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NOTE ON CERTAIN GENERATING FUNCTIONS FOR JACOBI AND LAGUERRE POLYNOMIALS*

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1. Put

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{1}$$

where $P_n^{(\alpha,\beta)}(x)$ is Szegö's notation [6, p. 68] for the Jacobi polynomial of order α, β and degree n in x.

Recently, Varma [7, p. 306] gave the generating relation

$$\sum_{n=0}^{\infty}\frac{(c-b)_n}{(c)_n}P_n^{(\alpha,\beta-n)}(x)\,t^n =\left(\frac{1+x}{2}\right)^{-\alpha-\beta-1}F_2\left[\alpha+1,\ \alpha+\beta+1,\ c-b;\ \ \alpha+1,\ c;\ \ \frac{x-1}{x+1},\ t\right], \tag{2}$$

where F_2 denotes the second type of Appell's hypergeometric functions of two variables defined by [3, p. 14]

$$F_2[\alpha,\beta,\beta';\ \gamma,\gamma';\ x,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}. \tag{3}$$

The series iteration technique used in [7] to derive (2) seems to be unnecessarily long and involved. As a matter of fact, the generating relation (2) is an immediate consequence of the definitions (1) and (3), and the familiar Gaussian hypergeometric transformation (cf., e.g., [3], p. 3)

$${}_2F_1[a,b;\ c;\ z]=(1-z)^{c-a-b}\,{}_2F_1[c-a,\ c-b;\ c;\ z],\quad |z|<1. \tag{4}$$

If we rewrite (1) as

$$P_n^{(\alpha,\beta-n)}(x)=\binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n{}_2F_1\left[-n,\ -\beta;\ \alpha+1;\ \frac{x-1}{x+1}\right], \tag{5}$$

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and apply (4) to the second member of (5), we shall readily obtain

$$(6)\qquad P_{n+\nu}^{(\alpha,\,\beta-n)}(x)=\binom{n+\nu+\alpha}{n+\nu}\left(\frac{1+x}{2}\right)^{-\alpha-\beta-\nu-1}$$

$$\cdot {}_2F_1[\alpha+n+\nu+1,\ \alpha+\beta+\nu+1;\ \ \alpha+1;\ \ (x-1)/(x+1)],$$

for every integer $\nu\geqq 0$.

Next we rewrite (3) in the form

$$(7)\qquad F_2[\alpha,\beta,\beta';\ \gamma,\gamma';\ x,y]=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta')_n}{n!\,(\gamma')_n}\,{}_2F_1[\alpha+n,\ \beta;\ \gamma;\ x]\,y^n,$$

which in conjunction with (6) would lead at once to a generalization of (2) given by

$$(8)\qquad \sum_{n=0}^{\infty}\binom{n+\nu}{n}\frac{(\lambda)_n}{(\mu)_n}P_{n+\nu}^{(\alpha,\,\beta-n)}(x)\,t^n=\binom{\nu+\alpha}{\nu}\left(\frac{1+x}{2}\right)^{-\alpha-\beta-\nu-1}$$

$$\cdot F_2[\alpha+\nu+1,\ \alpha+\beta+\nu+1,\ \lambda;\ \ \alpha+1,\ \mu;\ \ (x-1)/(x+1),\ t],$$

where $\nu=0,1,2,\ldots$.

Evidently (2) would follow from (8) in the special case $\nu=0$.

2. In view of the hypergeometric transformation [3, p. 32]

$$(9)\qquad F_2[\alpha,\beta,\beta';\ \gamma,\gamma';\ x,y]=(1-x)^{-\alpha}F_2\left[\alpha,\gamma-\beta,\beta';\ \gamma,\gamma';\ \frac{x}{x-1},\ \frac{y}{1-x}\right],$$

the generating function in (8), with β replaced by $\beta-\nu$, can be written in the form

$$(10)\qquad \sum_{n=0}^{\infty}\binom{n+\nu}{n}\frac{(\lambda)_n}{(\mu)_n}P_{n+\nu}^{(\alpha,\,\beta-n-\nu)}(x)\,t^n=\binom{\nu+\alpha}{\nu}\left(\frac{1+x}{2}\right)^{\nu-\beta}$$

$$\cdot F_2\left[\alpha+\nu+1,\ -\beta,\ \lambda;\ \ \alpha+1,\ \mu;\ \ \frac{1-x}{2},\ \frac{(1+x)t}{2}\right],\quad \nu=0,1,2,\ldots,$$

which is what our formula (9), p. 62 in [5] should read.

