

MATHEMATICS

A CERTAIN METHOD OF OBTAINING BILATERAL GENERATING FUNCTIONS

BY

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This paper presents a systematic introduction to and several applications of a certain method of obtaining bilinear or bilateral generating relations for a large variety of sequences of special functions. The main result, given by Theorem 1 below, is shown to apply, for instance, to the Bessel, Brafman, Charlier, Gegenbauer (or ultraspherical), Gould-Hopper, Jacobi, Hermite, Konhauser, Laguerre (or the modified Laguerre) and Srivastava-Singhal polynomials, while its generalization, given by Theorem 2, would apply to the Lauricella polynomials in several variables and to the familiar Lagrange polynomials which arise in certain problems in statistics. It is also shown how these results can be extended to yield bilateral generating relations for such other special functions as the Bessel functions.

1. INTRODUCTION

In her recent monograph [7] McBRIDE presented a systematic study of the various methods of obtaining generating functions. Every special function, which she considered as the coefficient set in a bilinear (or bilateral) generating relation, belongs to a class of functions $\{S_n(x) | n=0, 1, 2, \dots\}$ generated by

$$(1) \quad \sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x)t^n = f(x, t)\{g(x, t)\}^{-m} S_m(h(x, t)),$$

where $m \geq 0$ is an integer, the $A_{m,n}$ are arbitrary constants, and f, g, h are arbitrary functions of x and t . Thus it would seem worthwhile to investigate an effective method of obtaining bilateral generating functions for the $S_n(x)$ defined by (1). With this object in view, we first prove the following general result.

THEOREM 1. Corresponding to every sequence $\{S_n(x)\}$ generated by (1), let

$$(2) \quad F_q[x, t] = \sum_{n=0}^{\infty} a_n S_{qn}(x)t^n,$$

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where q is an arbitrary positive integer, and the $a_n \neq 0$ are arbitrary constants.

Then

$$(3) \quad \sum_{n=0}^{\infty} S_n(x) \sigma_n^a(y) t^n = f(x, t) F_q[h(x, t), y\{t/g(x, t)\}^a],$$

where $\sigma_n^a(y)$ is a polynomial of degree $[n/q]$ in y defined by

$$(4) \quad \sigma_n^a(y) = \sum_{k=0}^{[n/q]} a_k A_{qk, n- qk} y^k.$$

PROOF. If we substitute for the polynomials $\sigma_n^a(y)$ from (4) into the left-hand side of (3), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} S_n(x) \sigma_n^a(y) t^n &= \sum_{n=0}^{\infty} S_n(x) t^n \sum_{k=0}^{[n/q]} a_k A_{qk, n- qk} y^k \\ &= \sum_{k=0}^{\infty} a_k y^k t^{qk} \sum_{n=0}^{\infty} A_{qk, n} S_{qk+n}(x) t^n \\ &= f(x, t) \sum_{k=0}^{\infty} a_k S_{qk}(h(x, t)) [y\{t/g(x, t)\}^a]^k, \end{aligned}$$

by employing the generating relation (1). Now the theorem would follow at once if we interpret this last expression by means of (2).

2. APPLICATIONS TO CLASSICAL POLYNOMIALS

For $q=1$ the above theorem would readily yield a class of bilateral generating functions for the sequence $\{S_n(x)\}$ given recently by SINGHAL and SRIVASTAVA [10, p. 755]. In this section we show how Theorem 1 for integers $q \geq 1$ can be applied to derive bilinear or bilateral generating functions for those classical polynomial systems that satisfy a relationship like (1).

First of all we recall that the Hermite polynomials $H_n(x)$ defined by Rodrigues' formula

$$(5) \quad H_n(x) = (-1)^n \exp(x^2) D_x^n \{\exp(-x^2)\}, \quad D_x \equiv d/dx,$$

satisfy the relationship [7, p. 14]

$$(6) \quad \sum_{n=0}^{\infty} H_{m+n}(x) \frac{t^n}{n!} = \exp(2xt - t^2) H_m(x - t),$$

which is of type (1) with $A_{m,n} = 1/n!$, $f = \exp(2xt - t^2)$, $g = 1$, and $h = x - t$. Thus we are led to the following application of Theorem 1.

COROLLARY 1. If

$$(7) \quad F_q^{(1)}[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} H_{qn}(x) t^n,$$

then

$$(8) \quad \sum_{n=0}^{\infty} H_n(x) M_n^a(y) \frac{t^n}{n!} = \exp(2xt - t^2) F_q^{(1)}[x - t, yt^a],$$

where, as well as throughout this paper,

$$(9) \quad M_n^q(y) = \sum_{k=0}^{[n/q]} \binom{n}{qk} a_k y^k.$$

By suitably specializing the arbitrary coefficients a_n in (9), the polynomials $M_n^q(y)$ can easily be reduced to a number of familiar systems. For instance, if we put

$$(10) \quad a_n = \left\{ (qn)! \prod_{j=1}^r (\alpha_j)_n \right\} \left\{ n! \prod_{j=1}^s (\beta_j)_n \right\}^{-1}, \quad n=0, 1, 2, \dots,$$

and replace y by $y/(-q)^q$, the $M_n^q(y)$ will become identical with the BRAFMAN polynomials [1, p. 186]

$$(11) \quad B_n^q[\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; y] = {}_{q+r}F_s[\Delta(q; -n), \alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; y],$$

where, for convenience, $\Delta(q; \lambda)$ denotes the set of q parameters

$$\lambda/q, (\lambda+1)/q, \dots, (\lambda+q-1)/q, \quad q \geq 1.$$

Thus we obtain a special case of Corollary 1 given by

COROLLARY 1.1. If

$$(12) \quad F_q^{(2)}[x, t] = \sum_{n=0}^{\infty} \left\{ \prod_{j=1}^r (\alpha_j)_n \right\} \left\{ n! \prod_{j=1}^s (\beta_j)_n \right\}^{-1} H_{qn}(x) t^n,$$

then

$$(13) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} H_n(x) B_n^q[\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; y] \frac{t^n}{n!} \\ &= \exp(2xt - t^2) F_q^{(2)}[x-t, y(-t/q)^q]. \end{aligned} \right.$$

