GENERATING FUNCTIONS FOR JACOBI AND LAGUERRE POLYNOMIALS

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Let v be a function of t defined by

$$v = t(1+v)^{b+1}, \qquad v(0) = 0. \tag{1}$$

Then it follows from Lagrange's expansion formula [6, Vol. I, p. 126, Ex. 212] that

$$\frac{(1+v)^{a+1}}{1-bv} = \sum_{n=0}^{\infty} \binom{a+(b+1)n}{n} t^n. \tag{2}$$

Making use of the formula (2), Carlitz [2] has proved that the Laguerre polynomial $L_n^{(a+bn)}(x)$, where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \frac{x^k}{k!}, \tag{3}$$

satisfies a generating relation in the form

$$\sum_{n=0}^{\infty} L_n^{(a+bn)}(x) t^n = \frac{(1+v)^{a+1}}{1-bv} \exp(-xv), \tag{4}$$

where v is given by (1) and a, b are arbitrary complex numbers. Note that the special case of (4) when b is an arbitrary integer was proved earlier by Brown [1].

In terms of the generalized hypergeometric function

$${}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p; \\ & x \\ \beta_1, \cdots, \beta_q;\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{x^n}{n!}, \tag{5}$$

where

$$(\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), \quad n \geqq 1, \ (\lambda)_0 = 1, \tag{6}$$

the generating relation (4) assumes the form

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$$\sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}{}_1F_1\left[\begin{matrix}-n;\\ 1+a+bn;\end{matrix}\; x\right]t^n = \frac{(1+v)^{a+1}}{1-bv}\exp(-xv). \tag{7}$$

In (7) if we replace x by xz, multiply both sides by $z^{\lambda-1}$ and take their Laplace transforms with respect to the variable z, we shall readily obtain

$$\sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}{}_2F_1\left[\begin{matrix}-n, \lambda;\\ 1+a+bn;\end{matrix}\; x\right]t^n = \frac{(1+v)^{a+1}}{1-bv}(1+xv)^{-\lambda}, \tag{8}$$

where the binomial $(1+xv)^{-\lambda}$ may be written as an ${}_1F_0$.

The form of (8) suggests the existence of the general formula

$$\sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}{}_{p+1}F_{q+1}\left[\begin{matrix}-n, \alpha_1, \cdots, \alpha_p;\\ 1+a+bn, \beta_1, \cdots, \beta_q;\end{matrix}\; x\right]t^n = \frac{(1+v)^{a+1}}{1-bv}\,{}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p;\\ \beta_1, \cdots, \beta_q;\end{matrix}\; -xv\right], \tag{9}$$

where p, q are nonnegative integers, the α's and a, b take general values, real or complex, and

$$\beta_j \neq 0, -1, -2, \cdots, \qquad j = 1, 2, \cdots, q. \tag{10}$$

The derivation of (9) from (7) and (8) by the principle of multidimensional mathematical induction would require the Laplace and inverse Laplace transform techniques illustrated, for instance, by the author [7].

For a direct proof without using (7) and (8) we notice that, in view of the definition (5),

$$\sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}{}_{p+1}F_{q+1}\left[\begin{matrix}-n,\alpha_1,\cdots,\alpha_p;\\ 1+a+bn,\beta_1,\cdots,\beta_q;\end{matrix}\,x\right]t^n$$

$$=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}(-1)^k\binom{a+(b+1)n}{n-k}\frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\frac{x^k}{k!}$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\frac{x^k}{k!}\sum_{n=k}^{\infty}\binom{a+(b+1)n}{n-k}t^n$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\frac{x^kt^k}{k!}\sum_{n=0}^{\infty}\binom{a+(b+1)k+(b+1)n}{n}t^n$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{\prod_{j=1}^{p}(\alpha_j)_k}{\prod_{j=1}^{q}(\beta_j)_k}\frac{x^kt^k}{k!}\frac{(1+v)^{a+(b+1)k+1}}{1-bv},$$

by (2), and the formula (9) follows immediately. We can easily attribute a direct proof to the formula (8) which obviously corresponds to the special case $p=1$, $q=0$ of (9).

A similar generalization of Carlitz's formula [2, p. 827, Equation (16)] has the form

(11)
$$\sum_{n=0}^{\infty}\binom{-a-bn}{n}{}_{p+1}F_{q+1}\left[\begin{matrix}-n,\alpha_1,\cdots,\alpha_p;\\ 1-a-(b+1)n,\beta_1,\cdots,\beta_q;\end{matrix}\,x\right]t^n$$
$$=\frac{A(-t,a,b)}{1-B(-t,b)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p;\\ \beta_1,\cdots,\beta_q;\end{matrix}\,-\frac{xB(-t,b)}{1-B(-t,b)}\right],$$

where, for convenience,

(12)
$$B(t,b)=-\sum_{n=1}^{\infty}\binom{(b+1)n}{n-1}\frac{t^n}{n}$$

and

(13)
$$A(t,a,b)=\frac{[1-B(t,b)]^{a+1}}{1+bB(t,b)}\cdot$$

Indeed the formula (11) is obtainable from (9) by replacing a by $-a$ and b by $-(b+1)$.

