}, \ldots, e^{-2\pi i g_8/\omega_3}; \tilde{p}, \tilde{r})$. The latter function provides thus a new solution of the elliptic hypergeometric equation well defined for $|q| = 1$. It is evident that this function satisfies also a partner of equation (5.8) obtained from it by the permutation of ω_1 and ω_2. An example of a similar situation with two equations for one function at the q-hypergeometric level is given in [60].

Different formal degenerations of the V-function to q-hypergeometric integrals of the Mellin–Barnes or Euler type are briefly considered in [31, 58]. A detailed and rigorous analysis of the degeneration procedure is performed in [28, 61]. It is necessary to note [58] that the elliptic hypergeometric equation emerges as a particular case of the eigenvalue problem equation for the one particle Hamiltonian of the quantum model proposed by van Diejen [62] and investigated in detail by Komori and Hikami [63]. This Calogero–Sutherland type model represents a generalization of the Ruijsenaars [64] and Inozemtsev [65] systems.

6. Chains of symmetry transformations for functions

Symmetry transformations for the plain and q-hypergeometric series are built with the help of the Bailey chains [1, 66]. This technique was generalized to the elliptic level in [52, 67]. Let us sketch this generalization.

Two sequences of numbers $\alpha_n(a,k)$ and $\beta_n(a,k)$ by definition form an elliptic Bailey pair with respect to the parameters a and k, if

$$\beta_n(a,k) = \sum_{0\le m\le n} M_{nm}(a,k)\alpha_m(a,k), \tag{6.1}$$

where

$$M_{nm}(a,k) = \frac{\theta(k/a)_{n-m}\theta(k)_{n+m}}{\theta(q)_{n-m}\theta(aq)_{n+m}}\frac{\theta(aq^{2m};p)}{\theta(a;p)}a^{n-m}. \tag{6.2}$$

In the matrix form $\beta(a,k) = M(a,k)\alpha(a,k)$, where α and β denote the columns formed by α_n and β_n.

Let us introduce the diagonal matrix

$$D_{nm}(a;b,c) = D_m(a;b,c)\delta_{nm}, \qquad D_m(a;b,c) = \frac{\theta(b,c)_m}{\theta(aq/b,aq/c)_m}\left(\frac{aq}{bc}\right)^m. \tag{6.3}$$

Theorem 4. *Let $\alpha(a,t)$ and $\beta(a,t)$ form an elliptic Bailey pair with respect to the parameters a and t. Then the quantities*

$$\alpha'(a,k) = D(a;b,c)\alpha(a,t), \qquad \beta'(a,k) = K(t,k,b,c)\beta(a,t), \tag{6.4}$$
$$K(t,k,b,c) := D(k;qt/b,qt/c)M(t,k)D(t;b,c),$$

where $qat = kbc$ and b,c are two arbitrary new parameters, form a new elliptic Bailey pair with respect to a and k.

Proof. Substitution of (6.4) and of the relation $\beta = M\alpha$ in the required equality $\beta' = M\alpha'$ leads to the matrix identity

$$M(a,k)D(a;b,c)M(t,a) = D(k;qt/b,qt/c)M(t,k)D(t;b,c). \tag{6.5}$$

After substitution of the explicit expressions for matrices, one can see that it is equivalent to the Frenkel–Turaev summation formula (3.8). □

Because $M_{nm}(a,a) = \delta_{nm}$ and $D_{nm}(bc/q;b,c) = \delta_{nm}$, we find after setting $t = k$ in (6.5) that $M(a,k)M(k,a) = 1$. The inversion of the matrix M is reached thus by the permutation of parameters a and k (in the $p = 0$ case this fact was established in [68]; a more detailed discussion of such matrix inversions is given in [53, 56, 69]). Therefore, $\tilde\alpha(a,k) = \beta(k,a)$ and $\tilde\beta(a,k) = \alpha(k,a)$ define new Bailey pairs to which one can apply the transformation (6.4). The described rules of composition of new Bailey pairs generate a binary tree of identities for different products of matrices M and D, which are equivalent to some nontrivial identities for (multiple) elliptic hypergeometric series.

¿From the matrix relation $M(a,f)M(f,a) = 1$ we find the simplest Bailey pairs $\alpha_n^{(i)}(a,f) = M_{ni}(f,a)$ and $\beta_n^{(i)}(a,f) = \delta_{ni}$. Let us set $\alpha'(a,t) = D(a;d,e)M(f,a)$ and $\beta'(a,t) = K(f,t,d,e)$, where $qaf = tde$ and it is assumed that α' and β' are the matrices whose columns form the Bailey pairs. Then the equality $\beta'(a,t) = M(a,t)\alpha'(a,t)$ is equivalent to (6.5). The relation $\beta''(a,k) = M(a,k)\alpha''(a,k)$ with $\alpha''(a,k) = D(a;b,c)\alpha'(a,t)$ and $\beta''(a,k) = K(t,k,b,c)\beta'(a,t)$, where $qat = kbc$, leads to the identity

$$ {}_{12}V_{11}(a;b,c,d,e,kq^n,q^{-n},af^{-1};q,p) = \frac{\theta(qa,k/t,qt/b,qt/c)_n}{\theta(k/a,qt,kb/t,kc/t)_n} \times {}_{12}V_{11}(t;b,c,td/a,te/a,kq^n,q^{-n},tf^{-1};q,p), \tag{6.6}$$

where $f = kbcde/(aq)^2$. This relation represents an elliptic analogue of the Bailey transformation for terminating ${}_{10}\varphi_9$-series [3], which was proved for the first time in [8] by a different method. There exists also a four term Bailey transformation for non-terminating ${}_{10}\varphi_9$-series [3]. Its elliptic generalization is given by the V-function transformation (5.2) (written in the integral form since its infinite series version is not well defined).

Integral analogues of the Bailey chains were discovered in the paper [30], where a number of symmetry transformations for the elliptic hypergeometric integrals has been built with the help of this technique. The functions $\alpha(z,t)$ and $\beta(z,t)$ form by definition an integral elliptic Bailey pair with respect to the parameter t, if they are connected to each other by the relation

$$\beta(w,t) = \kappa \int_{\mathbb{T}} \Gamma(tw^{\pm 1}z^{\pm 1};p,q)\,\alpha(z,t)\frac{dz}{z}. \tag{6.7}$$

Theorem 5. *Let $\alpha(z,t)$ and $\beta(z,t)$ form an integral elliptic Bailey pair with respect to the parameter t, $|t|<1$. We take the parameters w,u,s satisfying the conditions $w\in\mathbb{T}$ and $|s|,|u|<1, |pq|<|t^2s^2u|$. Then the functions*

$$\alpha'(w,st) = \frac{\Gamma(tuw^{\pm 1};p,q)}{\Gamma(ts^2uw^{\pm 1};p,q)}\,\alpha(w,t),$$
$$\beta'(w,st) = \kappa\frac{\Gamma(t^2s^2,t^2suw^{\pm 1};p,q)}{\Gamma(s^2,t^2,suw^{\pm 1};p,q)}\int_{\mathbb{T}}\frac{\Gamma(sw^{\pm 1}x^{\pm 1},ux^{\pm 1};p,q)}{\Gamma(x^{\pm 2},t^2s^2ux^{\pm 1};p,q)}\,\beta(x,t)\frac{dx}{x}$$

define a new integral Bailey pair with respect to the parameter st, and the functions

$$\alpha'(w,t) = \kappa\frac{\Gamma(s^2t^2,uw^{\pm 1};p,q)}{\Gamma(s^2,t^2,w^{\pm 2},t^2s^2uw^{\pm 1};p,q)}\int_{\mathbb{T}}\frac{\Gamma(t^2sux^{\pm 1},sw^{\pm 1}x^{\pm 1}p,q)}{\Gamma(sux^{\pm 1};p,q)}\alpha(x,st)\frac{dx}{x},$$
$$\beta'(w,t) = \frac{\Gamma(tuw^{\pm 1};p,q)}{\Gamma(ts^2uw^{\pm 1};p,q)}\beta(w,st)$$

define a new integral Bailey pair with respect to the parameter t.

In order to prove the first statement, it is sufficient to substitute relation (6.7) in the definition of $\beta'(w,st)$, to change the order of integrations, and to use the elliptic beta integral (3.1). The second statement is proved in an analogous way.

In the paper [36] it is shown that under certain restrictions the integral transformation $\alpha(x,t)\to\beta(x,t)$ has a very simple inversion. Let $p,q,t\in\mathbb{C}$ are such that $|p|,|q|<|t|^2<1$. For a fixed $w\in\mathbb{T}$, we denote as C_w a contour inside the annulus $A=\{z\in\mathbb{C}|\ |t|-\epsilon<|z|<|t|^{-1}+\epsilon\}$ for some infinitesimally small positive ϵ, such that the points $t^{-1}w^{\pm 1}$ are lying inside C_w. Let $f(z,t)$ be a holomorphic function in A satisfying $f(z,t)=f(z^{-1},t)$. We define an integral transformation

$$g(w,t) = \kappa\int_{C_w}\delta(z,w;t^{-1})f(z,t)\frac{dz}{z}, \tag{6.8}$$

where

$$\delta(z,w;t^{-1}) = \frac{\Gamma(t^{-1}w^{\pm 1}z^{\pm 1};p,q)}{\Gamma(t^2,z^{\pm 2};p,q)}.$$

Then for $|t|<|z|<|t|^{-1}$, we have

$$f(z,t) = \kappa\int_{\mathbb{T}}\delta(w,z;t)g(w,t)\frac{dw}{w}. \tag{6.9}$$

This relation coincides in essence with the definition of elliptic integral Bailey pairs. It can be shown that the two ways of building chains of integral Bailey pairs indicated above are related to each other by the described inversion of integral transformation (6.8).

With the help of Theorem 5 one can build infinite sequences of Bailey pairs starting from a given initial pair. The simplest pair can be built with the help of the elliptic beta integral (3.1). Each new application of the substitutions indicated above brings in two new parameters. Equality (6.7), being applied to the appearing new Bailey pairs, leads to a binary tree of identities for multiple elliptic hypergeometric integrals with many parameters.

As an illustration, we describe a chain of nontrivial relations for the integrals

$$I^{(m)}(t_1,\ldots,t_{2m+6})=\kappa\int_{\mathbb{T}}\frac{\prod_{j=1}^{2m+6}\Gamma(t_jz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z},\qquad \prod_{j=1}^{2m+6}t_j=(pq)^{m+1},\quad(6.10)$$

where $|t_j|<1$. With the help of the elliptic beta integral (3.1) it is easy to verify validity of the recursion

$$I^{(m+1)}(t_1,\ldots,t_{2m+8})=\frac{\prod_{2m+5\le k<l\le 2m+8}\Gamma(t_kt_l;p,q)}{\Gamma(\rho_m^2;p,q)}\qquad(6.11)$$
$$\times\,\kappa\int_{\mathbb{T}}\frac{\prod_{k=2m+5}^{2m+8}\Gamma(\rho_m^{-1}t_kw^{\pm1};p,q)}{\Gamma(w^{\pm2};p,q)}I^{(m)}(t_1,\ldots,t_{2m+4},\rho_mw,\rho_mw^{-1})\frac{dw}{w},$$

where $\rho_m^2=\prod_{k=2m+5}^{2m+8}t_k/pq$. This equality gives a concrete realization of the Bailey pairs $\alpha\sim I^{(m)}$ and $\beta\sim I^{(m+1)}$ after a change of parameters. For $m=0$, substitution of the explicit expression for $I^{(0)}$ (3.1) to the right-hand side of (6.11) leads to the identity (5.2). Another important consequence of recursion (6.11) is considered in the next section. In general equality (6.11) yields an m-tuple integral representation for $I^{(m)}$ analogous to the Euler representation for the ${}_{m+1}F_m$-function.

7. Biorthogonal functions of the hypergeometric type

7.1. Discrete biorthogonal functions with the continuous measure.

Let us denote the series $\mathcal{E}(\mathbf{t}):={}_{12}V_{11}(t_0^2;t_0t_1,\ldots,t_0t_7;q,p)$ with the balancing condition $\prod_{m=0}^{7}t_m=q^2$ and the termination condition $t_0t_k=q^{-n}$ for some k. The contiguous with $\mathcal{E}(\mathbf{t})$ functions, which are obtained by the change of parameters t_i and t_k to t_iq and t_kq^{-1}, will be denoted as $\mathcal{E}(t_i^+,t_k^-)$. Then, using the addition formula for theta functions, it is not difficult to check the equality [9, 46]

$$\mathcal{E}(\mathbf{t})-\mathcal{E}(t_6^-,t_7^+)$$
$$=\frac{\theta(qt_0^2,q^2t_0^2,qt_7/t_6,t_6t_7/q;p)}{\theta(qt_0/t_6,q^2t_0/t_6,t_0/t_7,t_7/qt_0;p)}\prod_{r=1}^{5}\frac{\theta(t_0t_r;p)}{\theta(qt_0/t_r;p)}\mathcal{E}(t_0^+,t_6^-).\qquad(7.1)$$

Substitution of an elliptic analogue of the Bailey transformation and of its iterations in (7.1) allows one to build many formulae of such a type. One of them has the form

$$\frac{\theta(t_0t_7;p)}{\theta(t_6/qt_0,t_6/q^2t_0,t_6/t_7;p)}\prod_{r=1}^5\theta(t_rt_6/q;p)\,\mathcal{E}(t_0^+,t_6^-)$$
$$+\frac{\theta(t_0t_6;p)}{\theta(t_7/qt_0,t_7/q^2t_0,t_7/t_6;p)}\prod_{r=1}^5\theta(t_rt_7/q;p)\,\mathcal{E}(t_0^+,t_7^-)$$
$$=\frac{1}{\theta(qt_0^2,q^2t_0^2;p)}\prod_{r=1}^5\theta(qt_0/t_r;p)\,\mathcal{E}(\mathbf{t}). \tag{7.2}$$

Let us replace in (7.1) the parameter t_6 by qt_6, and t_7 by t_7/q, and substitute $\mathcal{E}(t_0^+,t_7^-)$ from the resulting equality and $\mathcal{E}(t_0^+,t_6^-)$ from (7.1) in (7.2). This leads to the equation

$$\frac{\theta(t_0t_7,t_0/t_7,qt_0/t_7;p)}{\theta(qt_7/t_6,t_7/t_6;p)}\prod_{r=1}^5\theta(q/t_6t_r;p)\left(\mathcal{E}(t_6^-,t_7^+)-\mathcal{E}(\mathbf{t})\right)$$
$$+\frac{\theta(t_0t_6,t_0/t_6,qt_0/t_6;p)}{\theta(qt_6/t_7,t_6/t_7;p)}\prod_{r=1}^5\theta(q/t_7t_r;p)\left(\mathcal{E}(t_6^+,t_7^-)-\mathcal{E}(\mathbf{t})\right)$$
$$+\theta(q/t_6t_7;p)\prod_{r=1}^5\theta(t_0t_r;p)\,\mathcal{E}(\mathbf{t})=0. \tag{7.3}$$

These relations are analogues of equations (5.6), (5.7), and (5.8) for the V-function, and they can be obtained from them by the residue analysis for some sequences of poles of the integrand. Let us replace in (7.3) the $\mathcal{E}$-function by

$$R_n(x;q,p)={}_{12}V_{11}\left(\frac{\varepsilon_6}{\varepsilon_8};\frac{q}{\varepsilon_1\varepsilon_8},\frac{q}{\varepsilon_2\varepsilon_8},\frac{q}{\varepsilon_3\varepsilon_8},q^{-n},\frac{Aq^{n-1}}{\varepsilon_8},\frac{\varepsilon_6}{x},\varepsilon_6x;q,p\right), \tag{7.4}$$

where $n=0,1,\dots$ and $A=\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_6\varepsilon_8$, and denote $q^{-n}=pq/\varepsilon_4\varepsilon_8$, $Aq^{n-1}/\varepsilon_8=pq/\varepsilon_5\varepsilon_8$. Then, after the change of notation for parameters $t_0^2=\varepsilon_6/\varepsilon_8,\dots,t_0t_7=\varepsilon_6x$, we obtain equation (5.10) with $\varepsilon_7=\varepsilon_8/q$ and discrete values of one of the parameters. The function $R_n(x;q,p)$ defines thus a particular solution of the elliptic hypergeometric equation obeying the property $R_n(px;q,p)=R_n(x;q,p)$. (Notations of the paper [11], where this function was investigated, pass to ours after the changes $t_{0,1,2}\to\varepsilon_{1,2,3}$, $t_3\to\varepsilon_6$, $t_4\to\varepsilon_8$, $\mu\to\varepsilon_4\varepsilon_8/pq$ and $A\mu/qt_4\to pq/\varepsilon_5\varepsilon_8$.)