On the other hand, for the GOULD-HOPPER generalization of the classical Hermite polynomials, viz. [4, p. 58]

$$(14) \quad g_n^q(x, \lambda) = \sum_{k=0}^{[n/q]} \frac{n!}{k!(n-qk)!} \lambda^k x^{n-qk} = x^n {}_qF_0[\Delta(q; -n); -; \lambda(-q/x)^q],$$

which obviously are contained in the Brafman polynomials (11) with $r=s=0$, from (5) and Corollary 1 we readily get

COROLLARY 1.2. If

$$(15) \quad F_q^{(3)}[x, t] = \exp(x^2) \exp[t(-D_x)^q] \{ \exp(-x^2) \},$$

then

$$(16) \quad \sum_{n=0}^{\infty} H_n(x) g_n^q(y, \lambda) \frac{t^n}{n!} = \exp(2xyt - y^2t^2) F_q^{(3)}[x-yt, \lambda t^q].$$

Since

$$(17) \quad g_n^q(2x, -1) = H_n(x), \quad n=0, 1, 2, \dots,$$

a special case of (16) when $q=2$, $\lambda=-1$, and y is replaced by $2y$, would yield

$$(18) \quad \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} = \exp(4xyt - 4y^2t^2) F_2^{(3)}[x - 2yt, -t^2],$$

where $F_2^{(3)}[x, t]$ is given by (15) with $q=2$. In view of Glaisher's operational formula

$$(19) \quad \exp(t D_x^2) \{ \exp(-x^2) \} = (1 + 4t)^{-1/2} \exp[-x^2/(1 + 4t)],$$

we have

$$(20) \quad F_2^{(3)}[x, t] = (1 + 4t)^{-1/2} \exp[4x^2t/(1 + 4t)],$$

and we arrive immediately at the bilinear generating relation

$$(21) \quad \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} = (1 - 4t^2)^{-1/2} \exp[x^2 - (x - 2yt)^2/(1 - 4t^2)],$$

which is well known as Mehler's formula (cf., e.g., [7], p. 15 and [9], p. 198).

As a final application of Corollary 1 we derive a hitherto unnoticed bilateral generating relation for the Hermite polynomials given by

$$(22) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} H_n(x) C_n^{\nu}(y) \frac{t^n}{(2\nu)_n} &= \exp(2xyt - 2y^2t^2 + t^2) \cdot \\ &\cdot \Phi_3[\nu; \nu + 1/2; (y^2 - 1)t^2, (x - yt)^2(y^2 - 1)t^2], \end{aligned} \right.$$

where $C_n^{\nu}(x)$ are the Gegenbauer (or ultraspherical) polynomials [7, p. 3], and Φ_3 is a confluent hypergeometric function of two variables defined by [3, Vol. I, p. 225]

$$(23) \quad \Phi_3[\alpha; \gamma; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}.$$

Indeed if in Corollary 1 we set $q=2$, $a_n = (2n)! \{n!(\nu + 1/2)_n\}^{-1}$, and replace y by $(y^2 - 1)/4y^2$, the $M_n^2(y)$ given by (9) will, in view of the known relationship [9, p. 280]

$$(24) \quad C_n^{\nu}(x) = \frac{(2\nu)_n x^n}{n!} {}_2F_1[A(2; -n); \nu + 1/2; (x^2 - 1)/x^2],$$

correspond to

$$(25) \quad \frac{n!}{(2\nu)_n y^n} C_n^{\nu}(y).$$

On the other hand, since it is easily verified that

$$(26) \quad H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2), \quad n = 0, 1, 2, \dots,$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomial of order α and

degree n in x [7, p. 10, Eq. (12)], from equation (7) with the aforementioned values of q and a_n we have

$$(27) \quad \left\{ \begin{aligned} F_2^{(1)}[x, t] &= \sum_{n=0}^{\infty} \frac{t^n}{n!(\nu+1/2)_n} H_{2n}(x) \\ &= \sum_{n=0}^{\infty} \frac{(-4t)^n}{(\nu+1/2)_n} L_n^{(-1/2)}(x^2) \\ &= \exp(-4t) \Phi_3[\nu; \nu+1/2; 4t, 4x^2t], \end{aligned} \right.$$

by virtue of the formula

$$(28) \quad \sum_{n=0}^{\infty} \frac{t^n}{(\varrho)_n} L_n^{(\alpha)}(x) = \exp(t) \Phi_3[\varrho - \alpha - 1; \varrho; -t, -xt],$$

which follows fairly easily from the generating relation (3.1), p. 76 of SRIVASTAVA [14].