It may be of interest to remark that for $b=0$ and $b=-1$ the formula (9) yields Chaundy's results (25) and (27) respectively (see [4, p. 62]). For $b=-\frac{1}{2}$, (9) reduces to the generating relation (7), p. 264 of Brown's recent paper.[2]

For the Jacobi polynomial defined by

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{\alpha+n}{k}\binom{\beta+n}{n-k}\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^{k}, \tag{14}$$

it is easy to show from the identity (4.22.1) of [8, p. 63] that

$$P_n^{(\alpha-n,\beta-n)}(x) = \binom{n-\alpha-\beta-1}{n}\left(\frac{1-x}{2}\right)^{n}{}_2F_1\left[\begin{matrix}-n,\,-\alpha;\\ -\alpha-\beta;\end{matrix}\ \frac{2}{1-x}\right], \tag{15}$$

and therefore (8) gives us the elegant generating function

$$\sum_{n=0}^{\infty}P_n^{(\alpha-n,\beta-(b+1)n)}(x)t^n=(1+w)^{-\alpha-\beta}(1+bw)^{-1}\left(1+\frac{2w}{1-x}\right)^{\alpha}, \tag{16}$$

where

$$w=\tfrac{1}{2}(1-x)t(1+w)^{b+1}. \tag{17}$$

Evidently (16) reduces to the known formula [3, p. 88]

$$\sum_{n=0}^{\infty}P_n^{(\alpha-n,\beta-n)}(x)t^n=\left[1+\tfrac{1}{2}(x+1)t\right]^{\alpha}\left[1+\tfrac{1}{2}(x-1)t\right]^{\beta} \tag{18}$$

when $b=0$, and for $b=-1$ it leads us to Feldheim's result [5, p.120]

$$\sum_{n=0}^{\infty}P_n^{(\alpha-n,\beta)}(x)t^n=(1+t)^{\alpha}\left[1-\tfrac{1}{2}(x-1)t\right]^{-\alpha-\beta-1}. \tag{19}$$

Now from the definition (14) we readily have [8, p. 61]

$$P_n^{(\alpha,\beta)}(x)=\binom{\alpha+n}{n}{}_2F_1\left[\begin{matrix}-n,\,1+\alpha+\beta+n;\\ 1+\alpha;\end{matrix}\ \frac{1-x}{2}\right], \tag{20}$$

whence it follows at once that

[2] J. W. Brown, *New generating functions for classical polynomials*, Proc. Amer. Math. Soc. 20 (1969), 263–268.

$$(21)\quad P_n^{(\alpha+bn,\beta-(b+1)n)}(x) = \binom{\alpha+(b+1)n}{n}\,{}_2F_1\left[\begin{matrix}-n,\,1+\alpha+\beta;\\ \\ 1+\alpha+bn;\end{matrix}\ \frac{1-x}{2}\right].$$

Consequently, (8) gives us another class of generating functions for the Jacobi polynomial in the form

$$(22)\quad \sum_{n=0}^{\infty} P_n^{(\alpha+bn,\beta-(b+1)n)}(x)t^n = (1+v)^{\alpha+1}(1-bv)^{-1}[1-\tfrac{1}{2}(x-1)v]^{-\alpha-\beta-1},$$

where v is defined by (1) and b, α, β are unrestricted, in general.

For $b=-1$, (22) leads us again to Feldheim's formula (19); when $b=0$, it reduces to the generating relation

$$(23)\quad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta-n)}(x)t^n = (1-t)^{\beta}[1-\tfrac{1}{2}(x+1)t]^{-\alpha-\beta-1}$$

also due to Feldheim [**5**, p. 120].

Finally, we remark that the special case $b=-\frac{1}{2}$ of our formula (22) corresponds to

$$(24)\quad \sum_{n=0}^{\infty} P_n^{(\alpha-n/2,\beta-n/2)}(x)t^n = [1+u(t)]^{\alpha+1}[1+\tfrac{1}{2}u(t)]^{-1}[1-\tfrac{1}{2}(x-1)u(t)]^{-\alpha-\beta-1},$$

where

$$(25)\quad u(t) = \tfrac{1}{2}t[t+\sqrt{(t^2+4)}].$$

The formula (24) appears in Brown's recent paper referred to earlier.

ADDED IN PROOF. In a private communication to the author, Professor L. Carlitz suggests that following the method of proof of the formula (9) one can readily obtain its straightforward generalization in the form

$$(*)\quad \sum_{n=0}^{\infty}\binom{a+(b+1)n}{n}t^n\sum_{k=0}^{n}\frac{(-n)_k c_k}{(1+a+bn)_k}\,\frac{x^k}{k!} = \frac{(1+v)^{a+1}}{1-bv}\sum_{k=0}^{\infty} c_k\,\frac{(-xv)^k}{k!},$$

where the c_k are arbitrary constants and v is defined by (1). It seems worthwhile to remark here that further extensions of (*) form the subject-matter of our discussion in a forthcoming paper.

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