The parameters $\varepsilon_1,\dots,\varepsilon_6$ enter (5.10) symmetrically. Because of the balancing condition $\prod_{k=1}^8\varepsilon_k=p^2q^2$, the function $R_n(x;q,p)$ is invariant with respect to changes $\varepsilon_k\to p\varepsilon_k$, $k=1,\dots,5$. This guarantees the symmetry of (7.4) in $\varepsilon_1,\dots,\varepsilon_5$, and any of these parameters can be used for the series termination. Permutation of one of the parameters $\varepsilon_{1,2,3,5}$ with ε_6 and application of the elliptic Bailey transformation for terminating ${}_{12}V_{11}$-series leads to $R_n(x;q,p)$ up to some multiplier which does not depend on x.

We identify now the parameters in equation (7.3) in a different way:

$$t_0^2=\frac{\varepsilon_6}{\varepsilon_8},\ t_0t_{1,2,3}=\frac{q}{\varepsilon_{1,2,3}\varepsilon_8},\ t_0t_4=\frac{\varepsilon_6}{x},\ t_0t_5=\varepsilon_6x,\ t_0t_6=q^{-n},\ t_0t_7=\frac{A}{\varepsilon_8}q^{n-1}.$$

This leads to a three term recurrence relation in the index n:

$$\frac{\theta\left(\frac{Aq^{n-1}}{\varepsilon_8}, \frac{\varepsilon_6 q^{2-n}}{A}, \frac{\varepsilon_6 q^{1-n}}{A}; p\right)}{\theta\left(\frac{Aq^{2n-1}}{\varepsilon_8}, \frac{Aq^{2n}}{\varepsilon_8}; p\right)} \prod_{r=1}^{3} \theta\left(\varepsilon_r \varepsilon_6 q^n; p\right) \theta\left(\frac{q^{n+1} x^{\pm 1}}{\varepsilon_8}; p\right) (R_{n+1} - R_n)$$
$$+ \frac{\theta\left(q^{-n}, \frac{\varepsilon_6 q^n}{\varepsilon_8}, \frac{\varepsilon_6 q^{1+n}}{\varepsilon_8}; p\right)}{\theta\left(\frac{q^{1-2n}\varepsilon_8}{A}, \frac{q^{2-2n}\varepsilon_8}{A}; p\right)} \prod_{r=1}^{3} \theta\left(\frac{\varepsilon_r \varepsilon_6 \varepsilon_8 q^{1-n}}{A}; p\right) \theta\left(\frac{q^{2-n} x^{\pm 1}}{A}; p\right) (R_{n-1} - R_n)$$
$$+ \theta\left(\frac{\varepsilon_6 q^n}{A}, \varepsilon_6 x^{\pm 1}; p\right) \prod_{r=1}^{3} \theta\left(\frac{q}{\varepsilon_r \varepsilon_8}; p\right) R_n = 0 \tag{7.5}$$

with the initial conditions $R_{-1} = 0$ and $R_0 = 1$.

Let us introduce the functions

$$z(x) = \frac{\theta(x\xi^{\pm 1}; p)}{\theta(x\eta^{\pm 1}; p)}, \qquad \alpha_n = z(q^n/\varepsilon_8), \qquad \beta_n = z(Aq^{n-1}),$$

where ξ and η are arbitrary gauge parameters, $\xi \neq \eta^{\pm 1} p^k$, $k \in \mathbb{Z}$. Then relation (7.5) can be rewritten in a more structured form

$$(z(x) - \alpha_{n+1})\rho(Aq^{n-1}/\varepsilon_8)\left(R_{n+1}(x; q, p) - R_n(x; q, p)\right)$$
$$+(z(x) - \beta_{n-1})\rho(q^{-n})\left(R_{n-1}(x; q, p) - R_n(x; q, p)\right)$$
$$+ \delta(z(x) - z(\varepsilon_6))R_n(x; q, p) = 0, \tag{7.6}$$

where

$$\rho(t) = \frac{\theta\left(t, \frac{\varepsilon_6}{\varepsilon_8 t}, \frac{q\varepsilon_6}{\varepsilon_8 t}, \frac{qt}{\varepsilon_1\varepsilon_2}, \frac{qt}{\varepsilon_2\varepsilon_3}, \frac{qt}{\varepsilon_1\varepsilon_3}, \frac{q^2 t\eta^{\pm 1}}{A}; p\right)}{\theta\left(\frac{qt^2\varepsilon_8}{A}, \frac{q^2 t^2 \varepsilon_8}{A}; p\right)},$$

$$\delta = \theta\left(\frac{q^2\varepsilon_6}{A}, \frac{q}{\varepsilon_1\varepsilon_8}, \frac{q}{\varepsilon_2\varepsilon_8}, \frac{q}{\varepsilon_3\varepsilon_8}, \varepsilon_6\eta^{\pm 1}; p\right).$$

The initial conditions $R_{-1} = 0$ and $R_0 = 1$ guarantee that $R_n(x; q, p)$ are rational functions of $z(x)$ with the poles at the points $\alpha_1, \ldots, \alpha_n$ (i.e., all the dependence on x enters R_n only through the variable $z(x)$).

Suppose that ϕ_λ is a solution of an abstract generalized eigenvalue problem $\mathcal{D}_1\phi_\lambda = \lambda\mathcal{D}_2\phi_\lambda$ for some operators $\mathcal{D}_{1,2}$. Let also a scalar product $\langle\psi|\phi\rangle$ is given, which defines the formal conjugated operators $\mathcal{D}_{1,2}^T$ by the standard rule $\langle\mathcal{D}_{1,2}^T\psi|\phi\rangle = \langle\psi|\mathcal{D}_{1,2}\phi\rangle$. Let ψ_λ denote solutions of the dual generalized eigenvalue problem $\mathcal{D}_1^T\psi_\lambda = \lambda\mathcal{D}_2^T\psi_\lambda$. Then $0 = \langle\psi_\mu|(\mathcal{D}_1 - \lambda\mathcal{D}_2)\phi_\lambda\rangle = (\mu - \lambda)\langle\mathcal{D}_2^T\psi_\mu|\phi_\lambda\rangle$, that is the function $\mathcal{D}_2^T\psi_\mu$ is orthogonal to ϕ_λ for $\mu \neq \lambda$. Consequences of this well known fact of the linear algebra were investigated in detail by Zhedanov [70] in the case when $\mathcal{D}_{1,2}$ are the Jacobi matrices (i.e., the tridiagonal matrices). In particular, such generalized eigenvalue problems were shown to be equivalent to the theory of biorthogonal rational functions generalizing the orthogonal polynomials. They are connected also to the recurrence relation of the R_{II}-type investigated in [71] and to the orthogonality relations appearing within the theory of multipoint Padé approximation [72, 73]. Recurrence relation (7.6) belongs to this class of problems and, therefore, there exists a linear functional $\mathcal{L}$ with the condition $\mathcal{L}\{T_m(x; q, p)R_n(x; q, p)\} = h_n\delta_{nm}$ for some rational functions T_m and normalization constants h_n.

The elliptic hypergeometric equation for R_n-functions can be rewritten in the form of a generalized eigenvalues problem [11]:

$$\mathcal{D}(\varepsilon_4', \varepsilon_5')R_n = \lambda_n \mathcal{D}(\varepsilon_4'', \varepsilon_5'')R_n,$$

where

$$\mathcal{D}(\varepsilon_4, \varepsilon_5) = A(x)(T_{q,x} - 1) + A(x^{-1})(T_{q,x}^{-1} - 1) + \nu, \quad T_{q,x}f(x) = f(qx),$$

denotes the operator with the help of which one can rewrite equation (5.10) as $\mathcal{D}(\varepsilon_4, \varepsilon_5)f(x) = 0$. The primed parameters are arbitrary under the restriction $\varepsilon_4'\varepsilon_5' = \varepsilon_4''\varepsilon_5'' = \varepsilon_4\varepsilon_5$, and other parameters remain untouched (that is why the dependence on them is not indicated). The spectral variable

$$\lambda_n = \frac{\theta(\varepsilon_4/\varepsilon_4', \varepsilon_4/\varepsilon_5'; p)}{\theta(\varepsilon_4/\varepsilon_4'', \varepsilon_4/\varepsilon_5''; p)}$$

is discrete because $\varepsilon_4 = pq^{n+1}/\varepsilon_8$. If we take as the functional $\mathcal{L}$ the integral whose kernel coincides with the kernel of elliptic beta integral,

$$\mathcal{L}\{\phi(x)\psi(x)\} = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1,2,3,6,8}\Gamma(\varepsilon_j x^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}, Ax^{\pm 1}; p, q)} \phi(x)\psi(x)\frac{dx}{x},$$

then the functions

$$T_n(x; q, p) = {}_{12}V_{11}\left(\frac{A\varepsilon_6}{q}; \frac{A}{\varepsilon_1}, \frac{A}{\varepsilon_2}, \frac{A}{\varepsilon_3}, \varepsilon_6 x, \frac{\varepsilon_6}{x}, q^{-n}, \frac{Aq^{n-1}}{\varepsilon_8}; q, p\right), \qquad (7.7)$$

serve as an analogue of $\mathcal{D}_2^T\psi_\mu$ for $R_n(x; q, p)$. $T_n(x; q, p)$ are the rational functions of $z(x)$ with the poles at the points $\beta_1, \ldots, \beta_n$ which are obtained from $R_n(x; q, p)$ after the parameter change $\varepsilon_8 \to pq/A$ (dependence on p in the parameters disappears because of the ellipticity of the ${}_{12}V_{11}$-series in them).

We denote $R_{nm}(x) := R_n(x; q, p)R_m(x; p, q)$ and $T_{nm}(x) := T_n(x; q, p)T_m(x; p, q)$, where all ${}_{12}V_{11}$-series terminate simultaneously because of the modified termination condition $\varepsilon_4\varepsilon_8 = p^{m+1}q^{n+1}$, $n, m = 0, 1, \ldots$. Since $R_m(qx; p, q) = R_m(x; p, q)$, the functions R_{nm} represent now solutions of not one, but two generalized eigenvalue problems differing from each other by permutation of p and q. Therefore the orthogonality relations in our case appear to be more complicated than for the biorthogonal rational functions.

Theorem 6. *The functions $R_{nm}(x)$ and $T_{nm}(x)$ satisfy the following two-index biorthogonality relations:*

$$\kappa \int_{C_{mn,kl}} T_{nl}(x)R_{mk}(x)\frac{\prod_{j\in S}\Gamma(\varepsilon_j x^{\pm 1}; p, q)}{\Gamma(x^{\pm 2}, Ax^{\pm 1}; p, q)}\frac{dx}{x} = h_{nl}\,\delta_{mn}\,\delta_{kl}, \qquad (7.8)$$

where $S = \{1, 2, 3, 6, 8\}$, $C_{mn,kl}$ denotes a contour separating the sequences of points $\varepsilon_j p^a q^b$ $(j = 1, 2, 3, 6)$, $\varepsilon_8 p^{a-k} q^{b-m}, p^{a+1-l}q^{b+1-n}/A$, $a, b = 0, 1, \ldots$, from their $x \to x^{-1}$ reciprocals, and the normalization constants have the form

$$h_{nl} = \frac{\prod_{j<k,\, j,k\in S}\Gamma(\varepsilon_j\varepsilon_k; p, q)}{\prod_{j\in S}\Gamma(A\varepsilon_j^{-1}; p, q)}\, h_n(q, p)\cdot h_l(p, q),$$

$$h_n(q, p) = \frac{\theta(A/q\varepsilon_8; p)\theta(q, q\varepsilon_6/\varepsilon_8, \varepsilon_1\varepsilon_2, \varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3, A\varepsilon_6)_n\, q^{-n}}{\theta(Aq^{2n}/q\varepsilon_8; p)\theta(1/\varepsilon_6\varepsilon_8, \varepsilon_1\varepsilon_6, \varepsilon_2\varepsilon_6, \varepsilon_3\varepsilon_6, A/q\varepsilon_6, A/q\varepsilon_8)_n}.$$

A direct proof of this statement by a straightforward computation of the integral on the left-hand side with the help of formula (3.1) and the Frenkel–Turaev sum is given in [11]. Appearance of the two-index orthogonality relations for univariate functions is a new phenomenon in the theory of special functions. It is worth of noting that $R_{nm}(x)$ and $T_{nm}(x)$ are meromorphic functions of $x \in \mathbb{C}^*$ with essential singularities at $x = 0, \infty$; only for $k = l = 0$ or $n = m = 0$ they become rational functions of some argument depending on x. For $k = l = 0$, one can take the limit $p \to 0$ with fixed parameters and obtain the functions $R_n(x; q, 0)$ and $T_n(x; q, 0)$, which coincide with the family of continuous ${}_{10}\varphi_9$ biorthogonal rational functions of Rahman [39]. A further degeneration of these functions leads to the Askey–Wilson polynomials [41]. Additional restrictions on one of the parameters in $R_n(x; q, p)$ and $T_n(x; q, p)$ leads to a finite-dimensional systems of biorthogonal rational functions of a discrete argument [9, 74], generalizing the Wilson functions [75]. An elementary approach to the analysis of these functions, related to the elliptic $6j$-symbols [8], was suggested by Rosengren in [76]. Some properties of the functions $R_n(x; q, p)$ are investigated in the recent paper [48].

One can build a relation analogous to (7.8) on the basis of the modified elliptic beta integral (3.10) [31]. For this it is necessary to use the parameterization of base variables in terms of the quasiperiods $\omega_{1,2,3}$ and pass to the functions $r_n(u; \omega_1, \omega_2, \omega_3) = R_n(e^{2\pi i u/\omega_2}; e^{2\pi i\omega_1/\omega_2}, e^{2\pi i\omega_3/\omega_2})$, where we have also substituted $\varepsilon_j = e^{2\pi i g_j/\omega_2}$. Analogously, it is necessary to redenote $T_n(x; q, p)$ as $s_n(u; \omega_1, \omega_2, \omega_3)$ and $h_n(q, p)$ as $h_n(\omega_1, \omega_2, \omega_3)$. The products $r_{nm}(u) = r_n(u; \omega_1, \omega_2, \omega_3)$ $r_m(u; \omega_2, \omega_1, \omega_3)$ and $s_{nm}(u) = s_n(u; \omega_1, \omega_2, \omega_3)s_m(u; \omega_2, \omega_1, \omega_3)$ are invariant with respect to the permutations $\omega_1 \leftrightarrow \omega_2$, $n \leftrightarrow m$. Then, for a specially chosen contour $\tilde{C}_{mn,kl}$, we have

$$\tilde{\kappa} \int_{\tilde{C}_{mn,kl}} s_{nl}(u) r_{mk}(u) \frac{\prod_{j\in S} G(g_j \pm u; \omega)}{G(\pm 2u, \mathcal{A} \pm u; \omega)} \frac{du}{\omega_2} = \tilde{h}_{nl}\, \delta_{mn}\, \delta_{kl}, \tag{7.9}$$

where $\mathcal{A} = \sum_{j\in S} g_j$ and

$$\tilde{h}_{nl} = \frac{\prod_{j<m,\, j,m\in S} G(g_j + g_m; \omega)}{\prod_{j\in S} G(\mathcal{A} - g_j; \omega)} h_n(\omega_1, \omega_2, \omega_3) h_l(\omega_2, \omega_1, \omega_3).$$

In distinction from the previous case, the limiting transition to the q-hypergeometric level $\mathrm{Im}(\omega_3/\omega_2), \mathrm{Im}(\omega_3/\omega_1) \to +\infty$ (that is, $p, r \to 0$) is well defined and preserves the two-index structure of biorthogonality relations. In particular, the $r_{nm}(u)$-function degenerates now to the product of two q-hypergeometric series

$$\begin{aligned} r_{nm}(u; \omega_1, \omega_2) = {}& {}_{10}W_9\Big(e^{2\pi i(g_6-g_8)/\omega_2}; e^{2\pi i(\omega_1-g_1-g_8)/\omega_2}, \ldots, e^{2\pi i(g_6+u)/\omega_2}; q, q\Big) \\ & \times {}_{10}W_9\Big(e^{2\pi i(g_6-g_8)/\omega_1}; e^{2\pi i(\omega_2-g_1-g_8)/\omega_1}, \ldots, e^{2\pi i(g_6+u)/\omega_1}; \tilde{q}^{-1}, \tilde{q}^{-1}\Big), \end{aligned}$$

whose basic variables are related by a modular transformation and the normalization of the measure is given by the integral (3.13).