Now we substitute in (8) the values of $M_n^2(y)$ and $F_2^{(1)}[x, t]$ given by (25) and (27), respectively, replace y by yt in the resulting equation, and this completes the derivation of the bilateral generating relation (22). Notice that (22) is contained also in the generating relation (13) with $q=2, r=s-1=0, \beta_1=\nu+1/2$.

Next we consider the classical Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ which satisfy the elegant relationships [10, p. 759]

$$(29) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha-m-n, \beta-m-n)}(x) t^n \\ &= \{1 + \frac{1}{2}(x+1)t\}^{\alpha-m} \{1 + \frac{1}{2}(x-1)t\}^{\beta-m} P_m^{(\alpha-m, \beta-m)}\left(x + \frac{1}{2}(x^2-1)t\right) \end{aligned} \right.$$

and

$$(30) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} P_{m+n}^{(\alpha, \beta-m-n)}(x) t^n \\ &= (1-t)^{\beta-m} \{1 - \frac{1}{2}(x+1)t\}^{-\alpha-\beta-1} P_m^{(\alpha, \beta-m)}(X), \end{aligned} \right.$$

where, for convenience,

$$(31) \quad X = \{x - \frac{1}{2}(x+1)t\} \{1 - \frac{1}{2}(x+1)t\}^{-1}.$$

On comparing (29) and (30) with the generating relation (1) we obtain the following special cases of Theorem 1.

COROLLARY 2. *If*

$$(32) \quad F_q^{(4)}[x, t] = \sum_{n=0}^{\infty} a_n P_{\alpha n}^{(\alpha-\alpha n, \beta-\alpha n)}(x) t^n,$$

then

$$(33) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} P_n^{(\alpha-n, \beta-n)}(x) M_n^q(y) t^n = \{1 + \frac{1}{2}(x+1)t\}^{\alpha} \{1 + \frac{1}{2}(x-1)t\}^{\beta} \\ &\quad \cdot F_q^{(4)}[x + \frac{1}{2}(x^2-1)t, yt^q / \{1 + \frac{1}{2}(x+1)t\}^q \{1 + \frac{1}{2}(x-1)t\}^q], \end{aligned} \right.$$

where, as before, the $M_n^q(y)$ are given by (9).

COROLLARY 3. If

$$(34) \quad F_q^{(5)}[x, t] = \sum_{n=0}^{\infty} a_n P_{qn}^{(\alpha, \beta-qn)}(x)t^n,$$

then

$$(35) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} P_n^{(\alpha, \beta-n)}(x) M_n^q(y)t^n \\ &= (1-t)^\beta \left\{ 1 - \frac{1}{2}(x+1)t \right\}^{-\alpha-\beta-1} F_q^{(5)}[X, yt^q/(1-t)^q], \end{aligned} \right.$$

where X is given by (31).

Notice, however, that Corollaries 2 and 3 are essentially equivalent, since we have

$$(36) \quad P_n^{(\alpha, \beta-n)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-n-1, \beta-n)}\left(\frac{x+3}{x-1}\right),$$

which is a rather immediate consequence of the known formula (7), p. 255 in [9]. Also, since [op. cit., p. 256]

$$(37) \quad P_n^{(\alpha, \beta)}(x) = (-1)^n P_n^{(\beta, \alpha)}(-x), \quad n = 0, 1, 2, \dots,$$

formula (35) can easily be transformed into a bilateral generating relation for the Jacobi polynomials $P_n^{(\alpha-n, \beta)}(x)$.

Similar applications of our theorem will lead to the following results involving other classical polynomials.

COROLLARY 4. If

$$(38) \quad F_q^{(6)}[x, t] = \sum_{n=0}^{\infty} a_n C_{qn}^v(x)t^n,$$

then

$$(39) \quad \sum_{n=0}^{\infty} C_n^v(x) M_n^q(y)t^n = R^{-2p} F_q^{(6)}[(x-t)/R, y(t/R)^q],$$

where

$$(40) \quad R = (1 - 2xt + t^2)^{1/2}.$$

COROLLARY 5. If

$$(41) \quad F_q^{(7)}[x, t] = \sum_{n=0}^{\infty} a_n L_{qn}^{(\alpha)}(x)t^n,$$

then

$$(42) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) M_n^q(y)t^n \\ &= (1-t)^{-\alpha-1} \exp[-xt/(1-t)] F_q^{(7)}[x/(1-t), yt^q/(1-t)^q]. \end{aligned} \right.$$

COROLLARY 6. *If*

$$(43) \quad F_q^{(8)}[x, t] = \sum_{n=0}^{\infty} a_n L_{qn}^{(\alpha-an)}(x)t^n,$$

then

$$(44) \quad \sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x) M_n^q(y)t^n = (1+t)^\alpha \exp(-xt) F_q^{(8)}[x(1+t), yt^q/(1+t)^q].$$

Since [7, pp. 68-69]

$$(45) \quad f_n^{-\alpha}(x) = (-1)^n L_n^{(\alpha-n)}(x) = \frac{x^n}{n!} c_n(\alpha; x),$$

Corollary 6 may alternatively be stated as follows in terms of the modified Laguerre polynomials $f_n^\alpha(x)$ defined by [op. cit., p. 4]

$$(46) \quad (1-t)^{-\alpha} \exp(xt) = \sum_{n=0}^{\infty} f_n^\alpha(x)t^n,$$

or the Charlier polynomials $c_n(x; \alpha)$ defined by [op. cit., p. 68]

$$(47) \quad c_n(x; \alpha) = \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{x}{k} k! \alpha^{-k}, \quad \alpha > 0, \quad x = 0, 1, 2, \dots$$

COROLLARY 7. *If*

$$(48) \quad F_q^{(9)}[x, t] = \sum_{n=0}^{\infty} a_n f_{qn}^\alpha(x)t^n,$$

then

$$(49) \quad \sum_{n=0}^{\infty} f_n^\alpha(x) M_n^q(y)t^n = (1-t)^{-\alpha} \exp(xt) F_q^{(9)}[x(1-t), yt^q/(1-t)^q],$$

where, as before, the $M_n^q(y)$ are defined by (9).

