A Unified Presentation of Certain Classical Polynomials

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Abstract. This paper attempts to present a unified treatment of the classical orthogonal polynomials, viz. Jacobi, Laguerre and Hermite polynomials, and their generalizations introduced from time to time. The results obtained here include a number of linear, bilinear and bilateral generating functions and operational formulas for the polynomials $\{T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r) \mid n = 0, 1, 2, \cdots\}$, defined by Eq. (3) below.*

1. Introduction and Definition. In an attempt to give a unified presentation of the classical orthogonal polynomials, viz. Jacobi, Laguerre and Hermite polynomials, Fujiwara [4] studied the polynomials defined by the generalized Rodrigues' formula

$$(1) \qquad p_n(x) = \frac{(-c)^n}{n!} (x - a)^{-\alpha}(b - x)^{-\beta} D^n\{(x - a)^{n+\alpha}(b - x)^{n+\beta}\},$$

where $D = d/dx$.

The polynomials $p_n(x)$ are orthogonal with respect to the weight function $(x - a)^\alpha(b - x)^\beta$, where $\alpha, \beta > -1$, over the interval $[a, b]$. In fact, as pointed out by Szegö himself [10, p. 58], they can be rewritten as (cf. also [2])

$$(2) \qquad p_n(x) = c^n(a - b)^n P_n^{(\alpha,\beta)}\left(2\left\{\frac{x - a}{a - b}\right\} + 1\right),$$

where $P_n^{(\alpha,\beta)}(x)$ is the classical Jacobi polynomial, orthogonal with respect to the weight function $(1 - x)^\alpha(1 + x)^\beta$, where $\alpha, \beta > -1$, over the interval $[-1, 1]$. By recourse to certain limiting processes, it is easy to verify that the $p_n(x)$, as also the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, give rise to Hermite and Laguerre polynomials.

It may be of interest to study here the polynomial system $\{T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)\}$ defined by

$$(3) \quad \begin{aligned} &T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r) \\ &= \frac{(ax + b)^{-\alpha}(cx + d)^{-\beta}}{n!} \exp(px^r) \, D^n\{(ax + b)^{n+\alpha}(cx + d)^{n+\beta} \exp(-px^r)\}. \end{aligned}$$

Apart from being more general than $p_n(x)$, the polynomial system defined by (3) has a distinct advantage in that it can be specialized to almost all classes of orthogonal polynomials without using any limiting processes. In particular, we mention the following obvious connections.

(4) $T_n^{(\alpha,\beta)}(x, a, -a, c, c, 0, r) = T_n^{(\alpha,\beta)}(x, a, -a, c, c, p, 0) = (2ca)^n P_n^{(\alpha,\beta)}(x).$

(5) $T_n^{(\alpha,\beta)}(x, a, 0, 0, d, 1, 1) = (ad)^n L_n^{(\alpha)}(x).$

(6) $T_n^{(\alpha,\beta)}(x, 0, b, c, 0, 1, 1) = (bc)^n L_n^{(\beta)}(x).$

(7) $T_n^{(\alpha,\beta)}(x, 0, b, 0, d, 1, 2) = (-bd)^n H_n(x).$

(8) $T_n^{(\alpha,0)}(x, a, 0, c, 0, 2, -1) = (2ac)^n Y_n^{(\alpha)}(x).$

(9) $T_n^{(\alpha-n,\beta)}(x, a, 0, 0, d, p, r) = \dfrac{(-adx)^n}{n!} H_n^r(x, \alpha, p).$

Here, $Y_n^{(\alpha)}(x)$ denotes the generalized Bessel polynomial of Krall and Frink [6] defined by

(10) $Y_n^{(\alpha)}(x) = {}_2F_0[-n, n + \alpha + 1; -; -\tfrac{1}{2}x],$

and $H_n^r(x, \alpha, p)$ is the generalized Hermite polynomial

(11) $H_n^r(x, \alpha, p) = (-1)^n x^{-\alpha} \exp(px^r) D^n\{x^\alpha \exp(-px^r)\},$

introduced earlier by Gould and Hopper [5].

