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An Explicit Formula for the Generalized Bernoulli Polynomials

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The object of the present note is to prove a new explicit formula for the generalized Bernoulli polynomials. The main result (3) below provides an interesting extension of a representation for the generalized Bernoulli numbers given recently by P. G. Todorov [*C.R. Acad. Sci. Paris Sér. I Math.* **301** (1985), 665–666]. © 1988 Academic Press, Inc.

1. INTRODUCTION

In the usual notations, let $B_n^{(\alpha)}(x)$ denote the generalized Bernoulli polynomial of degree n in x, defined by

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z|<2\pi;\ 1^\alpha=1) \tag{1}$$

for an arbitrary (real or complex) parameter α. Clearly,

$$B_n^{(\alpha)}(x) = (-1)^n B_n^{(\alpha)}(\alpha - x),$$

so that

$$B_n^{(\alpha)}(\alpha) = (-1)^n B_n^{(\alpha)}(0) \equiv (-1)^n B_n^{(\alpha)}$$

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in terms of the generalized Bernoulli numbers $B_n^{(\alpha)}$ (cf. [3, p. 227, Exercise 18]). From the generating relation (1), it is fairly straightforward to deduce the addition theorem:

$$B_n^{(\alpha+\beta)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x)\,B_{n-k}^{(\beta)}(y),$$

which, for $x=\beta=0$, corresponds to the elegant representation:

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}x^{n-k} \tag{2}$$

for the generalized Bernoulli polynomials as a finite sum of the generalized Bernoulli numbers.

Recently, Srivastava, Lavoie, and Tremblay [5, p. 442, Eqs. (4.4) and (4.5)] gave two new classes of addition theorems for the generalized Bernoulli polynomials. In the present note we first prove the following explicit formula for these generalized Bernoulli polynomials:

$$\begin{aligned}B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{2k}(x+j)^{n-k}&\\ \times F[k-n,k-\alpha;2k+1;j/(x+j)],&\end{aligned} \tag{3}$$

where $F[a,b;c;z]$ denotes the Gaussian hypergeometric function defined by (cf., e.g., [2, Chap. 1])

$$F[a,b;c;z]=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots. \tag{4}$$

We shall also apply the representation (3) in order to derive certain interesting special cases considered earlier by Gould [4] and Todorov [6].

2. Proof of the Explicit Formula (3)

By Taylor's expansion and Leibniz's rule, the generating relation (1) yields

$$B_n^{(\alpha)}(x)=\sum_{s=0}^{n}\binom{n}{s}x^{n-s}D_z^s\left\{\left(\frac{z}{e^z-1}\right)^{\alpha}\right\}\bigg|_{z=0},\qquad D_z=\frac{d}{dz}. \tag{5}$$

Since

$$(1+w)^{-\alpha}=\sum_{l=0}^{\infty}\binom{\alpha+l-1}{l}(-w)^l\qquad(|w|<1),$$

setting $1+w=(e^z-1)/z$, and applying the binomial theorem, we find from (5) that

$$B_n^{(\alpha)}(x)=\sum_{s=0}^{n}\binom{n}{s}x^{n-s}\sum_{l=0}^{s}\binom{\alpha+l-1}{l}$$
$$\times\sum_{k=0}^{l}(-1)^k\binom{l}{k}D_z^s\left\{\left(\frac{e^z-1}{z}\right)^k\right\}\Big|_{z=0} \tag{6}$$

Now make use of the well-known formula (cf., e.g., [4, p. 48])

$$(e^z-1)^k=\sum_{r=k}^{\infty}\frac{z^r}{r!}\Delta^k 0^r, \tag{7}$$

where, for convenience,

$$\Delta^k a^r=\Delta^k x^r\,|_{x=a}=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(a+j)^r, \tag{8}$$