COROLLARY 8. *Let*

$$(50) \quad F_q^{(10)}[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} c_{qn}(\alpha; x)t^n,$$

where $x > 0$ and $\alpha = 0, 1, 2, \dots$

Then

$$(51) \quad \sum_{n=0}^{\infty} c_n(\alpha; x) M_n^q(y) \frac{t^n}{n!} = (1-t/x)^\alpha \exp(t) F_q^{(10)}[x-t, yt^q].$$

Yet another form of Corollary 6 would involve the generalized Bessel polynomials defined by (cf., e.g., [9], p. 294)

$$(52) \quad y_n(x, a, b) = {}_2F_0[-n, a-1+n; -; -x/b].$$

Indeed, it is easily verified that

$$(53) \quad y_n(x, a-n, b) = n!(-x/b)^n L_n^{(1-a-n)}(b/x),$$

which would enable one to rewrite the familiar result (13), p. 757 in [10] as

$$(54) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} y_{m+n}(x, a-m-n, b) \frac{t^n}{n!} \\ = (1-xt/b)^{1-a} \exp(t) y_m(x(1-xt/b)^{-1}, a-m, b). \end{aligned} \right.$$

{See also [4], p. 58, Eq. (5.6).}

Making use of the relationship (53) in Corollary 6, or else by comparing (54) and (1), we obtain

COROLLARY 9. If

$$(55) \quad F_q^{(11)}[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} y_{qn}(x, a-qn, b)t^n,$$

then

$$(56) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} y_n(x, a-n, b) M_n^q(y) \frac{t^n}{n!} \\ = (1-xt/b)^{1-a} \exp(t) F_q^{(11)}[x(1-xt/b)^{-1}, yt^q]. \end{aligned} \right.$$

For the simple Bessel polynomials [7, p. 47]

$$(57) \quad y_n(x) = {}_2F_0[-n, n+1; -; -x/2] = y_n(x, 2, 2),$$

it is known that (cf., e.g., [10], p. 758)

$$(58) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} y_{m+n}(x) \frac{t^n}{n!} &= (1-2xt)^{-(m+1)/2} \exp[x^{-1}\{1-(1-2xt)^{\frac{1}{2}}\}] \\ &\cdot y_m(x(1-2xt)^{-\frac{1}{2}}), \end{aligned} \right.$$

which may be compared with (1) to get

COROLLARY 10. If

$$(59) \quad F_q^{(12)}[x, t] = \sum_{n=0}^{\infty} \frac{a_n}{(qn)!} y_{qn}(x)t^n,$$

then

$$(60) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} y_n(x) M_n^q(y) \frac{t^n}{n!} &= (1-2xt)^{-\frac{1}{2}} \exp[x^{-1}\{1-(1-2xt)^{\frac{1}{2}}\}] \\ &\cdot F_q^{(12)}[x(1-2xt)^{-\frac{1}{2}}, yt^q(1-2xt)^{-q/2}]. \end{aligned} \right.$$

Evidently this last Corollary 10 is not contained in Corollary 9.

We remark in passing that Corollaries 2 through 10 can also be applied to the Brafman polynomials (11), the Gould-Hopper polynomials (14),

and several other polynomial systems, in the same manner as we deduced Corollaries 1.1 and 1.2, and the generating relations (21) and (22), from Corollary 1.

3. OTHER APPLICATIONS

For the sequence of functions $\Omega_n^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x]$ defined by

$$(61) \quad \left\{ \begin{aligned} &\Omega_n^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x] \\ &= {}_{p+u}F_v[\Delta(p; \lambda + n), \gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x], \quad n = 0, 1, 2, \dots, \end{aligned} \right.$$

where p is an arbitrary positive integer, it is easy to establish the relationship

$$(62) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda + m)_n}{n!} \Omega_{m+n}^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x] t^n \\ &= (1-t)^{-\lambda-m} \Omega_m^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x/(1-t)^p], \end{aligned} \right.$$

which evidently is of type (1) with $A_{m,n} = (\lambda + m)_n/n!$, $f = (1-t)^{-\lambda}$, $g = 1-t$, and $h = x/(1-t)^p$.

Thus Theorem 1 when applied to this sequence would yield

COROLLARY 11. If

$$(63) \quad F_q^{(13)}[x, t] = \sum_{n=0}^{\infty} \frac{a_n(\lambda) q^n}{(qn)!} \Omega_{qn}^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x] t^n,$$

then

$$(64) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Omega_n^{(\lambda, p)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x] M_n^q(y) t^n \\ &= (1-t)^{-\lambda} F_q^{(13)} [x/(1-t)^p, y\{t/(1-t)\}^q], \end{aligned} \right.$$

where the $M_n^q(y)$ are given by (9).