7.2. **A terminating continued fraction.**

A terminating continued fraction related to the rational functions $R_n(x; q, p)$ is computed in the paper [46]. Let U_n and V_n denote two sequences of numbers satisfying the three term recurrence relation

$$\psi_{n+1} = \xi_n \psi_n + \eta_n \psi_{n-1}, \quad n = 1, 2, \ldots \tag{7.10}$$

with some coefficients ξ_n and η_n and the initial conditions $U_0 = 0,\ U_1 = 1$ and $V_0 = 1,\ V_1 = \xi_0$. It is well known that their ratio is related to the finite continued fraction

$$\frac{U_n}{V_n} = \cfrac{1}{\xi_0 + \cfrac{\eta_1}{\xi_1 + \cdots + \cfrac{\eta_{n-1}}{\xi_{n-1}}}}, \qquad n = 1, 2, \ldots. \tag{7.11}$$

Let us define polynomials of $z(x)$ of the n-th degree:

$$P_n(z(x)) = \kappa_n \prod_{k=1}^{n} (z - \alpha_k)\, R_n(x; q, p), \quad \kappa_n = \prod_{j=0}^{n-1} \rho(aq^{j-1})$$

and set $P_0(z(x)) = 1$. Replacing $R_n(x; q, p)$ by $P_n(z(x))$ in (7.6), we obtain the following recurrence relation

$$P_{n+1}(z) + (v_n - \rho_n z)P_n(z) + u_n(z - \alpha_n)(z - \beta_{n-1})P_{n-1}(z) = 0 \tag{7.12}$$

with the initial conditions $P_{-1} = 0,\ P_0 = 1$ and the recurrence coefficients

$$u_n = \rho(q^{-n})\rho(Aq^{n-2}/\varepsilon_8), \quad \rho_n = \rho(Aq^{n-1}/\varepsilon_8) + \rho(q^{-n}) - \delta,$$
$$v_n = \alpha_{n+1}\rho(Aq^{n-1}/\varepsilon_8) + \beta_{n-1}\rho(q^{-n}) - \delta z(\varepsilon_6). \tag{7.13}$$

In this case $V_n = P_n(z)$, and $U_n = P^{(1)}_{n-1}(z)$ are the associated polynomials of the degree $n - 1$ in z.

Let us suppose that the polynomial $P_{N+1}(z)$ has only simple zeros, that is $P_{N+1}(z_s) = 0,\ z_s \equiv z_s^{(N+1)},\ s = 0, 1, \ldots, N,\ z_s \neq z_{s'}$ for $s \neq s'$. Then the corresponding continued fraction can be expanded into the partial fraction (as a rational function of z)

$$\frac{P_N^{(1)}(z)}{P_{N+1}(z)} = \sum_{s=0}^{N} \frac{g_s}{z - z_s}, \qquad g_s = \frac{P_N^{(1)}(z_s)}{P'_{N+1}(z_s)}. \tag{7.14}$$

The Casoratian of any two solutions U_n and V_n of (7.10) satisfies the relation

$$U_{n+1}V_n - U_nV_{n+1} = (-1)^n \eta_1 \cdots \eta_n (U_1V_0 - U_0V_1),$$

which yields

$$P_n(z)P_n^{(1)}(z) - P_{n+1}(z)P_{n-1}^{(1)}(z) = h_n A_n(z) B_n(z), \tag{7.15}$$

where $h_n = u_1 u_2 \cdots u_n$ and

$$A_n(z) = \prod_{i=1}^{n} (z - \alpha_i), \qquad B_n(z) = \prod_{i=1}^{n} (z - \beta_{i-1}).$$

Fixing $n = N$ and setting $z = z_s,\ s = 0, 1, \ldots, N$, in (7.15), we can express $P_N^{(1)}(z_s)$ in terms of $P_N(z_s)$, h_N, $A_N(z_s)$ and $B_N(z_s)$. This leads to the following convenient for computations expression for the residues of the poles g_s in (7.14):

$$g_s = \frac{h_N A_N(z_s) B_N(z_s)}{P'_{N+1}(z_s) P_N(z_s)}. \tag{7.16}$$

In the limit

$$\varepsilon_3 \varepsilon_6 = q^{-N+\epsilon}, \quad N = 0, 1, \ldots, \quad \epsilon \to 0, \tag{7.17}$$

we find that $u_{N+1} \to 0$, and the continued fraction terminates automatically. It appears that this fraction can be computed in the closed form using formula

(7.14). For $\epsilon \to 0$, the rational function $R_{N+1}(x;q,p)$ diverges, since the elliptic Pochhammer symbol $\theta(\varepsilon_3\varepsilon_6)_{N+1}$ in the denominator of the last term of the ${}_{12}V_{11}$-series tends to zero. However, we have simultaneously $\kappa_{N+1} \to 0$, so that the polynomial $P_{N+1}(z)$ takes the finite value, and its zeros are found explicitly: $z_s = z(\varepsilon_6 q^s)$, $s = 0,\dots,N$. It appears also that the polynomial $P_N(z_s)$ is computable in the closed form owing to the Frenkel–Turaev summation formula. The other quantities defining the residues g_s are found sufficiently easily, although they are given by rather cumbersome expressions.

Suppose that the conditions of the simplicity of zeros z_s are satisfied as well as other restrictions on parameters guaranteeing that $u_k \neq 0$ for $k = 1,\dots,N$, the descriptions of which we skip. Then the terminating elliptic hypergeometric continued fraction has the following explicit representation:

$$\frac{1}{\rho_0 z - v_0 - \cfrac{u_1(z-\alpha_1)(z-\beta_0)}{\rho_1 z - v_1 - \dots - \cfrac{u_N(z-\alpha_N)(z-\beta_{N-1})}{\rho_N z - v_N}}} \tag{7.18}$$
$$= \frac{1}{(z(\varepsilon_6)-z(x))\delta}\,{}_{12}V_{11}\left(\frac{q\varepsilon_6}{\varepsilon_8}; q, \varepsilon_6\varepsilon_1, \varepsilon_6\varepsilon_2, \frac{qx}{\varepsilon_8}, \frac{q}{\varepsilon_8 x}, q^{-N}, \frac{\varepsilon_6 q^{N+2}}{\varepsilon_1\varepsilon_2\varepsilon_8}; q, p\right),$$

where in the expressions for all recurrence coefficients, including δ, it is necessary to substitute $\varepsilon_3 = q^{-N}/\varepsilon_6$ and $z(x) = \theta(x\xi^{\pm1};p)/\theta(x\eta^{\pm1};p)$. In [46] this result was presented in the different (additive) system of notation.

This formula describes the most general terminating continued fraction of the hypergeometric type, which was found to the present moment. For fixed values of parameters, in the limit $p \to 0$ one obtains the terminating continued fraction of Gupta and Masson [77] (see Corollary 3.3) described by a very-well poised balanced ${}_{10}\varphi_9$-series. Further specification of parameters leads to the continued fraction of Watson which, in its turn, is a q-analogue of the famous Ramanujan continued fraction (see the details in [78]).

7.3. **Continuous biorthogonality of the V-function.**

The relations described above (7.8) correspond to discrete values of one of the parameters in the elliptic hypergeometric equation. In paper [79], it was shown that the V-function with general set of continuous parameters obeys also some biorthogonality relations characteristic to the continuous spectra.

Let us consider the $m = 2$ case in recursion (6.11). After imposing the constraints on parameters $t_5t_7 = t_6t_8 = pq$ the integral on the left-hand side becomes explicitly computable. Then, after a number of notational changes and application of transformation (5.2), there appears the equality

$$\phi(x;c,d|\xi;s) = \kappa\int_{\mathbb{T}} R(c,d,a,b;x,w|s)\phi(w;a,b|\xi;s)\frac{dw}{w}, \tag{7.19}$$

where the basis vectors have the form

$$\phi(w;a,b|\xi;s) = \Gamma(sa\xi^{\pm1}, sb\xi^{\pm1}, \sqrt{\frac{pq}{ab}}w^{\pm1}\xi^{\pm1};p,q) \tag{7.20}$$

and

$$R(c,d,a,b;x,w|s)=\frac{1}{\Gamma(\frac{pq}{ab},\frac{ab}{pq},w^{\pm 2};p,q)}$$
$$\times V\left(sc,sd,\sqrt{\frac{pq}{cd}}x,\varepsilon\sqrt{\frac{pq}{cd}}x^{-1},\frac{pq}{as},\frac{pq}{bs},\sqrt{\frac{ab}{pq}}w,\sqrt{\frac{ab}{pq}}w^{-1}\right) \tag{7.21}$$

Here the parameters a,b,c,d,s and x,ξ are arbitrary, but their choice should match with the condition that taken contours of integration separate converging to zero and diverging to infinity sequences of poles of the integrands. The variable ξ enters only the basis vectors $\phi(w;a,b|\xi;s)$ and the kernel R can be considered as a "rotation matrix" with continuous indices x and w, which permits to change arbitrarily the parameters a and b. Equality (7.21) represents an integral generalization of the relation which was used by Rosengren in [76] for derivation of properties of the elliptic $6j$-symbols.

Denoting $w=e^{i\theta}$, $y=\cos\theta$, and using the equality

$$\int_{\mathbb{T}} f(\cos\theta)\frac{dw}{iw}=\int_0^{2\pi} f(\cos\theta)d\theta=2\int_{-1}^1 f(y)\frac{dy}{\sqrt{1-y^2}},$$

we can write

$$\phi(x;c,d|\xi;s)=2i\kappa\int_{-1}^1 R(c,d,a,b;x,e^{i\theta}|s)\phi(e^{i\theta};a,b|\xi;s)\frac{dy}{\sqrt{1-y^2}}. \tag{7.22}$$

In the limit $c\to a$ and $d\to b$, there appears the following relation in the distributional sense

$$\lim_{c\to a,d\to b} R(c,d,a,b;e^{i\varphi},e^{i\theta})=\frac{2\pi\sqrt{1-y^2}}{(p;p)_\infty(q;q)_\infty}\,\delta(v-y), \tag{7.23}$$

where $v=\cos\varphi$.

A double application of relation (7.19) with different parameters leads in an evident way to the self-reproducing property for the kernel

$$\kappa\int_{\mathbb{T}} R(a,b,c,d;x,w|s)R(c,d,e,f;w,z|s)\frac{dw}{w}=R(a,b,e,f;x,z|s) \tag{7.24}$$

and the biorthogonality relation

$$\int_{-1}^1 R(a,b,c,d;e^{i\varphi},e^{i\theta}|s)R(c,d,a,b;e^{i\theta},e^{i\varphi'}|s)\frac{dy}{\sqrt{1-y^2}}=\frac{\sqrt{1-v^2}}{(2i\kappa)^2}\,\delta(v-v'), \tag{7.25}$$

where $v=\cos\varphi$ and $v'=\cos\varphi'$. Substitution of the expression for R-function (7.21) in (7.25) results in the equality

$$\int_{-1}^1 \frac{1}{\Gamma(e^{\pm 2i\theta};p,q)}V\left(sa,sb,\sqrt{\frac{pq}{ab}}e^{i\varphi},\sqrt{\frac{pq}{ab}}e^{-i\varphi},\frac{pq}{c},\frac{pq}{d},\sqrt{\frac{cd}{\rho}}e^{i\theta},\sqrt{\frac{cd}{\rho}}e^{-i\theta}\right)$$
$$\times V\left(sc,sd,\sqrt{\frac{pq}{cd}}e^{i\theta},\sqrt{\frac{pq}{cd}}e^{-i\theta},\frac{pq}{as},\frac{pq}{bs},\sqrt{\frac{ab}{pq}}e^{i\varphi'},\sqrt{\frac{ab}{pq}}e^{-i\varphi'}\right)\frac{dy}{\sqrt{1-y^2}}$$
$$=\Gamma\left(\frac{ab}{pq},\frac{pq}{ab},\frac{cd}{pq},\frac{pq}{cd},e^{\pm 2i\varphi};p,q\right)\frac{\sqrt{1-v^2}}{(2i\kappa)^2}\,\delta(v-v'). \tag{7.26}$$

The parameters v and v' can be considered as continuous spectral variables in the operator formulation of the elliptic hypergeometric equation. Therefore relation (7.26) should follow from the latter equation, but the precise connection between them is not established yet.

8. Connection with the Sklyanin algebra

In [34], Rains introduced an interesting finite difference operator connected with the root system BC_n. For $n = 1$, it can be represented in the form

$$D(a, b, c, d; p; q) = \frac{\theta(az, bz, cz, dz; p)}{z\theta(z^2; p)} T_{z,q}^{1/2} + \frac{\theta(az^{-1}, bz^{-1}, cz^{-1}, dz^{-1}; p)}{z^{-1}\theta(z^{-2}; p)} T_{z,q}^{-1/2}, \tag{8.1}$$

where $T_{z,q}^{\pm 1/2} f(z) = f(q^{\pm 1/2} z)$ is the q-shift operator and a, b, c, d are arbitrary parameters. Later on Rains noticed also [35, 80] that this operator is equivalent to the general linear combination of four generators of the Sklyanin algebra $S_0, \ldots, S_3$ [81, 82].