For $\nu=0$, the generating relation (10) was given earlier by Al-Salam [2, p. 138, Eq. (6.12)]. Thus Varma's result (2), which evidently is equivalent to (10) with $\nu=0$, would indeed follow fairly easily from the aforementioned formula of Al-Salam [loc. cit.].

On the other hand, it is well known that [3, p. 35 and p. 30]

$$(11)\qquad F_2[\alpha,\beta,\beta';\ \alpha,\lambda;\ x,y]=(1-x)^{-\beta}F_1\left[\beta',\ \beta,\ \alpha-\beta;\ \gamma;\ \frac{y}{1-x},\ y\right]$$

$$=(1-x)^{-\beta}(1-y)^{-\beta'}F_1\left[\beta',\ \gamma-\alpha,\ \beta;\ \gamma;\ \frac{y}{y-1},\ \frac{xy}{(1-x)(1-y)}\right],$$

where F_1 is the Appell function of the first kind defined by [op. cit., p. 14]

$$F_1[\alpha, \beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{12}$$

Therefore, Varma's result (2) is also equivalent to

$$\sum_{n=0}^{\infty} \frac{(c-b)_n}{(c)_n} P_n^{(\alpha, \beta-n)}(x)\, t^n = (1-t)^{b-c} F_1\left[c-b,\ c-\alpha-1,\ \alpha+\beta+1;\ c;\ \frac{t}{t-1},\ \frac{(1-x)t}{2(t-1)}\right]. \tag{13}$$

On replacing b by $\alpha - a + b + 1$, and c by $\alpha + b + 1$, this last formula (13) yields

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(\alpha+b+1)_n} P_n^{(\alpha, \beta-n)}(x)\, t^n = (1-t)^{-a} F_1\left[a,\ b,\ \alpha+\beta+1;\ \alpha+b+1;\ \frac{t}{t-1},\ \frac{(1-x)t}{2(t-1)}\right], \tag{14}$$

which is the main result (3.1), p. 2 in another paper by Varma [8]. This evidently shows that the generating functions in (2) and (14) are equivalent; in fact, these are trivial variations of what is already contained in Al-Salam's work [2, p. 138, Eq. (6.12)].

Thus it would seem fairly obvious that the main results in Varma's papers [7] and [8] are substantially the same special case of the generating relation (8), or its equivalent form (10), each of which has been shown to follow as a rather immediate consequence of the definitions (1) and (3).

We remark in passing that, by means of the known relationships [6, p. 59]

$$P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x) \tag{15}$$

and [op. cit., p. 64]

$$P_n^{(\alpha, \beta)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-2n-1,\, \beta)}\left(\frac{x+3}{x-1}\right), \tag{16}$$

formulas (8) and (10), as also their special cases considered here, can be transformed fairly simply into generating relations for the Jacobi polynomials

$$P_n^{(\alpha-n, \beta)}(x) \quad \text{or} \quad P_n^{(\alpha-n, \beta-n)}(x).$$

3. Now we turn to the classical Laguerre polynomials defined by [6, p. 101]

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \tag{17}$$

Making use of the known relationship [op. cit., p. 103]

$$L_n^{(\alpha)}(x) = \lim_{\beta \to \infty} P_n^{(\alpha, \beta)}(1 - 2x/\beta), \quad n = 0, 1, 2, \ldots, \tag{18}$$

in our generating relation (10), it is easily verified that, for every non-negative integer ν,

$$(19)\quad \sum_{n=0}^{\infty}\binom{n+\nu}{n}\frac{(\lambda)_n}{(\mu)_n}L_{n+\nu}^{(\alpha)}(x)\,t^n=\binom{\nu+\alpha}{\nu}e^x\Psi_1[\alpha+\nu+1,\ \lambda;\ \mu,\ \alpha+1;\ t,\ -x],$$

where Ψ_1 is a (Humbert's) confluent hypergeometric function of two variables defined by [3, p. 126]

$$(20)\qquad \Psi_1[\alpha,\ \beta;\ \gamma,\ \gamma';\ x,\ y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_m(\gamma')_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