By making use of Leibniz' rule, $T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)$ can be expressed in the form

$$T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r) = (ad - bc)^n$$

(12)

$$\cdot \sum_{k=0}^n \frac{(-)^k}{k!} \left\{ \frac{(ax + b)(cx + d)}{ad - bc} \right\}^k P_{n-k}^{(\alpha+k,\beta+k)}\left(2a\,\frac{cx + d}{ad - bc} - 1\right) H_n^r(x, 0, p).$$

Moreover, it is readily seen that

(13) $T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r) = (ab)^{-n} T_n^{(\beta,\alpha)}\left(\dfrac{bcx}{ad}, a, b, a^2d, b^2c, \left(\dfrac{ad}{bc}\right)^r p, r\right),$

which is a generalization of the familiar result [10, p. 59]

(14) $P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x)$

involving Jacobi polynomials.

2. Generating Functions.

From the definition (3), we have

$$\sum_{n=0}^\infty \left\{ \frac{t}{(ax + b)(cx + d)} \right\}^n T_n^{(\alpha-\lambda n,\beta-\mu n)}(x, a, b, c, d, p, r)$$

(15)

$$= (ax + b)^{-\alpha}(cx + d)^{-\beta} \exp(px^r)$$

$$\cdot \sum_{n=0}^\infty \frac{t^n}{n!} D^n\{(ax + b)^{n+\alpha-\lambda n}(cx + d)^{n+\beta-\mu n} \exp(-px^r)\}$$

and, on interpreting this last expression by means of Lagrange's theorem [7]

$$(16) \qquad \frac{f(y)}{1 - t\phi'(y)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D^n\{[\phi(x)]^n f(x)\}, \qquad y = x + t\phi(y),$$

we find that

$$\sum_{n=0}^{\infty} T_n^{(\alpha-\lambda n, \beta-\mu n)}(x, a, b, c, d, p, r)t^n$$

$$= \frac{\left(\dfrac{ay+b}{ax+b}\right)^{\alpha}\left(\dfrac{cy+d}{cx+d}\right)^{\beta} \exp\{px^r - py^r\}}{1 - at(1-\lambda)(cx+d)\left(\dfrac{ax+b}{ay+b}\right)^{\lambda}\left(\dfrac{cx+d}{cy+d}\right)^{\mu-1} - ct(1-\mu)(ax+b)\left(\dfrac{ax+b}{ay+b}\right)^{\lambda-1}\left(\dfrac{cx+d}{cy+d}\right)^{\mu}},$$

where

$$y = x + t(ax + b)(cx + d)\left(\frac{ax + b}{ay + b}\right)^{\lambda-1}\left(\frac{cx + d}{cy + d}\right)^{\mu-1}.$$

If we put

$$(ax + b)/(ay + b) = 1 + u, \qquad (cx + d)/(cy + d) = 1 + v,$$

then we shall get the generating function

$$(17) \qquad \sum_{n=0}^{\infty} T_n^{(\alpha-\lambda n, \beta-\mu n)}(x, a, b, c, d, p, r)t^n$$

$$= \frac{(1 + u)^{-\alpha}(1 + v)^{-\beta}}{1 + (1 - \lambda)u + (1 - \mu)v} \exp\left\{px^r - p\left(\frac{ax - bu}{a + au}\right)^r\right\},$$

where u and v are given by

$$(18) \qquad \begin{cases} u = -at(cx + d)(1 + u)^{\lambda}(1 + v)^{\mu-1}, \\ v = -ct(ax + b)(1 + u)^{\lambda-1}(1 + v)^{\mu}. \end{cases}$$