Defining relations of the Sklyanin algebra have the form

$$\begin{aligned} S_\alpha S_\beta - S_\beta S_\alpha &= i(S_0 S_\gamma + S_\gamma S_0), \\ S_0 S_\alpha - S_\alpha S_0 &= i J_{\beta\gamma}(S_\beta S_\gamma + S_\gamma S_\beta), \end{aligned} \tag{8.2}$$

where $J_{\beta\gamma}$ are the structure constants of the algebra and (α, β, γ) is an arbitrary cyclic permutation of the triple $(1, 2, 3)$. A representation of S_a as finite difference operators has been found in [82]:

$$S_a = i^{\delta_{a,2}} \frac{\theta_{a+1}(\eta|\tau)}{\theta_1(2u|\tau)} \left(\theta_{a+1}(2u - 2g|\tau) e^{\eta\partial_u} - \theta_{a+1}(-2u - 2g|\tau) e^{-\eta\partial_u} \right),$$

where $e^{\pm\eta\partial_u} f(u) = f(u \pm \eta)$, and under the quantization condition $g = \ell\eta$, $\ell = 0, 1/2, 1, \ldots$, their action in the space of theta functions of the order 4ℓ was described. The combination of the generators

$$\begin{aligned} 2\Delta(a_1, a_2, a_3, a_4) := &\frac{\prod_{j=1}^3 \theta_1(a_j + a_4 + 2g)}{\theta_1(\eta)} S_0 - \frac{\prod_{j=1}^3 \theta_1(a_j + a_4 + 2g + \frac{1}{2})}{\theta_1(\eta + \frac{1}{2})} S_1 \\ &-ie^{\pi i(\frac{\tau}{2} + 2a_4 + 2g - \eta)} \frac{\prod_{j=1}^3 \theta_1(a_j + a_4 + 2g + \frac{1+\tau}{2})}{\theta_1(\eta + \frac{1+\tau}{2})} S_2 \\ &+e^{\pi i(\frac{\tau}{2} + 2a_4 + 2g - \eta)} \frac{\prod_{j=1}^3 \theta_1(a_j + a_4 + 2g + \frac{\tau}{2})}{\theta_1(\eta + \frac{\tau}{2})} S_3, \end{aligned}$$

with the normalization $\sum_{j=1}^4 a_j = -4g$, can be represented in the form [80]

$$\Delta(a_1, a_2, a_3, a_4) = \frac{\prod_{j=1}^4 \theta_1(a_j + u)}{\theta_1(2u)} e^{\eta\partial_u} + \frac{\prod_{j=1}^4 \theta_1(a_j - u)}{\theta_1(-2u)} e^{-\eta\partial_u}.$$

After the transition to multiplicative system of notation

$$(a, b, c, d) := e^{2\pi i a_{1,2,3,4}}, \quad \rho := abcd = e^{-8\pi i g}, \quad z := e^{2\pi i u}, \quad q := e^{4\pi i \eta}$$

there appears the operator described above (8.1):

$$\Delta(a_1, a_2, a_3, a_4) = \left(ip^{1/8}(p; p)_\infty \right)^3 e^{4\pi i g} D(a, b, c, d; p; q).$$

The standard eigenvalue problem $D\psi = \lambda\psi$ appears to be very complicated, since it represents a difference analogue of the Heun equation [79]. On the one hand, it is known that the eigenvalue problem for the one particle Hamiltonian of the Inozemtsev model [65] is equivalent to the Heun equation. On the other hand, the classical equations of motion for this Hamiltonian with the modular parameter τ considered as a time variable leads to the Painlevé VI equation [83]. The D-operator itself appears to be related to the van Diejen model [62, 79]. All this and the connection with elliptic hypergeometric functions described below demonstrate some mathematical universality of the operator (8.1).

Consider the generalized eigenvalue problem of the form

$$D(a,b,c,d;p;q)f(z;w;q^{1/2}a,q^{1/2}b;\rho) = \lambda(w)D(a,b,c',d';p;q)f(z;w;q^{1/2}a,q^{1/2}b;\rho), \tag{8.3}$$

where $cd = c'd'$. This equation is solved explicitly. Using the parameterization

$$\lambda(w) = \frac{\theta(w\sqrt{c/d}, w\sqrt{d/c};p)}{\theta(w\sqrt{c'/d'}, w\sqrt{d'/c'};p)},$$

for $|q| < 1$ we obtain [79]

$$f(z;w;a,b;\rho) = \Gamma\left(\frac{pq}{a}z^{\pm1}, \frac{pq}{b}z^{\pm1}, \sqrt{\frac{ab}{\rho}}w^{\pm1}z^{\pm1};p,q\right), \tag{8.4}$$

up to the multiplication by an arbitrary function $\varphi(z)$, $\varphi(qz) = \varphi(z)$. Evidently, function (8.4) coincides with the basis vector (7.20) after a change of parameters.

For functions invariant with respect to the transformation $z \to z^{-1}$, $\psi(z) = \psi(z^{-1})$, we define the scalar product

$$\langle \chi(z), \psi(z)\rangle = \kappa \int_{\mathbb{T}} \frac{\chi(z)\psi(z)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{z}. \tag{8.5}$$

Then the formally conjugated to D operator has the form

$$D^*(a,b,c,d;p;q) = \frac{cd}{q^{1/2}} D\left(\frac{pq^{1/2}}{a}, \frac{pq^{1/2}}{b}, \frac{q^{1/2}}{c}, \frac{q^{1/2}}{d};p;q\right).$$

The dual problem

$$D^*(a,b,c,d;p;q)g(z;v;a,b;\rho) = \lambda(v)D^*(a,b,c',d';p;q)g(z;v;a,b;\rho) \tag{8.6}$$

has a solution

$$g(z;v;a,b;\rho) = \Gamma\left(az^{\pm1}, bz^{\pm1}, \sqrt{\frac{\rho}{ab}}v^{\pm1}z^{\pm1};p,q\right), \tag{8.7}$$

which is also defined up to the multiplication by an arbitrary function $\varphi(z)$, $\varphi(qz) = \varphi(z)$. Now it is not difficult to see that the scalar product of functions (8.4) and (8.7) leads to the V-functions:

$$\begin{aligned}&\langle g(z;v;a,b;e), f(z;w;c,d;e)\rangle \\ &\quad = V\left(a,b,\sqrt{\frac{e}{ab}}v, \sqrt{\frac{e}{ab}}v^{-1}\frac{pq}{c}, \frac{pq}{d}, \sqrt{\frac{cd}{e}}w, \sqrt{\frac{cd}{e}}w^{-1};p,q\right).\end{aligned} \tag{8.8}$$

Thus, the elliptic analogue of the Gauss hypergeometric function appears to be directly related to the generalized eigenvalue problem for a linear combination of the Sklyanin algebra generators. It is necessary to note that our scalar product (8.5) is different from the Sklyanin invariant measure [82]. In [80], using the latter measure Rosengren has built an integral representation for the elliptic $6j$-symbols and proved the Sklyanin conjecture on the reproducing kernel for representations in the space of theta functions.

The non-uniqueness in the choices of functions f and g can be fixed by the requirement that these functions satisfy simultaneously the equations obtained from (8.3) and (8.6) by the permutation of p and q (because the equations $\varphi(qz) = \varphi(pz) = \varphi(z)$ lead to $\varphi(z) = const$). This means that we introduce into consideration a second copy of the Sklyanin algebra, obtained from the first one by the permutation of τ and 2η:

$$\tilde{S}_a = i^{\delta_{a,2}} \frac{\theta_{a+1}(\frac{\tau}{2}|2\eta)}{\theta_1(2u|2\eta)} \left(\theta_{a+1}(2u-2g|2\eta)e^{\frac{\tau}{2}\partial_u} - \theta_{a+1}(-2u-2g|2\eta)e^{-\frac{\tau}{2}\partial_u}\right).$$

For these two algebras the following cross-commutation relations are valid:

$$S_a\tilde{S}_b = \tilde{S}_b S_a, \quad a,b \in \{0,3\} \quad \text{or} \quad a,b \in \{1,2\},$$
$$S_a\tilde{S}_b = -\tilde{S}_b S_a, \quad a \in \{0,3\},\ b \in \{1,2\} \quad \text{or} \quad a \in \{1,2\},\ b \in \{0,3\}.$$

One can substitute in equations (8.3) and (8.6) the parameterization $z = e^{2\pi i u/\omega_2}$, $2\eta = \omega_1/\omega_2$, $\tau = \omega_3/\omega_2$, and to build their solutions well defined for $|q| = 1$ with the help of the modified elliptic gamma function $G(u;\omega)$. In this case the uniqueness of solutions can be reached by the requirement that they satisfy simultaneously to equations obtained from the original ones by the permutation of ω_1 and ω_2. This leads to another copy of the Sklyanin algebra, which is obtained from the first one by the transformations $\eta \to 1/(4\eta), u \to u/(2\eta), \tau \to \tau/(2\eta)$:

$$\tilde{S}_a = i^{\delta_{a,2}} \frac{\theta_{a+1}\left(\frac{1}{4\eta}\Big|\frac{\tau}{2\eta}\right)}{\theta_1\left(\frac{u}{\eta}\Big|\frac{\tau}{2\eta}\right)} \left(\theta_{a+1}\left(\frac{u-g}{\eta}\Big|\frac{\tau}{2\eta}\right)e^{\frac{1}{2}\partial_u} - \theta_{a+1}\left(\frac{-u-g}{\eta}\Big|\frac{\tau}{2\eta}\right)e^{-\frac{1}{2}\partial_u}\right).$$

In this case some of the generators S_a and $\tilde{S}_a$ anticommute with each other as well.

The described direct products of the Sklyanin algebra pairs can be considered as elliptic analogues of the Faddeev modular double $U_q(sl_2) \otimes U_{\tilde{q}^{-1}}(sl_2)$ [17, 84]. The latter double can be obtained from the second case in the limit $\mathrm{Im}(\tau) \to +\infty$ (in the first case this limit is not defined) [79].

9. Partial fraction decompositions and determinants

Expansions of different rational functions defined as ratios of two polynomials into partial fractions are used in the proofs of many identities for plain and q-hypergeometric series and integrals. At the elliptic level these rational functions are replaced by ratios of products of theta functions, and one searches for their expansions into sums of ratios of theta functions with the minimal number of poles. If the partial fraction expansion for an arbitrary rational function is a standard procedure, it is not so in the theta functions case. The first known relation of such a type follows from an identity given in [85] as an exercise.

Theorem 7. *Let $2n$ variables $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}^*$ satisfy the constraint $\prod_{k=1}^n a_k = \prod_{k=1}^n b_k$ and $a_j/a_k \neq p^k$, $k \in \mathbb{Z}$, for $j \neq k$. Then the following relation for theta functions is true:*

$$\sum_{k=1}^n \frac{\prod_{j=1}^n \theta(a_k/b_j;p)}{\prod_{j=1,\neq k}^n \theta(a_k/a_j;p)} = 0. \tag{9.1}$$

Proof. We replace in (9.1) n by $n+1$, denote $a_{n+1}=t$ and substitute $b_{n+1}=a_1\dots a_n t/b_1\dots b_n$. After taking out of the sum the $(n+1)$-st term, this relation can be rewritten in the form [86]

$$\prod_{k=1}^{n}\frac{\theta(t/b_k;p)}{\theta(t/a_k;p)}=\sum_{r=1}^{n}\frac{\theta(ta_1\cdots a_n/a_rb_1\cdots b_n;p)}{\theta(t/a_r,a_1\cdots a_n/b_1\cdots b_n;p)}\frac{\prod_{j=1}^n\theta(a_r/b_j;p)}{\prod_{j=1,\neq r}^n\theta(a_r/a_j;p)} \tag{9.2}$$

and interpreted as a partial fraction expansion over theta functions. Then the proof of this identity is rather elementary. For $n=2$ it is reduced to the addition formula (A.5). By induction it follows that the left-hand side can be decomposed into the sum

$$\sum_{r=1}^{n}c_r\frac{\theta(ta_1\cdots a_n/a_rb_1\cdots b_n;p)}{\theta(t/a_r;p)},$$

where the coefficients c_r are easily found after the multiplication by $\theta(t/a_r;p)$ and the choice $t=a_r$. □

Theorem 8. [87] *Let $2n$ variables $a_1,\dots,a_n,b_1,\dots,b_n\in\mathbb{C}^*$ satisfy the relations $a_ja_k, a_j/a_k\neq p^k$, $k\in\mathbb{Z}$, for $j\neq k$. Then the following identity for theta functions is true*

$$\sum_{k=1}^{n}\frac{a_k\prod_{j=1}^{n-2}\theta(a_kb_j^{\pm1};p)}{\prod_{j=1,\neq k}^n\theta(a_ka_j^{\pm1};p)}=0. \tag{9.3}$$

Proof. After the replacements $n\to n+1$ and $a_{n+1}\to t$ in (9.3) and singling the $(n+1)$-st term out of the sum, we obtain the partial fraction expansion of the form

$$\frac{\prod_{j=1}^{n-1}\theta(tb_j^{\pm1};p)}{\prod_{j=1}^{n}\theta(ta_j^{\pm1};p)}=\sum_{k=1}^{n}\frac{\prod_{j=1}^{n-1}\theta(a_kb_j^{\pm1};p)}{\theta(ta_k^{\pm1};p)\prod_{j=1,\neq k}^n\theta(a_ka_j^{\pm1};p)},$$

which is easily proved by induction. □

These expansions into “simple” fractions for theta functions were used in the papers [33, 36, 86] for the proof of some exact summation and integration formulae for elliptic hypergeometric functions. Let us describe also another expansion which was used recently in [59]:

$$\frac{\prod_{j=1}^{n+2}\theta(v_jz;p)}{z\theta(z^2;p)\prod_{i=1}^n\theta(u_iz^{-1};p)}+(z\to z^{-1})=\sum_{i=1}^{n}\frac{\prod_{j=1}^{n+2}\theta(u_iv_j;p)}{u_i\theta(u_iz^{\pm1};p)\prod_{k=1,\neq i}^n\theta(u_ku_i^{-1};p)},$$

where $\prod_{i=1}^n u_i\prod_{j=1}^{n+2}v_j=p^{n-1}$.

An elliptic analogue of the Cauchy determinant has the form

$$\det_{1\le i,j\le n}\left(\frac{1}{a_i^{-1}\theta(a_iz_j^{\pm1};p)}\right) \tag{9.4}$$
$$=(-1)^{n(n-1)/2}\frac{\prod_{1\le i<j\le n}a_i^{-1}\theta(a_ia_j^{\pm1};p)\prod_{1\le i<j\le n}z_i^{-1}\theta(z_iz_j^{\pm1};p)}{\prod_{1\le i,j\le n}a_i^{-1}\theta(a_iz_j^{\pm1};p)}.$$

The Frobenius determinant has the form

$$\det_{1\le i,j\le n}\left(\frac{\theta(ta_ib_j;p)}{\theta(t,a_ib_j;p)}\right)=\frac{\theta(t\prod_{i=1}^n a_ib_i;p)}{\theta(t;p)}\frac{\prod_{1\le i<j\le n}a_jb_j\theta(a_i/a_j,b_i/b_j;p)}{\prod_{1\le i,j\le n}\theta(a_ib_j;p)}. \tag{9.5}$$

In [53], Warnaar suggested a new determinant for theta functions

$$\det_{1\le i,j\le n}\left(\frac{\theta(ax_i, ac/x_i)_{n-j}}{\theta(bx_i, bc/x_i)_{n-j}}\right) \tag{9.6}$$
$$= a^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{1\le i<j\le n} x_j\theta(x_ix_j^{-1}, cx_i^{-1}x_j^{-1}; p)\prod_{i=1}^{n}\frac{\theta(b/a, abcq^{2n-2i})_{i-1}}{\theta(bx_i, bc/x_i)_{n-1}},$$

where $\theta(a)_n = \prod_{j=0}^{n-1}\theta(aq^j; p)$. For $p \to 0$, it reduces to the Krattenthaler determinant [88].

Formulae (9.4)–(9.6) are used in [11, 34, 53, 89, 90, 91] and some other papers as auxiliary tools for proving necessary elliptic hypergeometric identities. Partial fraction decompositions and determinants are somewhat equivalent to each other. For instance, if one expands determinant (9.5) along the last row and evaluates each term by the same formula (9.5) in the smaller dimension $n-1$, then there appears an identity equivalent to (9.2). Therefore formula (9.5) follows from (9.2) by induction on n, and vice versa. In a similar way, formula (9.4) is equivalent to (9.3) [91]. Applications of determinants at the level of q-hypergeometric functions are described, for example, in [20]. In the paper [92], Rosengren and Schlosser have systematically considered determinants of theta functions on root systems (in particular, this paper contains a detailed list of references on this subject) and constructed a number of new exactly computable cases, which we skip for brevity. For applications of elliptic determinants to some problems of the number theory, combinatorics, and statistical mechanics, see [93, 94].

10. The elliptic beta integrals on root systems

10.1. Integrals for the root system C_n.

There are two different generalizations of the elliptic beta integral (3.1) to multiple integrals for the root system C_n (or BC_n), suggested by van Diejen and the author [32, 33]. We describe first the multiparameter integral of type I.

