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Polynomials of Binomial Type and Compound Poisson Processes

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Let q_n and s_n, $n \in \mathbb{N}$, respectively, be a set of polynomials of binomial type and a Sheffer set related to it, both having positive coefficients. Then $q_n(x)$, $x > 0$ is connected with the probability that a compound Poisson process starting at zero is in state n at time τx and $q_n(x)/q_n(1)$ is the probability generating function of the number of jumps of this process in $[0, \tau]$ given that it is in state n at time τ. The s_n admit similar interpretations when the initial distribution of the compound Poisson process is not concentrated at zero. The possible limits for $n \to \infty$ of $q_n(x)/q_n(1)$ and $s_n(x)/s_n(1)$ are studied. © 1988 Academic Press, Inc.

1. INTRODUCTION

The theory of polynomials of binomial type was developed in [22, 25]. The sequence of polynomials

$$q_n(x) = \sum_{k=0}^{n} q_{nk} x^k, \tag{1.1}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$ is of binomial type if $q_{00} = 1$, $q_{nn} \neq 0$, $n \geqq 1$, and

$$q_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} q_k(x)\, q_{n-k}(y), \qquad x, y \in \mathbb{R},\ n \in \mathbb{N}. \tag{1.2}$$

Then $q_{n0} = 0$, $n \geqq 1$. The sequence (1.1) is of binomial type iff (if and only if) it is the sequence of basic polynomials for a delta operator $Q\colon \Pi \to \Pi$, where Π is the set of polynomials $\mathbb{R} \to \mathbb{R}$, i.e., iff

$$q_0(x) \equiv 1, \qquad q_{n0} = 0, \qquad n \geqq 1, \tag{1.3}$$

$$Qq_n = nq_{n-1}, \qquad n \geqq 1. \tag{1.4}$$

A delta operator is a linear operator $L\colon \Pi \to \Pi$ that is shift-invariant,

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i.e., $LE^a = E^a L$ with $E^a f(x) = f(x+a)$, and such that $L1 = 0$. By the first expansion theorem [22; 25, Theorem 2] we have $Q = q(D)$, where $Df = f'$ and the formal power series q has $q(0) = 0$, $q'(0) \neq 0$. Then q has a compositional inverse

$$g(u) = \sum_{k=1}^{\infty} g_k u^k, \qquad g_1 \neq 0, \tag{1.5}$$

and we have the formal expansion, see [22, Section 4; 25, Section 3],

$$\sum_{n=0}^{\infty} u^n q_n(x)/n! = \exp(xg(u)), \tag{1.6}$$

which is also sufficient for the q_n to be a basic set for Q. From [25, Corollaries 1 and 2 to Theorem 5] we have

$$g_k = q_{k1}/k!, \qquad k \geqq 1. \tag{1.7}$$

The sequence of polynomials s_n, $n \in \mathbb{N}$, is a Sheffer set for the delta operator Q, see [25, Section 5], if $s_0(x) \equiv c \neq 0$, if s_n has exact degree n and $Qs_n = ns_{n-1}$, $n \geqq 1$. This holds iff

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_k(0)\, q_{n-k}(x), \qquad x \in \mathbb{R},\ n \in \mathbb{N}, \tag{1.8}$$

with $s_0(0) \neq 0$, and then for x, $y \in \mathbb{R}$, $n \in \mathbb{N}$,

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} s_k(x)\, q_{n-k}(y). \tag{1.9}$$

In Section 14(5) of [25] a connection between polynomials of binomial type and compound Poisson processes was suggested. The aim of this paper is to follow this suggestion. An interpretation different from ours, in terms of Poisson point processes, was given by Cerasoli [2]. From (1.7) we see that $q_{n1} \geqq 0$, $n \geqq 1$, iff $g_k \geqq 0$, $k \geqq 1$, and (1.6) shows that then $q_{nk} \geqq 0$ for all n and k. This will be assumed, and also that the series (1.5) has radius of convergence $\rho > 0$. In Section 2 the probability distribution $q_{nk}/q_n(1)$, $k = 0, ..., n$, with probability generating function (pgf) $q_n(x)/q_n(1)$ and also the q_{nk} and $q_n(x)$ for $x \geqq 0$ will be interpreted in terms of a compound Poisson process and its jumps. For the Sheffer set (1.8) a related interpretation holds if $s_k(0) \geqq 0$, $k \geqq 0$. Umbral composition will be seen to have a probabilistic meaning.

In Section 3 we will study the possible limits as $n \to \infty$ of $q_n(x)/q_n(1)$ and $s_n(x)/s_n(1)$. Section 4 gives an example of applying the local central limit theorem to obtain the asymptotic behavior of the q_{nk} and describes a method to find explicit expressions for $g(u)$ and $q_n(x)$, used recently in probability theory to derive new distributions.