Alternatively, by letting $u = -\xi/(1 + \xi)$ and $v = -\eta/(1 + \eta)$, the generating function (17) can be written as

$$(19) \qquad \sum_{n=0}^{\infty} T_n^{(\alpha-\lambda n, \beta-\mu n)}(x, a, b, c, d, p, r)t^n$$

$$= \frac{(1 + \xi)^{\alpha+1}(1 + \eta)^{\beta+1}}{1 + \lambda\xi + \mu\eta - (1 - \lambda - \mu)\xi\eta} \exp\left\{px^r - p\left(x + \frac{\xi}{a}(ax + b)\right)^r\right\},$$

where

$$(20) \qquad \begin{cases} \xi = at(cx + d)(1 + \xi)^{1-\lambda}(1 + \eta)^{1-\mu}, \\ \eta = ct(ax + b)(1 + \xi)^{1-\lambda}(1 + \eta)^{1-\mu}. \end{cases}$$

In view of the relationships (4) through (9), the results given above would readily yield a large number of generating functions for the polynomials of Jacobi, Hermite, Laguerre, Gegenbauer, and many others. For instance, we have the following special cases:

(i) If $\lambda = \mu = 0$, then both (17) and (19) would give us the elegant generating function

$$\sum_{n=0}^{\infty} T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r)t^n$$

$$(21) \qquad = 2^{\alpha+\beta}\rho^{-1}\{1 + (bc - ad)t + \rho\}^{-\alpha}\{1 - (bc - ad)t + \rho\}^{-\beta}$$

$$\cdot \exp\left\{px^r - \frac{p}{a^r}\left[\frac{2(ax + b)}{1 + (bc - ad)t + \rho} - b\right]^r\right\},$$

where $\rho = \{1 - 2t(bc + 2acx + ad) + t^2(bc - ad)^2\}^{1/2}$.

(ii) For p or $r = 0$ and $-a/b = c/d = 1$, (17) and (19) correspond to our main results in [9], while (21) would give us the familiar generating function ([10, p. 69]; see also [7, Vol. 1, pp. 127, 303, Problem 219])

$$(22) \qquad \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)t^n = 2^{\alpha+\beta}R^{-1}(1 - t + R)^{-\alpha}(1 + t + R)^{-\beta},$$

in which $R = (1 - 2xt + t^2)^{1/2}$.

(iii) On combining (8) with (17) and (18), we get the known generating function [1, p. 654]

$$(23) \qquad \sum_{n=0}^{\infty} Y_n^{(\alpha+\lambda n)}(x)\frac{t^n}{n!} = (1 + u)^{-\alpha}\{1 + (2 + \lambda)u\}^{-1}\exp\{t(1 + u)^{-\lambda-1}\},$$

where $-\frac{1}{2}xt = u(1 + u)^{\lambda+1}$.

(iv) On using (9) with (19) and (20), we are led to the generating function

$$(24) \qquad \sum_{n=0}^{\infty} H_n^r(x, \alpha - \lambda n, p)\frac{t^n}{n!} = (1 + \xi)^{\alpha+1}(1 + \lambda\xi + \xi)^{-1}\exp[px^r\{1 - (1 + \xi)^r\}],$$

where $\xi = -x^{-1}t(1 + \xi)^{-\lambda}$.

The last result (24) does not seem to have been noticed earlier.

3. Operational Formulas. In this section, we shall make use of the differential operator $\delta = xd/dx$ which possesses the following interesting properties:

$$(25) \qquad x^n D^n = \delta(\delta - 1) \cdots (\delta - n + 1),$$

$$(26) \qquad f(\delta)\exp\{g(x)\}h(x) = \exp\{g(x)\}f\{\delta + xg'\}h(x).$$

Assuming Y to be a sufficiently differentiable function and using the properties (25) and (26), we observe that

$$(ax + b)^{-\alpha}(cx + d)^{-\beta}\exp(px^r) D^n\{(ax + b)^{n+\alpha}(cx + d)^{n+\beta}\exp(-px^r) Y\}$$

$$(27)$$

$$= \left\{\frac{(ax + b)(cx + d)}{x}\right\}^n \prod_{j=1}^{n}\left[\delta + \frac{(n + \alpha)ax}{ax + b} + \frac{(n + \beta)cx}{cx + d} - prx^r - j + 1\right] Y,$$

whereas, by employing Leibniz' rule, the left-hand side of (27) can also be expressed in the form

$$n! \sum_{k=0}^{n}\frac{(ax + b)^k(cx + d)^k}{k!} T_{n-k}^{(\alpha+k,\beta+k)}(x, a, b, c, d, p, r) D^k Y.$$