In terms of the Brafman polynomials (11), a special case of (64) with $p=q$, a_n given by (10), and y replaced on each side by $y/(-q)^q$, will result in the generating relation

$$(65) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} \Omega_n^{(\lambda, q)} [\gamma_1, \dots, \gamma_u; \delta_1, \dots, \delta_v; x] B_n^q[\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; y] t^n \\ &= (1-t)^{-\lambda} F \left[\frac{\Delta(q; \lambda): \gamma_1, \dots, \gamma_u; \alpha_1, \dots, \alpha_r; \frac{x}{(1-t)^q}}{\delta_1, \dots, \delta_v; \beta_1, \dots, \beta_s; (1-t)^q}, \frac{y t^q}{(1-t)^q} \right], \end{aligned} \right.$$

where $F[x, y]$ denotes a generalized Appell function of two variables in the contracted notation of BURCHNALL and CHAUNDY [2, p. 112].

Bilinear relations of type (65) have indeed been known to the first author for a long time. For hypergeometric forms of (65) with $q=1$ one may refer, for instance, to pages 43 and 82 of SRIVASTAVA's papers [11]

and [14], respectively, and also to the other papers cited there. Some interesting special cases of (65) with $q=2$ are worthy of mention.

If in (65) we set $q=2$, $\lambda=2\nu+m$, $r=s-1=0$, $u=v-1=0$, $\beta_1=\varrho+1/2$, $\delta_1=\nu+1/2$, m being an arbitrary nonnegative integer, apply the familiar hypergeometric transformation [3, Vol. I, p. 64]

$$(66) \quad {}_2F_1[a, b; c; z] = (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c; z], \quad |z| < 1,$$

to $\Omega_n^{(2\nu+m, 2)}[-; \nu+1/2: x]$, replace x, y, t by $(x^2-1)/x^2$, $(y^2-1)/y^2$ and yt/x , respectively, and apply the known formula (24), we shall obtain the elegant bilinear generating relation (cf. [6], p. 254, Eq. (10))

$$(67) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} \frac{(m+n)!}{(2\varrho)_n} C_{m+n}^{\nu}(x) C_n^{\varrho}(y)t^n &= (2\nu)_m (x-yt)^{-2\nu-m} \\ &\cdot F_4 \left[\nu + \frac{1}{2}m, \nu + \frac{1}{2}m + \frac{1}{2}; \nu + \frac{1}{2}, \varrho + \frac{1}{2}; \frac{x^2-1}{(x-yt)^2}, \frac{(y^2-1)t^2}{(x-yt)^2} \right], \end{aligned} \right.$$

where F_4 denotes the fourth type of Appell's double hypergeometric functions defined by [3, Vol. I, p. 224]

$$(68) \quad F_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m, n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{(\gamma)_m (\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}.$$

On the other hand, if we set $q=2$, $\lambda=2\nu+m$, $r=s=0$, $u=v-1=0$, $\delta_1=\nu+1/2$, formula (65) will similarly lead to a (divergent) bilateral generating function, viz.

$$(69) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} \binom{m+n}{n} C_{m+n}^{\nu}(x) H_n(y)t^n &\cong \binom{2\nu+m-1}{m} (x-2yt)^{-2\nu-m} \\ &\cdot F_4^* \left[\nu + \frac{1}{2}m, \nu + \frac{1}{2}m + \frac{1}{2}; \nu + \frac{1}{2}; \frac{x^2-1}{(x-2yt)^2}, \frac{-4t^2}{(x-2yt)^2} \right], \end{aligned} \right.$$

where F_4^* is a double hypergeometric function closely related to the Appell function F_4 defined by (68); indeed we have

$$(70) \quad F_4^*[\alpha, \beta; \gamma; x, y] = \lim_{\delta \rightarrow \infty} F_4[\alpha, \beta; \gamma, \delta; x, \delta y].$$

Note also that in the special case when $x=0$, the bilinear relation (65) would reduce to the generating function

$$(71) \quad \left\{ \begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} B_n^q[\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; y] t^n \\ = (1-t)^{-\lambda} {}_{q+r}F_s[\Delta(q; \lambda), \alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; yt^q/(t-1)^q], \end{aligned} \right.$$

which is substantially the same as the known result (55), p. 187 due to BRAFMAN [1].

Next we recall a generalization of the Hermite polynomials, viz.

$$(72) \quad H_n^r(x, a, p) = (-1)^n x^{-a} \exp(px^r) D_x^n \{x^a \exp(-px^r)\},$$

which was considered recently by GOULD and HOPPER [4, p. 52, Eq. (2.1)]. Since [op. cit., p. 57]

$$(73) \quad (1-t/x)^\alpha \exp [px^r\{1-(1-t/x)^r\}] H_m^r(x-t, a, p) = \sum_{n=0}^\infty H_{m+n}^r(x, a, p) \frac{t^n}{n!},$$

Theorem 1 would apply to these Gould-Hopper polynomials, and we obtain

COROLLARY 12. If

$$(74) \quad F_q^{(14)}[x, t] = \sum_{n=0}^\infty \frac{a_n}{(qn)!} H_{qn}^r(x, a, p)t^n,$$

then

$$(75) \quad \left\{ \begin{aligned} &\sum_{n=0}^\infty H_n^r(x, a, p) M_n^q(y) \frac{t^n}{n!} \\ &= (1-t/x)^\alpha \exp [px^r\{1-(1-t/x)^r\}] F_q^{(14)}[x-t, yt^q]. \end{aligned} \right.$$