Δ being the difference operator defined by (cf. [3, p. 13 *et seq.*])

$$\Delta f(x)=f(x+1)-f(x),$$

so that, in general,

$$\Delta^k f(x)=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}f(x+j).$$

Formula (7) readily yields

$$D_z^s\left\{\left(\frac{e^z-1}{z}\right)^k\right\}\Big|_{z=0}=\frac{s!}{(s+k)!}\Delta^k 0^{s+k}, \tag{9}$$

and upon substituting this value in (6), if we rearrange the resulting triple series, we have

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}(-1)^k\binom{\alpha+k-1}{k}\sum_{s=0}^{n-k}\binom{n}{s+k}\frac{(s+k)!}{(s+2k)!}x^{n-s-k}\Delta^k 0^{s+2k}$$
$$\times\sum_{l=0}^{s}\binom{\alpha+k+l-1}{l}. \tag{10}$$

The innermost sum in (10) can be evaluated by appealing to the elementary combinatorial identity:

$$\sum_{l=0}^{s}\binom{\lambda+l-1}{l}=\binom{\lambda+s}{s},$$

and if we further substitute for $\Delta^k 0^{s+2k}$ from the definition (8) with $a=0$, we obtain

$$B_n^{(\alpha)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{\alpha+k-1}{k}\frac{k!}{(2k)!}x^{n-k}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{2k}$$
$$\times F[k-n,\alpha+k+1;2k+1;-j/x] \tag{11}$$

in terms of the Gaussian hypergeometric function defined by (4).

Finally, we apply the known transformation [1, p. 3, Eq. (6)]

$$F[a,b;c;z]=(1-z)^{-a}F[a,c-b;c;z/(z-1)],$$

and (11) leads us immediately to the explicit formula (3).

3. Applications

By Vandermonde's theorem [2, p. 3], we have

$$F[-N,b;c;1]=\binom{c-b+N-1}{N}\binom{c+N-1}{N}^{-1} \qquad (N=0,1,2,...),$$

which (for $N=n-k$, $b=k-\alpha$, and $c=2k+1$) readily yields

$$F[k-n,k-\alpha;2k+1;1]=\binom{\alpha+n}{n-k}\frac{(n-k)!\,(2k)!}{(n+k)!} \qquad (0\leqslant k\leqslant n). \tag{12}$$

In view of (12), the special case of our formula (3) when $x=0$ gives us the following representation for the generalized Bernoulli numbers:

$$B_n^{(\alpha)}=\sum_{k=0}^{n}\binom{\alpha+n}{n-k}\binom{\alpha+k-1}{k}\frac{n!}{(n+k)!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^{n+k} \tag{13}$$

or, equivalently,

$$B_n^{(\alpha)}=\sum_{k=0}^{n}(-1)^k\binom{\alpha+n}{n-k}\binom{\alpha+k-1}{k}\frac{n!}{(n+k)!}\Delta^k 0^{n+k}, \tag{14}$$

in terms of the difference operation exhibited by (8).

Alternatively, since [3, p. 204, Theorem A]

$$S(n,k)=\frac{1}{k!}\Delta^k 0^n,$$

where $S(n, k)$ denotes the Stirling number of the second kind, defined by [3, p. 207, Theorem B]

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k!\, S(n, k),$$

this last representation (13) or (14) can be written also as

$$B_n^{(\alpha)} = \sum_{k=0}^{n} (-1)^k \binom{\alpha+n}{n-k} \binom{\alpha+k-1}{k} \binom{n+k}{k}^{-1} S(n+k, k). \tag{15}$$

Formula (15), given recently by Todorov [6, p. 665, Eq. (3)], provides an interesting generalization of the following known result for the Bernoulli numbers $B_n \equiv B_n^{(1)}$:

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \binom{n+k}{k}^{-1} \frac{\Delta^k 0^{n+k}}{k!},$$

which was considered, for example, by Gould [4, p. 49, Eq. (17)].

We should like to conclude by remarking that, in view of Todorov's result (15) and the representation (2), it is not difficult to construct an alternative proof of our explicit formula (3).

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