Equivalence of the two expressions yields the operational formula

$$
(28) \quad \prod_{j=1}^{n} \left[\delta + \frac{a(n+\alpha)x}{ax+b} + \frac{c(n+\beta)x}{cx+d} - prx^r - j + 1 \right] Y
$$

$$
= n! \left\{ \frac{x}{(ax+b)(cx+d)} \right\}^n \sum_{k=0}^{n} \frac{(ax+b)^k(cx+d)^k}{k!}
$$

$$
\cdot T_{n-k}^{(\alpha+k,\beta+k)}(x, a, b, c, d, p, r) \, D^k Y.
$$

For $Y = 1$, (27) would give us

$$
(29) \quad \prod_{j=1}^{n} \left[\delta + \frac{a(n+\alpha)x}{ax+b} + \frac{c(n+\beta)x}{cx+d} - prx^r - j + 1 \right] \cdot 1
$$

$$
= n! \left\{ \frac{x}{(ax+b)(cx+d)} \right\}^n T_n^{(\alpha,\beta)}(x, a, b, c, d, p, r).
$$

If in (29) we replace n by $n + m$ and then interpret the left-hand side with the help of (28) and (29), we shall get the interesting formula

$$
(30) \quad \binom{m+n}{n} T_{m+n}^{(\alpha,\beta)}(x, a, b, c, d, p, r) = \sum_{k=0}^{\infty} \frac{(ax+b)^k(cx+d)^k}{k!}
$$

$$
\cdot T_{n-k}^{(\alpha+k,\beta+k)}(x, a, b, c, d, p, r) \, D^k T_m^{(\alpha+n,\beta+n)}(x, a, b, c, d, p, r).
$$

As an application of (30), we first replace α, β by $\alpha - n$ and $\beta - n$, respectively, multiply both sides by t^n, take their sums from $n = 0$ to $n = \infty$ and then interpret its right-hand side with the help of Taylor's theorem and formula (18). We are thus led to the following generating function:

$$
(31) \quad \sum_{n=0}^{\infty} \binom{m+n}{n} T_{m+n}^{(\alpha-n,\beta-n)}(x, a, b, c, d, p, r)t^n
$$

$$
= \{1 + at(cx+d)\}^{\alpha}\{1 + ct(ax+b)\}^{\beta}
$$

$$
\cdot \exp\{px^r - p(x + t(ax+b)(cx+d))^r\}
$$

$$
\cdot T_m^{(\alpha,\beta)}(x + t(ax+b)(cx+d), a, b, c, d, p, r).
$$

On the other hand, by making use of the definition (3) and Lagrange's theorem (16), we observe that

$$
\sum_{n=0}^{\infty} \binom{m+n}{n} T_{m+n}^{(\alpha-n,\beta)}(x, a, b, c, d, p, r)\left(\frac{t}{ax+b}\right)^n
$$

$$
= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{(1-ct)^{m+1}} \exp(px^r) \frac{1}{m!} \left(\frac{d}{dy}\right)^m \{(ay+b)^{\alpha+m}(cy+d)^{\beta+m} \exp(-py^r)\},
$$

where $y = (x + dt)(1 - ct)^{-1}$.