The polynomials $G_n^{(\lambda)}(x, r, p, \alpha)$, which were introduced by SRIVASTAVA and SINGHAL [12] in an attempt to provide an elegant unification of the various recent generalizations of the classical Hermite and Laguerre polynomials, satisfy the relationship [op. cit., p. 79].

$$(76) \quad \left\{ \begin{aligned} &\sum_{n=0}^\infty \binom{m+n}{n} G_{m+n}^{(\lambda)}(x, r, p, \alpha)t^n \\ &= (1-\alpha t)^{-m-\lambda/\alpha} \exp [px^r\{1-(1-\alpha t)^{-r/\alpha}\}] G_m^{(\lambda)}(x(1-\alpha t)^{-1/\alpha}, r, p, \alpha), \end{aligned} \right.$$

which is of type (1). Thus we have

COROLLARY 13. If

$$(77) \quad F_q^{(15)}[x, t] = \sum_{n=0}^\infty a_n G_{qn}^{(\lambda)}(x, r, p, \alpha)t^n,$$

then

$$(78) \quad \left\{ \begin{aligned} &\sum_{n=0}^\infty G_n^{(\lambda)}(x, r, p, \alpha) M_n^q(y)t^n = (1-\alpha t)^{-\lambda/\alpha} \exp [px^r\{1-(1-\alpha t)^{-r/\alpha}\}] \\ &\cdot F_q^{(15)}[x(1-\alpha t)^{-1/\alpha}, yt^q/(1-\alpha t)^q]. \end{aligned} \right.$$

In view of the relationships (1.4) through (1.9) in [12, p. 76], Corollary 13 can be shown to incorporate, as its special cases, a fairly large number of bilateral generating functions including those that are contained in Corollaries 1, 5 and 12.

For the Konhauser biorthogonal polynomials $Y_n^\alpha(x; k)$, where $\alpha > -1$ and k is a positive integer, it is easy to show that

$$(79) \quad Y_n^\alpha(x; 1) = L_n^{(\alpha)}(x), \quad n = 0, 1, 2, \dots,$$

and that (cf., e.g., [8], p. 803)

$$(80) \left\{ \begin{aligned} &\sum_{n=0}^{\infty} \binom{m+n}{n} Y_{m+n}^{\alpha}(x; k) t^n \\ &= (1-t)^{-m-(\alpha+1)/k} \exp [x\{1-(1-t)^{-1/k}\}] Y_m^{\alpha}(x(1-t)^{-1/k}; k), \end{aligned} \right.$$

which evidently would lead to a generalization of Corollary 5, involving Laguerre polynomials, given by

COROLLARY 14. If

$$(81) \quad F_q^{(16)} [x, t] = \sum_{n=0}^{\infty} a_n Y_{qn}^{\alpha}(x; k) t^n,$$

then

$$(82) \left\{ \begin{aligned} &\sum_{n=0}^{\infty} Y_n^{\alpha}(x; k) M_n^{\alpha}(y) t^n = (1-t)^{-(\alpha+1)/k} \exp [x\{1-(1-t)^{-1/k}\}] \\ &\cdot F_q^{(16)} [x(1-t)^{-1/k}, yt^q/(1-t)^q]. \end{aligned} \right.$$

On comparing (80) with (76) it follows at once that

$$(83) \quad Y_n^{\alpha}(x; k) = k^{-n} G_n^{(\alpha+1)}(x, 1, 1, k), \quad \alpha > -1, k = 1, 2, 3, \dots,$$

whence Corollary 14, which would reduce to Corollary 5 when $k=1$, is contained in Corollary 13.

Finally, we consider a unification of the classical orthogonal polynomials of Hermite, Jacobi, and Laguerre, the Bessel polynomials (52), the Gould-Hopper polynomials (72), and several other polynomial systems studied in recent literature, defined by the Rodrigues formula [13, p. 969]

$$(84) \left\{ \begin{aligned} &T_n^{(\alpha, \beta)}(x, a, b, c, d, p, r) = \frac{(ax+b)^{-\alpha} (cx+d)^{-\beta}}{n!} \exp (px^r) \\ &\cdot D_x^n \{ (ax+b)^{n+\alpha} (cx+d)^{n+\beta} \exp (-px^r) \}. \end{aligned} \right.$$

Making use of the known generating relations [13, p. 973, Eq. (31) and (32)] in Theorem 1 above, we shall arrive at the following generalizations of Theorems 1 and 2 in [op. cit., p. 974].

COROLLARY 15. If

$$(85) \quad F_q^{(17)} [x, t] = \sum_{n=0}^{\infty} a_n T_{qn}^{(\alpha-qn, \beta)}(x, a, b, c, d, p, r) t^n,$$

then

$$(86) \left\{ \begin{aligned} &\sum_{n=0}^{\infty} T_n^{(\alpha-n, \beta)}(x, a, b, c, d, p, r) M_n^{\alpha}(y) t^n = \{1 + (ad-bc)t\}^{\alpha} \\ &\cdot \{1 - c(ax+b)t\}^{-\alpha-\beta-1} \exp \left[px^r - p \left(\frac{x+d(ax+b)t}{1-c(ax+b)t} \right) \right] \\ &\cdot F_q^{(17)} \left[\frac{x+d(ax+b)t}{1-c(ax+b)t}, \frac{yt^q}{\{1+(ad-bc)t\}^{\alpha}} \right]. \end{aligned} \right.$$