For $\nu=0$, (19) yields the generating function

$$(21)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\mu)_n}L_n^{(\alpha)}(x)\,t^n=e^x\Psi_1[\alpha+1,\ \lambda;\ \mu,\ \alpha+1;\ t,\ -x],$$

which, for $\lambda=c-b$, $\mu=c$, and $t=y$, would provide us with the corrected version of Varma's formula (4.2), p. 308 in [7]. Notice, as an obvious limiting case of (11), the transformation

$$(22)\quad \Psi_1[\alpha,\ \beta;\ \gamma,\ \alpha;\ x,\ y]=(1-x)^{-\beta}e^y\Phi_1\left[\beta,\ \gamma-\alpha;\ \gamma;\ \frac{x}{x-1},\ \frac{xy}{1-x}\right],$$

where Φ_1 is another (Humbert's) confluent hypergeometric function of two variables defined by [3, p. 126]

$$(23)\qquad \Phi_1[\alpha,\ \beta;\ \gamma;\ x,\ y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}.$$

By virtue of (22), the generating function in (21), and hence also the corrected version of Varma's formula (4.2), p. 308 in [7], can be rewritten in their equivalent form

$$(24)\quad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{(\mu)_n}L_n^{(\alpha)}(x)\,t^n=(1-t)^{-\lambda}\Phi_1\left[\lambda,\ \mu-\alpha-1;\ \mu;\ \frac{t}{t-1},\ \frac{xt}{t-1}\right],$$

which is essentially the same as the generating relation (4.3), p. 5 in Varma's paper [8]. In view of our remark in the preceding section, concerning the main results of Varma's papers [7] and [8], this equivalence was well anticipated.

Another interesting special case of (19) would occur when $\lambda=\mu$; indeed we have

$$\begin{aligned}
&\Psi_1[\alpha+\nu+1,\ \lambda;\ \lambda,\ \alpha+1;\ t,\ -x]\\
&=\sum_{m,n=0}^{\infty}\frac{(\alpha+\nu+1)_{m+n}\,t^m(-x)^n}{m!\,n!\,(\alpha+1)_n}\\
&=\sum_{n=0}^{\infty}\frac{(\alpha+\nu+1)_n(-x)^n}{n!\,(\alpha+1)_n}\sum_{m=0}^{\infty}\binom{m+n+\nu+\alpha}{m}t^m\\
&=(1-t)^{-\alpha-\nu-1}\,{}_1F_1[\alpha+\nu+1;\ \alpha+1;\ x/(t-1)]\\
&=(1-t)^{-\alpha-\nu-1}e^{x/(t-1)}\,{}_1F_1[-\nu;\ \alpha+1;\ x/(1-t)],
\end{aligned}$$

by Kummer's first theorem, and using the definition (17) once again, from (19) we thus arrive at the elegant generating relation

$$\sum_{n=0}^{\infty}\binom{n+\nu}{n} L_{n+\nu}^{(\alpha)}(x)\, t^n = (1-t)^{-\alpha-\nu-1} e^{xt/(t-1)} L_{\nu}^{(\alpha)}\left(\frac{x}{1-t}\right), \tag{25}$$

which is fairly well known (cf., e.g., [4], p. 757, Eq. (10)).

Finally, we record a new generating function, analogous to (19), in the form

$$\sum_{n=0}^{\infty}\binom{n+\nu}{n}\frac{(\lambda)_n}{(\mu)_n} L_{n+\nu}^{(\alpha-n)}(x)\, t^n = \binom{\nu+\alpha}{\nu}(1+t)^{\alpha} e^{-xt}\Phi_1^*[\mu-\lambda,\ -\nu,\ -\alpha,\ \mu;\ t/(1+t),\ xt,\ -x(1+t)], \tag{26}$$

where Φ_1^* denotes a generalization of the Φ_1 function given by (23). Indeed we have

$$\Phi_1^*[\alpha,\ \beta,\ \gamma,\ \delta;\ x,\ y,\ z] = \sum_{m,\,n,\,p=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_p(\gamma)_{m-p}}{(\delta)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{27}$$

whence

$$\Phi_1^*[\alpha,\ 0,\ \gamma,\ \delta;\ x,\ y,\ z] = \Phi_1[\alpha,\ \gamma;\ \delta;\ x,\ y] = \Phi_1^*[\alpha,\ \beta,\ \gamma,\ \delta;\ x,\ y,\ 0], \tag{28}$$

and

$$\Phi_1^*[0,\ \beta,\ \gamma,\ \delta;\ x,\ y,\ z] = {}_1F_1[\beta;\ 1-\gamma;\ -z] = \Phi_1^*[\alpha,\ \beta,\ \gamma,\ \delta;\ 0,\ 0,\ z]. \tag{29}$$