Theorem 9. *Let $z_1, \dots, z_n \in \mathbb{T}$ and complex parameters $t_1, \dots, t_{2n+4}$ and p, q satisfy the constraints $|p|, |q|, |t_j| < 1$ and $\prod_{j=1}^{2n+4} t_j = pq$. Then*

$$\kappa_n\int_{\mathbb{T}^n}\prod_{1\le j<k\le n}\frac{1}{\Gamma(z_j^{\pm1}z_k^{\pm1}; p, q)}\prod_{j=1}^{n}\frac{\prod_{m=1}^{2n+4}\Gamma(t_mz_j^{\pm1}; p, q)}{\Gamma(z_j^{\pm2}; p, q)}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$
$$= \prod_{1\le m<s\le 2n+4}\Gamma(t_mt_s; p, q), \qquad \kappa_n = \frac{(p;p)_\infty^n(q;q)_\infty^n}{(4\pi i)^n n!}. \tag{10.1}$$

Proof. We consider the function

$$\rho(z, t; C_n) = \prod_{1\le i<j\le n}\frac{1}{\Gamma(z_i^{\pm1}z_j^{\pm1}; p, q)}\prod_{i=1}^{n}\frac{\prod_{m=1}^{2n+3}\Gamma(t_mz_i^{\pm1}; p, q)}{\Gamma(z_i^{\pm2}, Az_i^{\pm1}; p, q)}$$
$$\times\frac{\prod_{m=1}^{2n+3}\Gamma(At_m^{-1}; p, q)}{\prod_{1\le m<s\le 2n+3}\Gamma(t_mt_s; p, q)}, \tag{10.2}$$

where $A = \prod_{m=1}^{2n+3} t_m$. For all z_i there are poles of (10.2) in the points

$$\mathcal{P} = \{t_mq^ap^b,\ A^{-1}q^{a+1}p^{b+1}\}_{m=1,\dots,2n+3,\, a,b=0,1,\dots}$$

converging to zero. The coordinates of the poles going to infinity form the set $\mathcal{P}^{-1}$. Then the statement of the theorem may be rewritten in the form

$$\int_{C^n} \rho(z,t;C_n)\frac{dz}{z} = \frac{(4\pi i)^n n!}{(q;q)_\infty^n (p;p)_\infty^n}, \qquad \frac{dz}{z} := \prod_{j=1}^n \frac{dz_j}{z_j}, \tag{10.3}$$

where the contour C in an arbitrary deformation of $\mathbb{T}$ separating $\mathcal{P}$ and $\mathcal{P}^{-1}$.

The integral kernel $\rho(z,t;C_n)$ satisfies the equation analogous to (3.4):

$$\begin{aligned}\rho(z,qt_1,t_2,\ldots,t_{2n+3};C_n) - \rho(z,t;C_n)\\ = \sum_{i=1}^n \big(g_i(z_1,\ldots,q^{-1}z_i,\ldots,z_n,t) - g_i(z,t)\big),\end{aligned} \tag{10.4}$$

where

$$g_i(z,t) = \rho(z,t;C_n) \prod_{j=1,\neq i}^n \frac{\theta(t_1 z_j^{\pm 1};p)}{\theta(z_i z_j^{\pm 1};p)} \frac{\prod_{m=1}^{2n+3}\theta(t_m z_i;p)}{\prod_{m=2}^{2n+3}\theta(t_1 t_m;p)} \frac{\theta(t_1 A;p)}{\theta(z_i^2, Az_i;p)} \frac{t_1}{z_i}. \tag{10.5}$$

Dividing equation (10.4) by $\rho(z,t;C_n)$, we obtain

$$\begin{aligned}\prod_{i=1}^n \frac{\theta(t_1 z_i^{\pm 1};p)}{\theta(Az_i^{\pm 1};p)} \prod_{m=2}^{2n+3} \frac{\theta(At_m^{-1};p)}{\theta(t_1 t_m;p)} - 1 = \frac{t_1\theta(t_1 A;p)}{\prod_{m=2}^{2n+3}\theta(t_1 t_m;p)} \sum_{i=1}^n \frac{1}{z_i\theta(z_i^2;p)}\\ \times \prod_{j=1,\neq i}^n \frac{\theta(t_1 z_j^{\pm 1};p)}{\theta(z_i z_j^{\pm 1};p)} \left(z_i^{2n+2} \frac{\prod_{m=1}^{2n+3}\theta(t_m z_i^{-1};p)}{\theta(Az_i^{-1};p)} - \frac{\prod_{m=1}^{2n+3}\theta(t_m z_i;p)}{\theta(Az_i;p)}\right)\end{aligned} \tag{10.6}$$

Both sides of this equality are invariant under the transformation $z_1 \to pz_1$ and have equal sets of poles (singularities at the points $z_1 = z_j, z_j^{-1}, j = 2,\ldots,n$, and $z_1 = \pm p^{k/2},\ k \in \mathbb{Z}$ on the right-hand side are cancelled) with their residues. Therefore the functions on both sides of the equality (10.6) differ only by an additive constant, independent on z_1. This constant equals to zero which follows from a trivial check of equality (10.6) at $z_1 = t_1$.

Integrating (10.4) over the variables $z \in C^n$, we obtain

$$I(qt_1,t_2,\ldots,t_{2n+3}) - I(t) = \sum_{i=1}^n \left(\int_{C^{i-1}\times(q^{-1}C)\times C^{n-i}} - \int_{C^n}\right) g_i(z,t)\frac{dz}{z}, \tag{10.7}$$

where $I(t) = \int_{C^n}\rho(z,t;C_n)dz/z$ and $q^{-1}C$ denotes the contour C dilated with respect to the zero point.

Poles of the function (10.5) in variable z_i converge to zero along the point $z_i = t_m q^a p^b,\ A^{-1}q^a p^{b+1}$ and diverge to infinity at $z_i = t_m^{-1}q^{-1-a}p^{-b}, Aq^{-a}p^{-b-1}$, where $m = 1,\ldots,2n+3,\ a,b = 0,1,\ldots$. For $|t_m| < 1$ and $|p| < |A|$ the region $1 \le |z_i| \le |q|^{-1}$ does not contain the poles, so that we can set $C = \mathbb{T}$, deform back $q^{-1}\mathbb{T}$ to $\mathbb{T}$ in (10.7) and obtain the equality $I(qt_1,t_2,\ldots,t_{2n+3}) = I(t)$.

Repeating almost literally the procedure of analytical continuation used in the $n = 1$ case, we find that I is a constant, which depends only on p and q. Its value is found by considering the limits $t_j t_{j+n} \to 1,\ j = 1,\ldots,n$, analogous to the $n = 1$ case, which yields the right-hand side of (10.3). □

Formula (10.1) was suggested and partially justified in [33], and it was proved completely by different methods in [34, 43, 59]. In a special limit $p \to 0$, it is reduced to one of the integration formulae of Gustafson [95].

The root system C_n consists of the set of vectors from $\mathbb{R}^n$ of the form $X(C_n) = \{\pm 2e_i, \pm e_i \pm e_j,\, i<j\}\big|_{i,j=1,\ldots,n}$, where e_i is an orthonormal basis of $\mathbb{R}^n$. Denoting $z_i = \exp(e_i)$, we see that the denominator of integral's kernel (10.1) contains a product over roots of C_n of the form

$$\prod_{\alpha\in X(C_n)} \Gamma(\mathrm{e}^{\alpha};p,q) = \prod_{i=1}^{n} \Gamma(z_i^{\pm 2};p,q) \prod_{1\le i<j\le n} \Gamma(z_i^{\pm1} z_j^{\pm1};p,q).$$

The root system BC_n contains additionally the vectors $\{\pm e_1,\ldots,\pm e_n\}$, but the general rules of the appearance of these vectors in integrals' kernels are not established yet. The Weyl group of these systems $S_n \times \mathbb{Z}_2^n$ is a symmetry of the integral kernel.

The C_n-elliptic beta integral of type II is built with the help of formula (10.1) by a purely algebraic means [33].

Theorem 10. *Let complex parameters $t, t_m (m = 1,\ldots,6), p$ and q satisfy conditions $|p|, |q|, |t|, |t_m| < 1$, and $t^{2n-2}\prod_{m=1}^{6} t_m = pq$. Then*

$$\kappa_n \int_{\mathbb{T}^n} \prod_{1\le j<k\le n} \frac{\Gamma(t z_j^{\pm1} z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1} z_k^{\pm1};p,q)} \prod_{j=1}^{n} \frac{\prod_{m=1}^{6}\Gamma(t_m z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)} \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}$$

$$= \prod_{j=1}^{n} \left(\frac{\Gamma(t^j;p,q)}{\Gamma(t;p,q)} \prod_{1\le m<s\le 6} \Gamma(t^{j-1} t_m t_s;p,q) \right). \tag{10.8}$$

Proof. We denote the integral on the left-hand side of (10.8) as $I_n(t,t_1,\ldots,t_5)$ and consider the $(2n-1)$-tuple integral

$$\kappa_n\kappa_{n-1} \int_{\mathbb{T}^{2n-1}} \prod_{1\le j<k\le n} \frac{1}{\Gamma(z_j^{\pm1} z_k^{\pm1};p,q)} \prod_{j=1}^{n} \frac{\prod_{r=0}^{5}\Gamma(t_r z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}$$
$$\times \prod_{\substack{1\le j\le n\\ 1\le k\le n-1}} \Gamma(t^{1/2} z_j^{\pm1} w_k^{\pm1};p,q) \prod_{1\le j<k\le n-1} \frac{1}{\Gamma(w_j^{\pm1} w_k^{\pm1};p,q)}$$
$$\times \prod_{j=1}^{n-1} \frac{\Gamma(w_j^{\pm1} t^{n-3/2}\prod_{s=1}^{5} t_s;p,q)}{\Gamma(w_j^{\pm2}, w_j^{\pm1} t^{2n-3/2}\prod_{s=1}^{5} t_s;p,q)} \frac{dw_1}{w_1}\cdots\frac{dw_{n-1}}{w_{n-1}} \frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n}, \tag{10.9}$$

with p, q, t and t_r, $r = 0,\ldots,5$, lying inside the unit circle such that $t^{n-1}\prod_{r=0}^{5} t_r = pq$. Integration over the variables w_j with the help of formula (10.1) brings the expression (10.9) to the form $\Gamma^n(t;p,q)\Gamma^{-1}(t^n;p,q)\ I_n(t,t_1,\ldots,t_5)$ (where it is assumed that $t_6 = pqt^{2-2n}/\prod_{j=1}^{5} t_j$). Because the integrand is bounded on the contour of integration, we can change the order of integrations. Then the integration over the z_k-variables with the help of formula (10.1) converts expression (10.9) to

$$\Gamma^{n-1}(t;p,q) \prod_{0\le r<s\le 5} \Gamma(t_r t_s;p,q)\, I_{n-1}(t, t^{1/2}t_1,\ldots,t^{1/2}t_5).$$

As a result, we obtain a recurrence relation connecting integrals of different dimension n:

$$I_n(t,t_1,\ldots,t_5) = \frac{\Gamma(t^n;p,q)}{\Gamma(t;p,q)} \prod_{0\le r<s\le 5} \Gamma(t_r t_s;p,q)\ I_{n-1}(t, t^{1/2}t_1,\ldots,t^{1/2}t_5).$$

Using known initial condition at $n = 1$ (3.1), we find (10.8) by recursion. $\square$

If one expresses t_6 via other parameters and removes the multipliers pq in the arguments of the elliptic gamma functions with the help of the reflection formula, then it is easy to pass to the limit $p \to 0$ for fixed parameters. This leads to a multiple q-beta integral of Gustafson [96]. A number of other limiting transitions in parameters leads to the Selberg integral – a fundamentally important integral with a large number of applications in mathematical physics [97]:

$$\int_0^1 \cdots \int_0^1 \prod_{1\le j\le n} x_j^{\alpha-1}(1-x_j)^{\beta-1} \prod_{1\le j<k\le n} |x_j - x_k|^{2\gamma}\, dx_1 \cdots dx_n$$
$$= \prod_{1\le j\le n} \frac{\Gamma(\alpha+(j-1)\gamma)\Gamma(\beta+(j-1)\gamma)\Gamma(1+j\gamma)}{\Gamma(\alpha+\beta+(n+j-2)\gamma)\Gamma(1+\gamma)}, \tag{10.10}$$

where $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\gamma) > -\min(1/n, \mathrm{Re}(\alpha)/(n-1), \mathrm{Re}(\beta)/(n-1))$.

Therefore formula (10.8) represents an elliptic analogue of the Selberg integral. It can also be interpreted as an elliptic generalization of the Macdonald–Morris constant term identities for the BC_n-root system [98]. The given proof is taken from the work [33]. It models the proof of the Selberg integral suggested by Anderson [99] and represents a generalization of the method used by Gustafson in [96] for proving the corresponding q-beta integral.

10.2. Integrals for the root system A_n.

Different elliptic beta integrals on the root system A_n have been proposed in papers [11, 36]. By analogy with the C_n-cases integration formulae depending on $2n+3$ parameters will be considered as the type I integrals. Exactly computable integrals with a smaller number of parameters, which are derived with the help of type I integrals, will be classified as the type II integrals. Let us list these formulae omitting their derivations.

Theorem 11. *Let $|p|, |q| < 1$ and $2n+4$ parameters t_m, s_m, $m = 1, \dots, n+2$, satisfy the constraints $|t_m|, |s_m| < 1$ and $ST = pq$, where $S = \prod_{m=1}^{n+2} s_m$ and $T = \prod_{m=1}^{n+2} t_m$. Then*

$$\kappa_n^A \int_{\mathbb{T}^n} \prod_{1\le j<k\le n+1} \frac{1}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \prod_{j=1}^{n+1} \prod_{m=1}^{n+2} \Gamma(s_m z_j, t_m z_j^{-1}; p, q) \frac{dz}{z}$$
$$= \prod_{m=1}^{n+2} \Gamma(S s_m^{-1}, T t_m^{-1}; p, q) \prod_{k,m=1}^{n+2} \Gamma(s_k t_m; p, q), \tag{10.11}$$

where $z_1 z_2 \cdots z_{n+1} = 1$ and

$$\kappa_n^A = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(2\pi i)^n (n+1)!}.$$

In this type I integral we have a split of $2n+4$ parameters (homogeneous in the C_n-case) with the fixed product into two groups with $n+2$ elements. This integration formula was proposed and partially justified in [11], various different complete proofs are given in [34, 43]. In the simplest possible $p \to 0$ limit there appears one of the Gustafson integrals [95].

The root system A_n consists of the vectors $X(A_n) = \{e_i - e_j,\ i \neq j\}\big|_{i,j=1,\dots,n+1}$, where e_i is an orthonormal basis of $\mathbb{R}^{n+1}$. These vectors lie in the hyperplane orthogonal to the vector $E = \sum_{i=1}^{n+1} e_i$. Setting $z_i = \exp(e_i - E/(n+1))$, we obtain $z_1 \cdots z_{n+1} = 1$. The denominator of the kernel of integral (10.11) contains a product of the form

$$\prod_{\alpha\in X(A_n)} \Gamma(\mathrm{e}^{\alpha}; p, q) = \prod_{1\leq i<j\leq n+1} \Gamma(z_i/z_j, z_j/z_i; p, q).$$

The full integral kernel is invariant with respect to the A_n-Weyl group S_{n+1}.

There exists an additional independent A_n-integral of type I [36].