COROLLARY 16. *If*

$$(87) \quad F_q^{(1\beta)}[x, t] = \sum_{n=0}^{\infty} a_n T_m^{(\alpha-m, \beta-m)}(x, a, b, c, d, p, r)t^n,$$

then

$$(88) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} T_n^{(\alpha-n, \beta-n)}(x, a, b, c, d, p, r) M_n^a(y)t^n \\ &= \{1 + a(cx + d)t\}^\alpha \{1 + c(ax + b)t\}^\beta \exp [px^r - p\{x + (ax + b)(cx + d)t\}^r] \\ &\cdot F_q^{(1\beta)}[x + (ax + b)(cx + d)t, yt^a / \{1 + a(cx + d)t\}^a \{1 + c(ax + b)t\}^a]. \end{aligned} \right.$$

In view of the relationships (4) through (9) in [13, p. 970], it can be verified fairly easily that Corollary 15 provides a unification of Corollaries 1, 3, 5, 6, 9 and 12, while Corollary 16 generalizes Corollaries 1, 2, 6 and 12. Note also that since [loc. cit.]

$$(89) \quad T_n^{(\alpha, \beta)}(x, a, b, c, d, p, r) = (ab)^{-n} T_n^{(\beta, \alpha)}(bcx/ad, a, b, a^2d, b^2c, p(ad/bc)^r, r),$$

which obviously is a generalization of the familiar result (37), Corollary 15 may be restated in terms of the polynomials

$$T_n^{(\alpha, \beta-n)}(x, a, b, c, d, p, r), \quad n = 0, 1, 2, \dots$$

4. EXTENSION IN SEVERAL VARIABLES

If we define a sequence of functions of several variables $\{S_n(x_1, \dots, x_r) \mid n = 0, 1, 2, \dots\}$ by means of the generating relation

$$(90) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x_1, \dots, x_r)t^n = \frac{f(x_1, \dots, x_r; t)}{\{g(x_1, \dots, x_r; t)\}^m} \\ &\cdot S_m(h_1(x_1, \dots, x_r; t), \dots, h_r(x_1, \dots, x_r; t)), \end{aligned} \right.$$

where f, g, h_1, \dots, h_r are arbitrary functions of x_1, \dots, x_r and t , and, as before, the $A_{m,n}$ are arbitrary constants, and $m = 0, 1, 2, \dots$, it is fairly straightforward to prove the following extension of Theorem 1.

THEOREM 2. *For arbitrary $a_n \neq 0, n \geq 0$, let*

$$(91) \quad F_{q,r}[x_1, \dots, x_r; t] = \sum_{n=0}^{\infty} a_n S_{qn}(x_1, \dots, x_r)t^n.$$

Then

$$(92) \quad \left\{ \begin{aligned} &\sum_{n=0}^{\infty} S_n(x_1, \dots, x_r)\sigma_n^a(y)t^n = f(x_1, \dots, x_r; t) \\ &\cdot F_{q,r}[h_1(x_1, \dots, x_r; t), \dots, h_r(x_1, \dots, x_r; t); y\{t/g(x_1, \dots, x_r; t)\}^a], \end{aligned} \right.$$

where the $\sigma_n^a(y)$ are given by (4).

For an interesting application of Theorem 2, we recall TOSCANO's formula [15, p. 240]

$$(93) \left\{ \begin{aligned} & \sum_{n=0}^{\infty} \frac{(\gamma+m)_n}{n!} F_D^{(r)}[-m-n, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r] t^n \\ & = (1-t)^{-\gamma-m} \prod_{j=1}^r \left(1 + \frac{x_j t}{1-t} \right)^{-\beta_j} \\ & \cdot F_D^{(r)} \left[-m, \beta_1, \dots, \beta_r; \gamma; \frac{x_1}{1-t+x_1 t}, \dots, \frac{x_r}{1-t+x_r t} \right], \end{aligned} \right.$$

where $F_D^{(r)}$ denotes the fourth type of Lauricella's hypergeometric functions of r variables defined by [5, p. 113]

$$(94) \left\{ \begin{aligned} & F_D^{(r)}[\alpha, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r] \\ & = \sum_{m_1, \dots, m_r=0}^{\infty} \frac{(\alpha)_{m_1+\dots+m_r} (\beta_1)_{m_1} \dots (\beta_r)_{m_r}}{(\gamma)_{m_1+\dots+m_r}} \frac{x_1^{m_1}}{m_1!} \dots \frac{x_r^{m_r}}{m_r!}. \end{aligned} \right.$$

Formula (93) is the same as (90) with $A_{m,n} = (\gamma+m)_n/n!$,

$$f = (1-t)^{-\gamma} \prod_{j=1}^r \left(1 + \frac{x_j t}{1-t} \right)^{-\beta_j}, \quad g = 1-t, \quad h_j = \frac{x_j}{1-t+x_j t}, \quad j=1, \dots, r,$$

and

$$(95) \quad S_n(x_1, \dots, x_r) = F_D^{(r)}[-n, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r], \quad n \geq 0.$$