Formula (26) can be derived, for instance, as a limiting case of the generating relation (8) or (10), since (15) and (18), together, would imply at once that

$$(-1)^n L_n^{(\beta)}(x) = \lim_{\alpha\to\infty} P_n^{(\alpha,\beta)}\left(\frac{2x}{\alpha}-1\right),\quad n=0,\ 1,\ 2,\ \dots\ . \tag{30}$$

Two special cases of the generating relation (26) are worthy of note. If we set $\lambda=\delta-\gamma$, $\mu=\delta$, and $\nu=0$, and apply the reduction formula (28), we get

$$\sum_{n=0}^{\infty}\frac{(\delta-\gamma)_n}{(\delta)_n} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^{\alpha} e^{-xt}\Phi_1[\gamma,\ -\alpha;\ \delta;\ t/(1+t),\ xt], \tag{31}$$

which appears slightly erroneously as formula (4.11), p. 57 in reference [1]. On the other hand, by an appeal to (29) a special case of (26) when $\lambda=\mu$ would readily give us

$$\sum_{n=0}^{\infty}\binom{n+\nu}{n} L_{n+\nu}^{(\alpha-n)}(x)\, t^n = \binom{\nu+\alpha}{\nu}(1+t)^{\alpha} e^{-xt} L_{\nu}^{(\alpha)}(x(1+t)), \tag{32}$$

which is also a known generating relation (cf., e.g., [4], p. 757, Eq. (13)).

It would seem worthwhile to remark that a generating relation, analogous to (25) and (32), does indeed hold also for the Jacobi polynomials. This formula would obviously follow from (8) or (10) with $\lambda=\mu$, and we thus have

$$(33)\qquad \sum_{n=0}^{\infty}\binom{n+\nu}{n}P_{n+\nu}^{(\alpha,\,\beta-n)}(x)\,t^n = (1-t)^{\beta}\left\{1-\frac{1}{2}(1+x)\,t\right\}^{-\alpha-\beta-\nu-1}P_{\nu}^{(\alpha,\,\beta)}(\xi),$$

where, for convenience,

$$(34)\qquad \xi=\left\{x-\frac{1}{2}(1+x)\,t\right\}\left\{1-\frac{1}{2}(1+x)\,t\right\}^{-1}.$$

Equivalently, one has the relatively more elegant result

$$(35)\qquad \sum_{n=0}^{\infty}\binom{n+\nu}{n}P_{n+\nu}^{(\alpha-n,\,\beta-n)}(x)\,t^n = \left\{1+\frac{1}{2}(x+1)\,t\right\}^{\alpha}\left\{1+\frac{1}{2}(x-1)\,t\right\}^{\beta}P_{\nu}^{(\alpha,\,\beta)}\left(x+\frac{1}{2}(x^2-1)\,t\right),$$

which follows fairly rapidly if we replace x, t and α on both sides of (33) by $(x+3)/(x-1)$, $(1-x)\,t/2$, and $-\alpha-\beta-2\nu-1$, respectively, and apply the relationship (16) twice.

Incidentally, both (33) and (35) were deduced in refeience [4, p. 759] as special cases of certain bilinear generating relations for the Jacobi polynomials.

REFERENCES

[1] N. Abdul-Halim and W. A Al-Salam, *Double Euler transformations of certain hypergeometric functions*, Duke Math. J. **30** (1963), 51—62.

[2] W. A. Al-Salam, *Operational representations for the Laguerre and other polynomials*, Duke Math. J. **31** (1964), 127—142.

[3] P. Appell et J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques. Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[4] J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math. **42** (1972), 755—762.

[5] H. M. Srivastava, *On a generating function for the Jacobi polynomial*, J. Math. Sci. **4** (1969), 61—68.

[6] G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, Third edition, Amer. Math. Soc., Providence, Rhode Island, 1967.

[7] V. K. Varma, *Appell's double hypergeometric function as a generating function of the Jacobi polynomial*, Riv. Mat. Univ. Parma (2) **8** (1967), 305—308.

[8] V. K. Varma, *Double hypergeometric functions as generating functions of the Jacobi and Laguerre polynomials*, J. Indian Math. Soc. (N. S.) **32** (1968), 1—5.

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