Theorem 12. *Let $|p|, |q| < 1$ and $2n+3$ parameters t_k, $k = 1, 2, \dots, n$, and s_m, $m = 1, 2, \dots, n+3$, satisfy the constraints $|t_k| < 1$, $|s_m| < 1$ and $|pq| < |St_k|$, where $S = \prod_{m=1}^{n+3} s_m$. Then the following explicit integration formula is true:*

$$\kappa_n^A \int_{\mathbb{T}^n} \prod_{1\leq i<j\leq n+1} \frac{\Gamma(Sz_i^{-1}z_j^{-1}; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \times \prod_{j=1}^{n+1} \frac{\prod_{k=1}^{n} \Gamma(t_k z_j; p, q) \prod_{m=1}^{n+3} \Gamma(s_m z_j^{-1}; p, q)}{\prod_{k=1}^{n} \Gamma(St_k z_j^{-1}; p, q)} \frac{dz}{z}$$
$$= \prod_{k=1}^{n} \prod_{m=1}^{n+3} \frac{\Gamma(t_k s_m; p, q)}{\Gamma(St_k s_m^{-1}; p, q)} \prod_{1\leq l<m\leq n+3} \Gamma(Ss_l^{-1}s_m^{-1}; p, q), \quad (10.12)$$

where $z_1 \cdots z_{n+1} = 1$.

Here we have a split of $2n+3$ independent variables into two homogeneous groups with n and $n+3$ elements. In the limit $p \to 0$ there appears a q-beta integral which was not considered in the literature until the derivation of the elliptic case, as well as its plain hypergeometric degeneration appearing in the $q \to 1$ limit.

There are several A_n-elliptic beta integrals of type II [11]. We denote

$$I^{II}(\mathbf{t}, \mathbf{s}; p, q; A_n) = \kappa_n^A \int_{\mathbb{T}^n} \prod_{1\leq i<j\leq n+1} \frac{\Gamma(tz_i z_j; p, q)}{\Gamma(z_i z_j^{-1}, z_i^{-1} z_j; p, q)} \times \prod_{j=1}^{n+1} \prod_{k=1}^{n+1} \Gamma(t_k z_j^{-1}; p, q) \prod_{i=1}^{4} \Gamma(s_i z_j; p, q) \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}, \quad (10.13)$$

where $|t_k|$, $|s_i| < 1$, $k = 1, \dots, n+1$, $i = 1, \dots, 4$, and

$$t^{n-1} \prod_{k=1}^{n+1} t_k \prod_{i=1}^{4} s_i = pq, \qquad \prod_{j=1}^{n+1} z_j = 1.$$

Theorem 13. *Under the described restrictions on the parameters the following A_n-integration formulae are true. For odd n, we have*

$$I^{II}(\mathbf{t}, \mathbf{s}; p, q; A_n) = \prod_{1\leq j<k\leq n+1} \Gamma(tt_j t_k; p, q) \prod_{k=1}^{n+1} \prod_{i=1}^{4} \Gamma(t_k s_i; p, q) \times \frac{\Gamma(t^{\frac{n+1}{2}}, A; p, q)}{\Gamma(t^{\frac{n+1}{2}} A; p, q)} \prod_{1\leq i<m\leq 4} \Gamma(t^{\frac{n-1}{2}} s_i s_m; p, q), \quad (10.14)$$

where $A=\prod_{k=1}^{n+1} t_k$*; for even* n

$$I^{II}(\mathbf{t},\mathbf{s};p,q;A_n)=\prod_{1\leq j<k\leq n+1}\Gamma(tt_jt_k;p,q)\prod_{k=1}^{n+1}\prod_{i=1}^{4}\Gamma(t_ks_i;p,q)$$
$$\times\Gamma(A;p,q)\prod_{i=1}^{4}\frac{\Gamma(t^{\frac{n}{2}}s_i;p,q)}{\Gamma(t^{\frac{n}{2}}As_i;p,q)}. \tag{10.15}$$

These formulae contain only $n+5$ free parameters. They can be derived as direct consequences of the elliptic beta integrals of type I (10.1) and (10.11) [11]. In the simplest limit $p\to 0$ they are reduced to the Gustafson-Rakha q-beta integrals [100].

In order to describe a different A_n-integral of type II, we need ten complex parameters $p,q,\, t,s,\, t_1,t_2,t_3,\, s_1,s_2,s_3$ with the constraints $(ts)^{n-1}\prod_{k=1}^{3}t_ks_k=pq$ and $|p|,|q|,|t|,|s|,|t_k|,|s_k|<1$. We define now the integral

$$I^{II}(\mathbf{t},\mathbf{s};p,q;A_n)= \tag{10.16}$$
$$\kappa_n^A\int_{\mathbb{T}^n}\prod_{1\leq i<j\leq n+1}\frac{\Gamma(tz_iz_j,sz_i^{-1}z_j^{-1};p,q)}{\Gamma(z_iz_j^{-1},z_i^{-1}z_j;p,q)}\prod_{j=1}^{n+1}\prod_{k=1}^{3}\Gamma(t_kz_j,s_kz_j^{-1};p,q)\frac{dz}{z}.$$

Theorem 14. *Under the indicated restrictions on the parameters the following integration formulae for the root system* A_n *are true. For odd* n*, we have*

$$I^{II}(\mathbf{t},\mathbf{s};p,q;A_n)=\Gamma(t^{\frac{n+1}{2}},s^{\frac{n+1}{2}};p,q)\prod_{1\leq i<k\leq 3}\Gamma(t^{\frac{n-1}{2}}t_it_k,s^{\frac{n-1}{2}}s_is_k;p,q)$$
$$\times\prod_{j=1}^{(n+1)/2}\prod_{i,k=1}^{3}\Gamma((ts)^{j-1}t_is_k;p,q) \tag{10.17}$$
$$\times\prod_{j=1}^{(n-1)/2}\left(\Gamma((ts)^j;p,q)\prod_{1\leq i<k\leq 3}\Gamma(t^{j-1}s^jt_it_k,t^js^{j-1}s_is_k;p,q)\right);$$

for even n

$$I^{II}(\mathbf{t},\mathbf{s};p,q;A_n)=\prod_{i=1}^{3}\Gamma(t^{\frac{n}{2}}t_i,s^{\frac{n}{2}}s_i;p,q)$$
$$\times\Gamma(t^{\frac{n}{2}-1}t_1t_2t_3,s^{\frac{n}{2}-1}s_1s_2s_3;p,q)\prod_{j=1}^{n/2}\Bigg(\Gamma((ts)^j;p,q) \tag{10.18}$$
$$\times\prod_{i,k=1}^{3}\Gamma((ts)^{j-1}t_is_k;p,q)\prod_{1\leq i<k\leq 3}\Gamma(t^{j-1}s^jt_it_k,t^js^{j-1}s_is_k;p,q)\Bigg).$$

These formulae can be derived as direct consequences of the C_n-elliptic beta integrals of type I and II and the A_n-integral of type I (10.11) [11]. In the simplest possible limit $p\to 0$ these integrals are reduced to the Gustafson q-hypergeometric integrals [96].

11. Some multiple series summation formulae

There are several known summation formulae for multiple elliptic hypergeometric series representing multivariate extensions of the Frenkel–Turaev sum [11, 34, 53, 86, 90, 101, 102]. A direct connection with the sums of residues of particular pole sequences of the kernels of elliptic beta integrals on root systems was established for some of them. Let us describe some of these formulae without proofs.

Theorem 15. *Let $|p|<1$ and the parameter $q\in\mathbb{C}$ is not equal to an integer power of p. Then the following summation formula is true*

$$\sum_{\substack{0\leq\lambda_j\leq N_j\\ j=1,\ldots,n}} q^{\sum_{j=1}^n j\lambda_j} \prod_{1\leq j<k\leq n} \frac{\theta(t_jt_kq^{\lambda_j+\lambda_k}, t_jt_k^{-1}q^{\lambda_j-\lambda_k};p)}{\theta(t_jt_k, t_jt_k^{-1};p)} \tag{11.1}$$
$$\times \prod_{1\leq j\leq n}\left(\frac{\theta(t_j^2q^{2\lambda_j};p)}{\theta(t_j^2;p)}\prod_{0\leq r\leq 2n+3}\frac{\theta(t_jt_r)_{\lambda_j}}{\theta(qt_jt_r^{-1})_{\lambda_j}}\right)$$
$$= \theta(qa^{-1}b^{-1}, qa^{-1}c^{-1}, qb^{-1}c^{-1})_{N_1+\cdots+N_n}$$
$$\times \prod_{1\leq j<k\leq n}\frac{\theta(qt_jt_k)_{N_j}\theta(qt_jt_k)_{N_k}}{\theta(qt_jt_k)_{N_j+N_k}}$$
$$\times \prod_{1\leq j\leq n}\frac{\theta(qt_j^2)_{N_j}}{\theta(qt_ja^{-1}, qt_jb^{-1}, qt_jc^{-1}, q^{1+N_1+\cdots+N_n-N_j}t_j^{-1}a^{-1}b^{-1}c^{-1})_{N_j}},$$

where $\theta(a)_\lambda=\Gamma(aq^\lambda;p,q)/\Gamma(a;p,q)$ *and*

$$q^{-1}\prod_{r=0}^{2n+3}t_r=1 \qquad \textit{(the balancing condition)},$$
$$q^{N_j}t_jt_{n+j}=1, \qquad j=1,\ldots,n \quad \textit{(the termination condition)},$$

with $N_j=0,1,\ldots$, $j=1,\ldots,n$, *and* $a=t_{2n+1}, b=t_{2n+2}, c=t_{2n+3}$.

This formula was deduced in [101] from the C_n-elliptic beta integral of type I (which was not proven at that moment yet). Its first recursive proof was obtained in [86]. In the limit $p\to 0$ it degenerates to a multivariate ${}_8\varphi_7$-sum, which was found in [103].

Theorem 16. *Let $N=0,1,\ldots$ and the parameters $p,q,t,t_0,\ldots,t_5\in\mathbb{C}$ satisfy the restrictions $|p|<1$ and*

$$\begin{aligned} q^{-1}t^{2n-2}\prod_{r=0}^5 t_r&=1 \quad \textit{(the balancing condition)},\\ q^Nt^{n-1}t_0t_4&=1 \quad \textit{(the termination condition)}. \end{aligned} \tag{11.2}$$

Then the following summation formula for a multiple elliptic hypergeometric series is true

$$\sum_{0\leq\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq N} q^{\sum_{j=1}^m\lambda_j}t^{2\sum_{j=1}^m(n-j)\lambda_j}\prod_{1\leq j<k\leq m}\left(\frac{\theta(\tau_k\tau_jq^{\lambda_k+\lambda_j}, \tau_k\tau_j^{-1}q^{\lambda_k-\lambda_j};p)}{\theta(\tau_k\tau_j, \tau_k\tau_j^{-1};p)}\right.$$
$$\left.\times\frac{\theta(t\tau_k\tau_j)_{\lambda_k+\lambda_j}}{\theta(qt^{-1}\tau_k\tau_j)_{\lambda_k+\lambda_j}}\frac{\theta(t\tau_k\tau_j^{-1})_{\lambda_k-\lambda_j}}{\theta(qt^{-1}\tau_k\tau_j^{-1})_{\lambda_k-\lambda_j}}\right)$$
$$\times\prod_{j=1}^m\left(\frac{\theta(\tau_j^2q^{2\lambda_j};p)}{\theta(\tau_j^2;p)}\prod_{r=0}^5\frac{\theta(t_r\tau_j)_{\lambda_j}}{\theta(qt_r^{-1}\tau_j)_{\lambda_j}}\right)$$
$$= \prod_{j=1}^n\frac{\theta(qt^{n+j-2}t_0^2)_N\prod_{1\leq r<s\leq 3}\theta(qt^{1-j}t_r^{-1}t_s^{-1})_N}{\theta(qt^{2-n-j}\prod_{r=0}^3t_r^{-1})_N\prod_{r=1}^3\theta(qt^{j-1}t_0t_r^{-1})_N}. \tag{11.3}$$

Here we use the notation $\tau_j=t_0t^{j-1}$, $j=1,\ldots,n$.

This summation formula was suggested by Warnaar [53]. It follows from the C_n-elliptic beta integral of the type II [32] and was proven for the first time recursively in the paper [45].

For the root system A_n the type I elliptic hypergeometric sum has the following form:

$$\sum_{\substack{0\leq\lambda_j\leq N_j\\ j=1,\ldots,n}} q^{\sum_{j=1}^n j\lambda_j}\prod_{j=1}^n\frac{\theta(t_jq^{\lambda_j+|\lambda|};p)}{\theta(t_j;p)}\prod_{1\leq i<j\leq n}\frac{\theta(t_it_j^{-1}q^{\lambda_i-\lambda_j};p)}{\theta(t_it_j^{-1};p)}$$
$$\times\prod_{i,j=1}^n\frac{\theta(t_it_j^{-1}q^{-N_j})_{\lambda_i}}{\theta(qt_it_j^{-1})_{\lambda_i}}\prod_{j=1}^n\frac{\theta(t_j)_{|\lambda|}}{\theta(t_jq^{1+N_j})_{|\lambda|}}$$
$$\times\frac{\theta(b,c)_{|\lambda|}}{\theta(q/d,q/e)_{|\lambda|}}\prod_{j=1}^n\frac{\theta(dt_j,et_j)_{\lambda_j}}{\theta(t_jq/b,t_jq/c)_{\lambda_j}}$$
$$=\frac{\theta(q/bd,q/cd)_{|N|}}{\theta(q/d,q/bcd)_{|N|}}\prod_{j=1}^n\frac{\theta(t_jq,t_jq/bc)_{N_j}}{\theta(t_jq/b,t_jq/c)_{N_j}},\tag{11.4}$$

where $|\lambda|=\lambda_1+\cdots+\lambda_n$, $|N|=N_1+\cdots+N_n$ and $bcde=q^{1+|N|}$. This formula was proven recursively in [86]. It follows also from the analysis of residues for the type I integral (10.11) [11] and for $p\to 0$ reduces to the multiple ${}_8\varphi_7$-sum of Milne [104].

Omitting a number of other established summation formulae for multiple elliptic hypergeometric series, we describe a hypothesis from [11].

Conjecture. *Let $|p|<1$, $N=0,1,\ldots,$ and n parameters $t_k\in\mathbb{C}$, $k=1,\ldots,n$, satisfy the balancing condition $\prod_{k=1}^n t_k=q^{-N}$. Then the following summation formula is true:*

$$\sum_{\substack{\lambda_k=0,\ldots,N\\ \lambda_1+\ldots+\lambda_n=N}}\frac{\prod_{1\leq i<j\leq n}\theta(tt_it_j)_{\lambda_i+\lambda_j}\prod_{i=1}^n\prod_{j=n+1}^{n+3}\theta(tt_it_j)_{\lambda_i}\prod_{i,j=1}^n\theta(t_it_j^{-1})_{-\lambda_j}}{\prod_{i,j=1;i\neq j}^n\theta(t_it_j^{-1})_{\lambda_i-\lambda_j}\prod_{j=1}^n\theta(t^{n+1}t_j^{-1}\prod_{k=1}^{n+3}t_k)_{-\lambda_j}}$$
$$=\begin{cases}\dfrac{\theta(1)_{-N}}{\theta(t^{n/2})_{-N}\prod_{n+1\leq i<j\leq n+3}\theta(t^{(n+2)/2}t_it_j)_{-N}}, & n\quad \text{even},\\[2ex] \dfrac{\theta(1)_{-N}}{\prod_{i=n+1}^{n+3}\theta(t^{(n+1)/2}t_i)_{-N}\theta(t^{(n+3)/2}\prod_{i=n+1}^{n+3}t_i)_{-N}}, & n\quad \text{odd}.\end{cases}\tag{11.5}$$

It is conjectured also that this formula is related to sums of residues for the A_n-integral (10.13). For $p\to 0$ there appears a Gustafson–Rakha summation formula for a multiple ${}_8\varphi_7$-series [100].