Consequently, Theorem 2 would yield a class of bilateral generating functions for the Lauricella polynomials in (95), and we have

$$(96) \left\{ \begin{aligned} & \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} F_D^{(r)}[-n, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r] M_n^q(y) t^n \\ & = (1-t)^{-\gamma} \prod_{j=1}^r \left(1 + \frac{x_j t}{1-t} \right)^{-\beta_j} \\ & \cdot F_{q,r}^{(1)} [x_1/(1-t+x_1 t), \dots, x_r/(1-t+x_r t); y t^q/(1-t)^q], \end{aligned} \right.$$

where the $M_n^q(y)$ are defined by (9), and

$$(97) \quad F_{q,r}^{(1)} [x_1, \dots, x_r; t] = \sum_{n=0}^{\infty} \frac{a_n(\gamma) q^n}{(qn)!} F_D^{(r)} [-qn, \beta_1, \dots, \beta_r; \gamma; x_1, \dots, x_r] t^n.$$

Evidently this last result (96) with $q=1$ and $a_n = (\lambda)_n/(\gamma)_n$, $n=0, 1, 2, \dots$, would reduce to a bilateral generating relation due to TOSCANO [15, p. 246, Eq. (14.1)].

Yet another application of Theorem 2 would result in a class of bilateral generating relations for the familiar Lagrange polynomials defined by (cf., e.g., [3], Vol. III, p. 267, Eq. (1))

$$(98) \quad \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y) t^n = (1-xt)^{-\alpha} (1-yt)^{-\beta};$$

these polynomials occur in certain statistical problems. Indeed it is easily verified from (98) that

$$(99) \quad \left\{ \begin{aligned} & \sum_{n=0}^{\infty} \binom{m+n}{n} g_{m+n}^{(\alpha, \beta)}(x, y) t^n \\ & = (1-xt)^{-\alpha} (1-yt)^{-\beta} g_m^{(\alpha, \beta)}\left(x/(1-xt), y/(1-yt)\right), \end{aligned} \right.$$

and Theorem 2 with $r=2$ yields

$$(100) \quad \left\{ \begin{aligned} & \sum_{n=0}^{\infty} g_n^{(\alpha, \beta)}(x, y) M_n^q(z) t^n = (1-xt)^{-\alpha} (1-yt)^{-\beta} \\ & \cdot F_{q,2}^{(2)}[x/(1-xt), y/(1-yt); zt^q], \end{aligned} \right.$$

where the $M_n^q(z)$ are given by (9) with y replaced by z , and

$$(101) \quad F_{q,2}^{(2)}[x, y; t] = \sum_{n=0}^{\infty} a_n g_{qn}^{(\alpha, \beta)}(x, y) t^n,$$

for arbitrary coefficients $a_n \neq 0, n \geq 0$, and $q=1, 2, 3, \dots$

Finally, it may be of interest to observe from (98) that

$$(102) \quad g_n^{(\alpha, \beta)}(x, y) = \frac{(\beta)_n y^n}{n!} {}_2F_1[-n, \alpha; 1-\beta-n; x/y],$$

and on comparing this result with the known hypergeometric representation of the Jacobi polynomials [9, p. 254, Eq. (3)], we readily have

$$(103) \quad g_n^{(\alpha, \beta)}(x, y) = y^n P_n^{(\alpha+\beta-1, -\beta-n)}\left(\frac{2x-y}{y}\right).$$

In view of this interesting relationship between Jacobi and Lagrange polynomials, the bilateral generating function in (100) would evidently follow also from Corollary 3 above.

5. FURTHER GENERALIZATIONS

Since the Bessel function $J_\mu(z)$ possesses a generating relation of the type (cf., e.g., [9], p. 121, Ex. 12)

$$(104) \quad \sum_{n=0}^{\infty} J_{\mu+n}(x) \frac{t^n}{n!} = (1-2t/x)^{-\mu/2} J_\mu\left(\sqrt{(x^2-2xt)t}\right),$$

it may be worthwhile to conclude by recording here a nontrivial generalization of Theorem 1 to hold for a set of functions $\Delta_\mu(x)$ of order μ generated by

$$(105) \quad \sum_{n=0}^{\infty} \gamma_{\mu,n} \Delta_{\mu+n}(x) t^n = \theta(x, t) \{\phi(x, t)\}^{-\mu} \Delta_\mu(\psi(x, t)),$$

where μ is an arbitrary complex number, the $\gamma_{\mu,n}$ are arbitrary constants, and θ, ϕ, ψ are arbitrary functions of x and t .

Indeed, if we let

$$(106) \quad \Phi_{q,\nu}[x, t] = \sum_{n=0}^{\infty} \delta_{\nu,n} \Delta_{\nu+qn}(x) t^n, \delta_{\nu,n} \neq 0,$$

where q is a positive integer, but ν is an arbitrary complex number, then it is readily seen that

$$(107) \quad \sum_{n=0}^{\infty} \Delta_{\nu+n}(x) R_{n,\nu}^q(y) t^n = \theta(x, t) \{\phi(x, t)\}^{-\nu} \Phi_{q,\nu}[\psi(x, t), y\{t/\phi(x, t)\}^q],$$

where $R_{n,\nu}^q(y)$ is a polynomial of degree $[n/q]$ in y defined by

$$(108) \quad R_{n,\nu}^q(y) = \sum_{k=0}^{[n/q]} \gamma_{\nu+qk, n-ak} \delta_{\nu,k} y^k.$$

Evidently, when $\mu = m$ and $\nu = 0$, this last result (107) would correspond to the bilateral generating relation (3) given by Theorem 1.

A similar generalization of Theorem 2, involving a set of functions of several variables, is fairly straightforward, and it may well be left to the interested reader.

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