Interpreting the right-hand side of the last equation by means of (3), we get another generating function in the form

$$
\sum_{n=0}^{\infty} \binom{m+n}{n} T_{m+n}^{(\alpha-n,\beta)}(x, a, b, c, d, p, r)t^n
$$

$$
(32) \quad = \{1 + t(ad-bc)\}^{\alpha}\{1 - ct(ax+b)\}^{-\alpha-\beta-m-1}
$$

$$
\cdot \exp\left\{px^r - p\left(\frac{x + dt(ax+b)}{1 - ct(ax+b)}\right)^r\right\} T_m^{(\alpha,\beta)}\left(\frac{x + dt(ax+b)}{1 - ct(ax+b)}, a, b, c, d, p, r\right).
$$

It is worthwhile to remark here that in view of relations (5) and (6), the generating function (32) provides us with a unification of a large number of results including, for instance, the known formulas (cf., e.g., [2, pp. 221–222])

$$(33) \qquad \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha)}(x) t^n = (1-t)^{-\alpha-m-1} \exp\left(-\frac{xt}{1-t}\right) L_m^{(\alpha)}\left(\frac{x}{1-t}\right)$$

and

$$(34) \qquad \sum_{n=0}^{\infty} \binom{m+n}{n} L_{m+n}^{(\alpha-n)}(x) t^n = (1+t)^{\alpha} \exp(-xt) L_m^{(\alpha)}\{x(1+t)\}.$$

4. Bilateral Generating Functions. In this section, we apply the generating relations (31) and (32) to prove the following theorems.

THEOREM 1. *If*

$$(35) \qquad F[x,t] = \sum_{n=0}^{\infty} a_n T_n^{(\alpha-n,\beta)}(x,a,b,c,d,p,r) t^n,$$

where the a_n are arbitrary constants, then

$$\{1 + t(ad-bc)\}^{\alpha} \{1 - ct(ax+b)\}^{-\alpha-\beta-1} \exp\left\{px^r - p\left(\frac{x+dt(ax+b)}{1-ct(ax+b)}\right)^r\right\}$$

$$(36)$$

$$\cdot F\left[\frac{x+dt(ax+b)}{1-ct(ax+b)}, \frac{yt}{1+t(ad-bc)}\right] = \sum_{n=0}^{\infty} T_n^{(\alpha-n,\beta)}(x,a,b,c,d,p,r) \sigma_n(y) t^n,$$

where $\sigma_n(y)$ is a polynomial of degree n in y given by

$$(37) \qquad \sigma_n(y) = \sum_{k=0}^{n} \binom{n}{k} a_k y^k.$$

THEOREM 2. *For arbitrary a_n, $n \geq 0$, let*

$$(38) \qquad G[x,t] = \sum_{n=0}^{\infty} a_n T_n^{(\alpha-n,\beta-n)}(x,a,b,c,d,p,r) t^n,$$

then

$$\{1 + at(cx+d)\}^{\alpha} \{1 + ct(ax+b)\}^{\beta} \exp\{px^r - p(x+t(ax+b)(cx+d))^r\}$$

$$(39)$$

$$\cdot G[x + t(ax+b)(cx+d), yt/\{1+at(cx+d)\}\{1+ct(ax+b)\}]$$

$$= \sum_{n=0}^{\infty} T_n^{(\alpha-n,\beta-n)}(x,a,b,c,d,p,r) \sigma_n(y) t^n,$$

where $\sigma_n(y)$ is given by (37).

To prove Theorem 1, we substitute the series expansion of $\sigma_n(y)$, given by (37), on the right-hand side of (36) and we get

$$\sum_{n=0}^{\infty} T_n^{(\alpha-n,\beta)}(x,a,b,c,d,p,r) \sigma_n(y) t^n$$

$$= \sum_{k=0}^{\infty} a_k y^k t^k \sum_{n=0}^{\infty} \binom{n+k}{n} T_{n+k}^{(\alpha-n-k,\beta)}(x,a,b,c,d,p,r) t^n.$$

On summing the inner series with the help of (32) and then interpreting the resulting expression by means of (35), we are led immediately to Theorem 1.

Similar is the proof of Theorem 2. Indeed, we make use of the generating relation (31) in place of (32).

Alternatively, Theorems 1 and 2 may be deduced as corollaries of a general result on bilateral generating functions given elsewhere by us [8].

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