12. Symmetry transformations for multiple integrals

General elliptic hypergeometric integrals of type I for the root system C_n have the form

$$I_n^{(m)}(t_1,\ldots,t_{2n+2m+4})$$
$$=\kappa_n\int_{\mathbb{T}^n}\prod_{1\leq i<j\leq n}\frac{1}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}\prod_{j=1}^n\frac{\prod_{i=1}^{2n+2m+4}\Gamma(t_iz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{z_j},$$

where $|t_j| < 1$,

$$\prod_{j=1}^{2n+2m+4} t_j = (pq)^{m+1}, \qquad \kappa_n = \frac{(p;p)_\infty^n (q;q)_\infty^n}{(4\pi i)^n n!}.$$

For $n = 0$ we set $I_0^{(m)} = 1$. The integral $I_1^{(1)}$ coincides with the V-function, and $I_n^{(0)}$ – with the elliptic beta integral (10.1). After degeneration of $I_n^{(m)}$-functions to the level of ordinary beta integral, they reduce to the Dixon integrals [28, 105]. In [34], Rains proved the following transformation formula:

$$I_n^{(m)}(t_1, \ldots, t_{2n+2m+4}) = \prod_{1\le r<s\le 2n+2m+4} \Gamma(t_r t_s; p, q)\, I_m^{(n)}\left(\frac{\sqrt{pq}}{t_1}, \ldots, \frac{\sqrt{pq}}{t_{2n+2m+4}}\right). \tag{12.1}$$

It represents a direct generalization of the third symmetry transformation for the V-function described in (5.4).

In [59], the following determinant representation has been found:

$$I_n^{(m)}(t_1, \ldots, t_{2n+2m+4}) = \prod_{1\le i<j\le n} \frac{1}{a_j\theta(a_i a_j^{\pm1}; p) b_j \theta(b_i b_j^{\pm1}; q)} \tag{12.2}$$

$$\times \det_{1\le i,j\le n}\left(\kappa \int_{\mathbb{T}} \frac{\prod_{r=1}^{2n+2m+4}\Gamma(t_r z^{\pm1}; p, q)}{\Gamma(z^{\pm2}; p, q)} \prod_{k\neq i} \theta(a_k z^{\pm1}; p) \prod_{k\neq j} \theta(b_k z^{\pm1}; q) \frac{dz}{z}\right),$$

where a_j, b_j are some arbitrary parameters. For the choice $a_i = t_j$, $b_j = t_{n+j}$, $j = 1, \ldots, n$, there appears the determinant of the matrix

$$T_q(t_i)^{-1} T_p(t_{n+j})^{-1} I_1^{(m+n-1)}(qt_1, \ldots, qt_n, pt_{n+1}, \ldots, pt_{2n}, t_{2n+1}, \ldots, t_{2n+2m+4}),$$

where $T_q(t_k)$ is the q-shift operator, $T_q(t_k)f(t_k) = f(qt_k)$. For $m = 0$, we obtain thus an exactly computable determinant of univariate elliptic hypergeometric integrals.

In order to prove (12.2), it is necessary to write

$$\prod_{1\le i<j\le n} \frac{1}{\Gamma(z_i^{\pm1} z_j^{\pm1}; p, q)} = \prod_{1\le i<j\le n} z_i^{-1}\theta(z_i z_j^{\pm1}; p) \prod_{1\le i<j\le n} z_i^{-1}\theta(z_i z_j^{\pm1}; q)$$

and connect both multipliers on the right-hand side with the elliptic analogue of the Cauchy determinant (9.4). Using the Heine formula

$$\frac{1}{n!}\int \det_{1\le i,j\le n} \phi_i(z_j) \det_{1\le i,j\le n} \psi_i(z_j) \prod_{1\le i\le n} d\mu(z_i) = \det_{1\le i,j\le n} \int \phi_i(z)\psi_j(z) d\mu(z)$$

with $\phi_i(z_j) = a_i/\theta(a_i z_j^{\pm1}; p)$ and $\psi_i(z_j) = b_i/\theta(b_i z_j^{\pm1}; q)$, one obtains the necessary result.

The following $(n+2)$-term recurrence relation takes place [33, 59]

$$\sum_{i=1}^{n+2} \frac{t_i}{\prod_{j=1,\neq i}^{n+2} \theta(t_i t_j^{\pm1}; p)} T_q(t_i) I_n^{(m)}(t_1, \ldots, t_{2n+2m+4}) = 0, \tag{12.3}$$

where $\prod_{j=1}^{2n+2m+4} t_j = (pq)^m p$. Indeed, if we write the same recurrence relation for the $I_n^{(m)}$-integral kernel, then it reduces to identity (9.3) after taking away a common multiplier. Integrating the latter relation over appropriate contours, we obtain (12.3).

The determinant representation permits one to derive another $(m+2)$-term recurrence relation

$$\sum_{k=1}^{m+2} \frac{\prod_{m+3\leq i\leq 2n+2m+4}\theta(t_it_k/q;p)}{t_k\prod_{1\leq i\leq m+2; i\neq k}\theta(t_i/t_k;p)} T_q(t_k)^{-1} I_n^{(m)}(t_1,\ldots,t_{2n+2m+4}) = 0, \qquad (12.4)$$

where $t_1\cdots t_{2m+2n+4} = (pq)^{m+1}q$. Note that transformation (12.1) maps equation (12.3) to (12.4), and vice versa. Therefore an analysis of common solutions of these equations provides an alternative way of proving transformation (12.1) [59].

Let us consider symmetry transformations for other multidimensional integrals. For nine parameters $t, t_1, \ldots, t_8 \in \mathbb{C}$ satisfying the restrictions $|t|, |t_j| < 1$ and $t^{2n-2}\prod_{j=1}^8 t_j = p^2q^2$, we define the integral

$$I(t_1,\ldots,t_8;t;p,q) = \prod_{1\leq j<k\leq 8}\Gamma(t_jt_k;p,q,t) \qquad (12.5)$$

$$\times \kappa_n \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n} \frac{\Gamma(tz_j^{\pm 1}z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1}z_k^{\pm 1};p,q)} \prod_{j=1}^n \frac{\prod_{k=1}^8\Gamma(t_kz_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)}\frac{dz_j}{z_j}.$$

Here

$$\Gamma(z;p,q,t) = \prod_{j,k,l=0}^{\infty}(1-zt^jp^kq^l)(1-z^{-1}t^{j+1}p^{k+1}q^{l+1})$$

is the elliptic gamma function of a higher order, which is connected to the Barnes multiple gamma function $\Gamma_4(u;\omega)$ and satisfies the equation

$$\Gamma(tz;p,q,t) = \Gamma(z;p,q)\Gamma(z;p,q,t).$$

Function (12.5) generalizes the V-function to the type II integrals for the root system BC_n. In [34], Rains has proved the following symmetry transformation formula:

$$I(t_1,\ldots,t_8;t;p,q) = I(s_1,\ldots,s_8;t;p,q),$$

where

$$\begin{cases} s_j = \rho^{-1}t_j, & j=1,2,3,4 \\ s_j = \rho t_j, & j=5,6,7,8 \end{cases}; \quad \rho = \sqrt{\frac{t_1t_2t_3t_4}{pqt^{1-n}}} = \sqrt{\frac{pqt^{1-n}}{t_5t_6t_7t_8}}, \quad |t|, |t_j|, |s_j| < 1.$$

It describes a generalization of the key relation for the E_7-group (5.2), i.e. integral (12.5) is invariant with respect to all transformations of this group. As shown in [58], the function $I(t_1,\ldots,t_8;t;p,q)$ emerges in the quantum multiparticle model of [62] under certain restrictions on the parameters (the balancing condition) as a normalization of a special eigenfunction of the Hamiltonian.

General elliptic hypergeometric integrals of type I on the root system A_n are defined as

$$I_n^{(m)}(s_1,\ldots,s_{n+m+2};t_1,\ldots,t_{n+m+2};A)$$

$$= \kappa_n^A \int_{\mathbb{T}^n} \prod_{1\leq j<k\leq n+1}\frac{1}{\Gamma(z_jz_k^{-1}, z_j^{-1}z_k;p,q)} \prod_{j=1}^{n+1}\prod_{l=1}^{n+m+2}\Gamma(s_lz_j^{-1}, t_lz_j;p,q)\,\frac{dz}{z},$$

where $|t_j|, |s_j| < 1$,

$$\prod_{j=1}^{n+1} z_j = 1, \qquad \prod_{l=1}^{n+m+2} s_lt_l = (pq)^{m+1},$$

and we set $I_0^{(m)} = \prod_{l=1}^{m+2}\Gamma(s_l,t_l;p,q)$. Using the result of exact computation of the $I_n^{(0)}$-integral (10.11), one can derive the following recursive relation for the $I_n^{(m)}$-integrals in the variable m:

$$\begin{aligned} I_n^{(m+1)}&(s_1,\ldots,s_{n+m+3};t_1,\ldots,t_{n+m+3};A) \\ &= \frac{\kappa_n^A}{\Gamma(v^{n+1};p,q)}\prod_{l=1}^{n+2}\frac{\Gamma(t_{n+m+3}s_l;p,q)}{\Gamma(v^{-n-1}t_{n+m+3}s_l;p,q)}\int_{\mathbb{T}^n}\prod_{1\le j<k\le n+1}\frac{1}{\Gamma(w_jw_k^{-1},w_j^{-1}w_k;p,q)} \\ &\quad\times\prod_{j=1}^{n+1}\Gamma(v^{-n}t_{n+m+3}w_j;p,q)\prod_{l=1}^{n+2}\Gamma(v^{-1}s_lw_j^{-1};p,q) \\ &\quad\times I_n^{(m)}(vw_1,\ldots,vw_{n+1},s_{n+3},\ldots,s_{n+m+3};t_1,\ldots,t_{n+m+2};A)\,\frac{dw_1}{w_1}\cdots\frac{dw_n}{w_n}, \end{aligned}$$

where

$$v^{n+1} = \frac{t_{n+m+3}}{pq}\prod_{k=1}^{n+2}s_k = \frac{(pq)^{m+1}}{\prod_{k=1}^{n+m+2}t_k\prod_{l=n+3}^{n+m+3}s_l}.$$

For $m=0$ the $I_n^{(0)}$-integral on the right-hand side is computable and yields the symmetry transformation [11]

$$\begin{aligned} I_n^{(1)}(s_1,\ldots,s_{n+3};t_1,\ldots,t_{n+3};A) &= \prod_{k=1}^{n+2}\Gamma\left(t_{n+3}s_k,\frac{\prod_{i=1}^{n+2}s_i}{s_k},s_{n+3}t_k,\frac{\prod_{i=1}^{n+2}t_i}{t_k};p,q\right) \\ &\quad\times I_n^{(1)}(v^{-1}s_1,\ldots,v^{-1}s_{n+2},v^ns_{n+3};vt_1,\ldots,vt_{n+2},v^{-n}t_{n+3};A), \end{aligned}$$

where $\prod_{k=1}^{n+3}t_ks_k=(pq)^2$ and

$$v^{n+1} = \frac{t_{n+3}}{pq}\prod_{k=1}^{n+2}s_k = \frac{pq}{s_{n+3}\prod_{k=1}^{n+2}t_k}.$$

Since the left-hand side of this relation is symmetric in the parameters t_k or s_k, the same $S_{n+3}\times S_{n+3}$ symmetry is valid for the right-hand side which leads to additional nontrivial transformations [34]. It is convenient to denote

$$\begin{aligned} I(v) &= \prod_{k=1}^{n+2}\Gamma(v^{-n-1}s_{n+3}s_k,v^{n+1}t_{n+3}t_k;p,q) \\ &\quad\times I_n^{(1)}(vs_1,\ldots,vs_{n+2},v^{-n}t_{n+3};v^{-1}t_1,\ldots,v^{-1}t_{n+2},v^ns_{n+3};A), \end{aligned}$$

where the arguments of $I_n^{(1)}$ lie inside the unit circle, $\prod_{k=1}^{n+3}t_k=\prod_{k=1}^{n+3}s_k=pq$, and v is an arbitrary free parameter (the total number of free parameters is equal to $2n+5$). Then the derived relation can be rewritten in the form $I(v)=I(v^{-1})$.

Another type of transformations for the $I_n^{(m)}$-integrals was found by Rains [34]. We denote $T=\prod_{j=1}^{n+m+2}t_j$, $S=\prod_{j=1}^{n+m+2}s_j$, so that $ST=(pq)^{m+1}$, and let all $|t_k|$, $|s_k|$, $|T^{\frac{1}{m+1}}/t_k|$, $|S^{\frac{1}{m+1}}/s_k|<1$. Then the following symmetry transformation is true:

$$I_n^{(m)}(t_1,\ldots,t_{n+m+2};s_1,\ldots,s_{n+m+2};A)=\prod_{j,k=1}^{n+m+2}\Gamma(t_js_k;p,q)$$
$$\times I_m^{(n)}\left(\frac{T^{\frac{1}{m+1}}}{t_1},\ldots,\frac{T^{\frac{1}{m+1}}}{t_{n+m+2}};\frac{S^{\frac{1}{m+1}}}{s_1},\ldots,\frac{S^{\frac{1}{m+1}}}{s_{n+m+2}};A\right). \tag{12.6}$$

This relation generalizes transformation (5.3), and there are no natural analogues of other E_7-reflections. We see thus that the V-function symmetries, generated by different elements of the Weyl group for the exceptional root system E_7, have some multidimensional analogues. However, different reflections are generalized to different integrals whose kernels obey symmetries in the integration variables related to different root systems.

13. Conclusion

Despite of a rather large volume, many problems were considered in this review only fragmentarily, a significant number of statements was given without proofs, and a number of interesting questions was not touched at all. Let us list some of the skipped achievements of the theory of elliptic hypergeometric functions and indicate several important open problems.

Suppose that there exists a finite difference operator of the first order which maps given rational functions to different rational functions with a smaller number of poles (the "lowering" operator). In the paper [106] it was shown that this is possible only under the condition that the poles of these rational functions are parameterized by a general elliptic function of the second order, and the problem itself is related to the Poncelet mapping.

We skipped description of the connection between biorthogonal rational functions and the Padé approximation with prescribed zeros and poles [70, 106, 107]. So, in the paper [107] it is shown that the ${}_{12}V_{11}$-elliptic hypergeometric series appears in the Padé interpolation tables of some functions.

It is natural to expect that the multiple elliptic beta integrals define a measure in biorthogonality relations for some functions of many variables, which would generalize the univariate relations (7.8). The first system of such multivariate functions, which is based on the elliptic analogue of the Selberg integral (10.8), was built by Rains [34, 35] with the help of raising and lowering operators. In certain limits, these functions degenerate to orthogonal polynomials of Macdonald [98], Koornwinder [108], or interpolating polynomials of Okounkov [109] (on the connection with the latter polynomials see also [102]). The results of the papers [34, 35] represent the most advanced achievements of the theory of elliptic hypergeometric functions of many variables. Another type of generalization of the Macdonald polynomials to the level of theta functions was suggested in [110].

There is a beautiful geometric interpretation of some of the elliptic hypergeometric functions in terms of the dynamics on algebraic surfaces. In [111], Sakai gave a classification of discrete Painlevé equations connected with the affine Weyl groups. On the top of this scheme one has the elliptic Painlevé equation related to the root system $\hat{E}_8$. In the paper [112] this equation was considered in detail and a reduction to the elliptic hypergeometric equation was found. Respectively, it was indicated that the elliptic hypergeometric series ${}_{12}V_{11}$ provides a particular solution of the elliptic Painlevé equation. A similar role is played by the general solution of the elliptic hypergeometric equation [31, 58] and by the BC_n-elliptic hypergeometric integral of type II for some special values of the parameters [91].

The Pochhammer and Horn approach to the functions of hypergeometric type [1, 2], which we used in the case of univariate functions, is not generalized yet to the level of elliptic hypergeometric functions of many variables. For that it is necessary to learn how to solve systems of difference equations of the first order for the kernels of multiple series or integrals with the coefficients which are elliptic functions of all summation or integration variables [10, 11]. This task is quite complicated, and all the examples, which were considered above, are built on the basis of different constructive ideas. On this route there appears a general problem of classification of all types of the elliptic beta integrals and of their multiparameter extensions to the higher order functions. It is expected, in particular, that there exist elliptic generalizations of multiple q-beta integrals for the exceptional root systems [113]. Let us remark also that under appropriate restrictions on the parameters all the integrals considered above are the integrals over polycycles. It would be interesting to consider integrals over more complicated regions of the complex integration variables.

In the paper [9], a nonlinear discrete integrable system was considered and a self-similar reduction of the corresponding equations was suggested, which leads to elliptic solutions with many parameters. Using this result, a system of discrete biorthogonal functions was built, which are expressed in terms of the ${}_{12}V_{11}$-series. Biorthogonal functions described in section 7.1 represent only a particular subcase of these more general functions, which are expressed as linear combinations of several ${}_{12}V_{11}$-series and contain three additional parameters. A detailed analysis of these functions is not performed yet. Let us mention among open problems related to the general solution of the elliptic hypergeometric equation a search for an explicit form of the non-terminating elliptic hypergeometric continued fraction and a buildup of elliptic analogues of the associated Askey–Wilson polynomials [114].

Different generalizations of the integral transformation (6.9) to multidimensional integrals on root systems were suggested in [36], but some of the corresponding inversion formulae are not proven yet. The problem of convergency of infinite elliptic hypergeometric series requires a deep analysis. It is necessary to understand in what sense such functions can exist. At this moment it is completely unclear what are the elliptic analogues of the number theoretical properties of the plain hypergeometric functions. It is necessary also to clarify whether it is possible to build nontrivial functions of the hypergeometric type for Riemann surfaces of a higher genus (the simplest example of such functions is given in [115]).

In conclusion, we can state that the main structural elements of the theory of plain and q-hypergeometric functions have their natural elliptic analogues. Moreover, various “old” hypergeometric notions acquire a new meaning connected with the properties of the elliptic functions. Sufficiently many applications of elliptic hypergeometric functions in mathematical physics are known at present: in the exactly solvable models of statistical mechanics related to the elliptic solutions of the Yang–Baxter equation [8] and the Sklyanin algebra [35, 79, 80, 116], in nonlinear integrable discrete time chains [9], in relativistic quantum multiparticle models of the Calogero–Sutherland type [58], and in nonlinear discrete equations of the Painlevé type [91, 112]. It is natural to expect that with time the number of such applications will grow and, besides that, there will appear new conceptual intersections with other parts of mathematics.

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Appendix A. Elliptic functions and the Jacobi theta functions

The periods of a periodic function are called primitive, if their linear combinations with integer coefficients yield all periods of this function. It is well known that nontrivial meromorphic functions cannot have more than two primitive periods [117]. Functions of a real variable can have only one primitive period. These statements were used in the derivation of the expression (2.21) and in the construction of elliptic analogues of the Meijer function. The meromorphic functions $f(u)$ with two primitive periods are called elliptic functions, that is there exist $\omega_1, \omega_2 \in \mathbb{C}$, $\mathrm{Im}(\omega_1/\omega_2) \neq 0$, and $f(u+\omega_1) = f(u+\omega_2) = f(u)$.

Primitive periods of an elliptic function $f(u)$ form parallelograms of periods and $f(u)$ is determined by its values inside of them and on a pair of adjacent edges. We call as a fundamental domain D the interior of one of such parallelograms chosen in such a way that on its boundary ∂D there are no divisor points of $f(u)$. Clearly, $f(u)$ (as a meromorphic function) has a finite number of zeros and poles inside D.

The integral $(2\pi i)^{-1}\int_{\partial D} f'(u)/f(u)du$ defines the number of zeros in a domain D for entire functions and the difference of the numbers of zeros and poles for meromorphic functions. Because of the periodicity, this integral is equal to zero for elliptic functions, that is the number of zeros equals to the number of poles in D. This number of zeros s (or poles) is called the order of elliptic function. The equality $f(u) = C$, where C is an arbitrarily chosen constant, is satisfied in D precisely s times (i.e., $f(u)$ is a s-sheeted function). This statement follows from the fact that the elliptic function $f(u) - C$ has s poles in D and, consequently, precisely s zeros.

The sum of residues of the poles of $f(u)$ in D is equal to zero, which follows from the equality $\int_{\partial D} f(u)du = 0$. Therefore there are no elliptic functions of the order $s = 1$, and an elliptic function of the zeroth order is constant (Liouville's theorem). This gives a method of proving elliptic functions' identities: if the difference of functions $f_1(u) - f_2(u)$ (or the ratio $f_1(u)/f_2(u)$) is elliptic and contains no more than than one pole in the fundamental domain, then $f_1(u) - f_2(u) = const$ (or $f_1(u)/f_2(u) = const$).

The well known elliptic function of Weierstrass $\wp(u|\omega_1,\omega_2)$ [117] has in the fundamental domain one pole of the second order, i.e. $s=2$. The pair $(x,y)=(\wp(u),\wp'(u))$ defines a uniformization of an elliptic curve $y^2=4x^3-g_2x-g_3$. Elliptic functions form a differential field, and any two elliptic functions $f(u)$ and $g(u)$ are related by an algebraic relation $P(f,g)=0$, where $P(f,g)$ is a polynomial of its arguments. For the choice $g(u)=f'(u)$, one sees that any elliptic function satisfies some nonlinear differential equation of the first order $P(f,f')=0$. Taking $g(u)=f(u+y)$, we obtain $P(f,g)=\sum_{k=0}^N p_k(f(u),y)\ f(u+y)^k=0$, where $p_k(f(u),y)$ are some polynomials in $f(u)$ with the coefficients depending on y. Permuting u and y, we see that $p_k(f(y),u)=p_k(f(u),y)$ are symmetric polynomials of $f(u)$ and $f(y)$ with constant coefficient. The condition $P(f,g)=0$ can be rewritten therefore as $Q(f(u),f(y),f(u+y))=0$, where Q is some polynomial of three variables. When such a condition is satisfied, one says that a function $f(u)$ obeys an algebraic addition theorem. As shown by Weierstrass, a meromorphic function $f(u)$ obeying such an addition theorem must be either an elliptic function or its degeneration to a trigonometric or a rational function.

For a more explicit representation of elliptic functions one needs theta functions. Arbitrary entire functions $f(u)$ are called (elliptic) theta functions, if

$$f(u+\omega_1)=e^{au+b}f(u),\qquad f(u+\omega_2)=e^{cu+d}f(u), \tag{A.1}$$

for some $a,b,c,d\in\mathbb{C}$ and $\mathrm{Im}(\omega_1/\omega_2)\neq 0$. Replacing u by $\omega_2 u$ and multiplying $f(u)$ by $e^{\alpha u^2+\beta u}$ with some specially chosen constants α and β, one can reach the equalities

$$f(u+1)=f(u),\qquad f(u+\tau)=e^{au+b}f(u).$$

In this appendix we use the parameterization $\tau=\omega_1/\omega_2$ and denote $q=e^{2\pi i\tau}$ (in difference from the main body of the review, where $\tau=\omega_3/\omega_2$ and $p=e^{2\pi i\tau}$). For $a\neq 0$, the parameter b can be removed by the shift $u\to u-b/a$. For a parallelogram D with the vertices $(0,1,1+\tau,\tau)$, we find $\int_{\partial D} f'(u)/f(u)du=-a$. Therefore $a=-2\pi i s$, where the quantity $s=0,1,\ldots$ determines the number of zeros of $f(u)$ in D. The key characteristics of a theta functions s is called its order.

The periodicity $f(u+1)=f(u)$ permits to expand $f(u)$ into the Fourier series $f(u)=\sum_{j=-\infty}^{\infty}c_je^{2\pi iju}$. Substituting it into the second equation with $a=-2\pi is$ and solving the emerging recurrence relation for the coefficients c_j, we find

$$f(u)=\sum_{l=0}^{s-1}c_lz^l\sum_{k\in\mathbb{Z}}q^{sk(k-1)/2}(q^lz^s)^k,\quad z=e^{2\pi iu}.$$

The coefficients $c_0,\ldots,c_{s-1}$ are arbitrary, i.e. theta functions of the order s form an s-dimensional vector space.

If we restore arbitrary quasiperiodicity multipliers, then a theta function without zeros is equal to $e^{P_2(u)}$, where $P_2(u)$ is a polynomial of the second order. Theta functions of the first order with one zero in the fundamental parallelogram of quasiperiods are called the Jacobi theta functions, and the functions with $s>1$ are called theta functions of the higher level. For $s=1$, it is convenient to work with four theta functions with characteristics

$$\theta_{ab}(u)=\sum_{k\in\mathbb{Z}}e^{\pi i\tau(k+a/2)^2}e^{2\pi i(k+a/2)(u+b/2)},$$

where the variables a and b take the values 0 or 1. The standard Jacobi theta functions are defined as [12]:

$$\begin{aligned}
\theta_1(u|\tau) &= \theta_1(u) = -\theta_{11}(u),\\
\theta_2(u|\tau) &= \theta_2(u) = \theta_{10}(u) = \theta_1(u+1/2),\\
\theta_3(u|\tau) &= \theta_3(u) = \theta_{00}(u) = e^{\pi i\tau/4+\pi iu}\theta_1(u+1/2+\tau/2),\\
\theta_4(u|\tau) &= \theta_4(u) = \theta_{01}(u) = -ie^{\pi i\tau/4+\pi iu}\theta_1(u+\tau/2).
\end{aligned}$$

Note that all of them have the form $\sum_{k\in\mathbb{Z}} c_k$ with $h(k) = c_{k+1}/c_k = q^k y$ for some constant y. Since $h(k)$ is rational in q^k, Jacobi theta functions represent a special class of q-hypergeometric functions.

The $\theta_1(u)$-function is odd, $\theta_1(-u) = -\theta_1(u)$, and satisfies the quasiperiodicity conditions

$$\theta_1(u+1) = -\theta_1(u), \qquad \theta_1(u+\tau) = -e^{-\pi i\tau-2\pi iu}\theta_1(u).$$

It is related to the shortened theta function $\theta(z;q) = (z;q)_\infty(qz^{-1};q)_\infty$ by the Jacobi triple product identity

$$\theta_1(u) = iq^{1/8}e^{-\pi iu}(q;q)_\infty\theta(e^{2\pi iu};q). \tag{A.2}$$

Transformation properties of the θ_1-function with respect to the $PSL(2,\mathbb{Z})$-group of modular transformations $\tau \to (a\tau+b)/(c\tau+d)$, $a,b,c,d\in\mathbb{Z}$, $ad-bc=1$, are determined by the relations [118]

$$\theta_1(u|\tau+1) = e^{\pi i/4}\theta_1(u|\tau), \quad \theta_1\left(\frac{u}{\tau}\Big|\frac{-1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\pi iu^2/\tau}\theta_1(u|\tau). \tag{A.3}$$

It is convenient to use notation $\theta_a(u_1,\ldots,u_k) := \theta_a(u_1)\cdots\theta_a(u_k)$ and $\theta_a(x\pm y) := \theta_a(x+y, x-y)$. Then the argument duplication formula has the form

$$\theta_1(2u) = \frac{iq^{1/8}}{(q;q)_\infty^3}\theta_1\left(u, u+\frac{1}{2}, u+\frac{\tau}{2}, u-\frac{1+\tau}{2}\right).$$

The addition theorem for theta functions, which is called sometimes a Riemann relation, uses products of four theta functions

$$\theta_1(u\pm a, v\pm b) - \theta_1(u\pm b, v\pm a) = \theta_1(a\pm b, u\pm v) \tag{A.4}$$

or

$$\theta(xw^{\pm1}, yz^{\pm1};p) - \theta(xz^{\pm1}, yw^{\pm1};p) = yw^{-1}\theta(xy^{\pm1}, wz^{\pm1};p). \tag{A.5}$$

The proof of this equality if elementary. The ratio of expressions standing on its left- and right-hand sides is a bounded function of $x\in\mathbb{C}^*$ (it is invariant with respect to the transformation $x\to px$, and it does not contain poles in the annulus $|p|\le|x|\le 1$). By the Liouville theorem this ratio does not depend on x, but for $x=w$ it equals to 1.

Any theta function $f(u)$ of the order s with the quasiperiods ω_1,ω_2 and coordinates of the zeros in the fundamental domain $a_1,\ldots,a_s$ can be represented in the form

$$f(u) = e^{P_2(u)}\prod_{k=1}^{s}\theta_1\left(\frac{u-a_k}{\omega_2}\Big|\frac{\omega_1}{\omega_2}\right), \tag{A.6}$$

where $P_2(u)$ is some polynomial of u of the second order. Indeed, the function

$$g(u) = \prod_{k=1}^{s} \theta_1 \left(\frac{u - a_k}{\omega_2} \Big| \frac{\omega_1}{\omega_2} \right)$$

is a theta function of the order s with the same zeros in the parallelogram of quasiperiods ω_1 and ω_2 as the function $f(u)$. Therefore the ratio $f(u)/g(u)$ is an entire function without zeros and poles, namely, a theta function of the zeros order, i.e. $e^{P_2(u)}$. In [118], a self-contained theory of Jacobi forms — the functions obeying transformation properties similar to those of functions (A.6), was formulated. Note that all vectors $f_j(u)$, $j = 1, \ldots, s$, of any basis of the space of theta functions of the order s can be represented in the indicated form with the matrix of zeros a_{jk} satisfying a number of constraints (e.g., $\sum_{k=1}^{s} a_{jk} = const$).

We call as meromorphic theta functions ratios of theta functions of an arbitrary finite order. It is easy to see that they define meromorphic solutions of equations (A.1). For this we denote as $a_1, \ldots, a_n$ coordinates of the zeros and as $b_1, \ldots, b_m$ coordinates of the poles of the corresponding function $f(u)$ in the fundamental domain. Then the ratio

$$\frac{f(u) \prod_{k=1}^{m} \theta_1 \left(\frac{u-b_k}{\omega_2} \Big| \frac{\omega_1}{\omega_2} \right)}{\prod_{k=1}^{n} \theta_1 \left(\frac{u-a_k}{\omega_2} \Big| \frac{\omega_1}{\omega_2} \right)}$$

is an entire function without zeros satisfying the equations $f(u+\omega_1) = e^{a'u+b'} f(u)$ and $f(u+\omega_2) = e^{c'u+d'}$ with some a', b', c', d', that is a theta function of the zeroth order $e^{P_2(u)}$.

Any elliptic function $f(u)$ of the finite order s with the periods ω_1, ω_2 can be represented in the form

$$f(u) = C \prod_{k=1}^{s} \frac{\theta_1 \left(\frac{u-a_k}{\omega_2} \Big| \frac{\omega_1}{\omega_2} \right)}{\theta_1 \left(\frac{u-b_k}{\omega_2} \Big| \frac{\omega_1}{\omega_2} \right)}, \tag{A.7}$$

where C is some constant, and a_k and b_k denote coordinates of some zeros and poles of $f(u)$ congruent to the zeros and poles in the fundamental parallelogram of periods. The following constraint should be satisfied by a_k and b_k:

$$a_1 + \ldots + a_s = b_1 + \ldots + b_s \mod \omega_2. \tag{A.8}$$

This follows from the fact that both parts of equality (A.7) are meromorphic and doubly periodic. Therefore their ratio defines a bounded entire function, i.e. a constant. The linear constraint on the values of a_k and b_k (A.8), which we call the balancing condition, follows from the requirement of cancellation of the quasiperiodicity multipliers of the θ_1-functions appearing from the $u \to u + \omega_1$ shift.

Substituting expression (A.2) in (A.7) and denoting $z = e^{2\pi i u/\omega_2}$, $t_k = e^{-2\pi i a_k/\omega_2}$, $w_k = e^{-2\pi i b_k/\omega_2}$, we obtain

$$f(u) = \pm C \prod_{k=1}^{s} \frac{\theta(t_k z; q)}{\theta(w_k z; q)}, \qquad \prod_{k=1}^{s} t_k = \prod_{k=1}^{s} w_k, \tag{A.9}$$

where the sign ambiguity appears from the factor $e^{-\pi i u}$ in (A.2).

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