2. The Compound Poisson Process

We take q_n, g, and s_n fixed as defined in Section 1. Let $\{Y_t \equiv Y(t),\ t \geqq 0\}$ with $Y_0 \equiv 0$ be a compound Poisson process, i.e., a stochastic process with stationary independent nonnegative increments. See [15, Vol. I, Chap. XII.2; 19, Chap. 16.9]. The pgf of $Y(t+a) - Y(t)$, $a \geqq 0$, has the form, for $|u| \leqq 1$,

$$Eu^{Y(t)} = \exp\left(-ct + ct \sum_{h=1}^{\infty} \pi_h u^h\right), \tag{2.1}$$

with $c > 0$ and $\pi_k \geqq 0$, $k \geqq 1$, $\sum \pi_k = 1$. Let M_t be the number of jumps of the Y-process occuring during the time interval $[0, t]$. Then $\{M_t, t \geqq 0\}$ is a Poisson process with intensity c, the jumps $X_1, X_2, \ldots$ are independent random variables, independent also of the M_t-process, with $P(X_i = k) = \pi_k$, $k \geqq 1$, $i \geqq 1$, and we have

$$Y_t = S(M_t), \tag{2.2}$$

$$S_0 \equiv S(0) = 0, \qquad S_n \equiv S(n) = X_1 + \cdots + X_n, \qquad n \geqq 1. \tag{2.3}$$

We take $c = 1$ and

$$\pi_k = \tau^{-1} g_k \eta^k, \qquad \eta > 0, \qquad \tau = g(\eta) < \infty, \tag{2.4}$$

with g_k and g as in (1.5). The dependence of probabilities and expectations on the parameter η will be shown by writing them as P_η and E_η. We have from (2.1) for $x \geqq 0$

$$E_\eta u^{Y(\tau x)} = \exp\{-\tau x + x g(\eta u)\}, \qquad |u| \leqq 1. \tag{2.5}$$

With (1.6) this gives

$$q_n(x)/n! = \eta^{-n} e^{\tau x} P_\eta(Y(\tau x) = n), \qquad x \geqq 0, \tag{2.6}$$

for all $\eta > 0$ with $g(\eta) < \infty$, so that $q_n(x)/n!$ for $x \geqq 0$, apart from the factor $\eta^{-n} \exp(\tau x)$ has the meaning of a probability in the Y-process. The relation (1.2) corresponds to the convolution property for the Y-process:

$P_\eta(Y_{x+y}=n)=\sum_{k=0}^{n} P_\eta(Y_x=k)\, P_\eta(Y_y=n-k)$, $x, y \geqq 0$. From (2.2) we have, by conditioning with respect to $M_{\tau x}$,

$$P_\eta(Y_{\tau x}=n)=\sum_{k=0}^{n} (\tau x)^k\, e^{-\tau x} P_\eta(S_k=n)/k!, \tag{2.7}$$

so with (2.6) and (1.1)

$$q_{nk}/n! = \eta^{-n} \tau^k P_\eta(S_k=n)/k!, \tag{2.8}$$

so that the coefficients of the polynomial q_n are connected to the distributions of sums of jumps of the Y-process.

From (2.8), (2.6) with $x=1$, (2.2) and the independence of M_τ and S_k,

$$\begin{aligned} q_{nk}/q_n(1) &= \tau^k (k!)^{-1}\, e^{-\tau} P_\eta(S_k=n)/P_\eta(Y_\tau=n) \\ &= P_\eta(M_\tau=k,\ S_k=n)/P_\eta(Y_\tau=n) \\ &= P_\eta(M_\tau=k,\ Y_\tau=n)/P_\eta(Y_\tau=n) = P_\eta(M_\tau=k \mid Y_\tau=n). \end{aligned} \tag{2.9}$$

So the probability distribution $q_{nk}/q_n(1)$, $k=0, ..., n$, is the conditional distribution given $Y_\tau=n$ of the number of jumps in $[0, \tau)$, for all $\eta>0$ with $g(\eta)<\infty$. A similar interpretation holds for $a^k q_{nk}/q_n(a)$. As applications we mention queues with Poisson group arrival and immigration where the process of immigrating families is Poisson, see [18, Chap. 1.2]. Then (2.9) gives the conditional number of groups or families given the number of customers arrived or persons immigrated.

From (1.1), (2.9), and (2.6),

$$\begin{aligned} q_n(x)/q_n(1) &= E_\eta(x^{M_\tau} \mid Y_\tau=n) \\ &= \exp(\tau x-\tau)\, P_\eta(Y_{\tau x}=n)/P_\eta(Y_\tau=n), \end{aligned} \tag{2.10}$$

where the second equality holds for $x \geqq 0$. The following more general results hold.

THEOREM 1. *Let $\{A_1, ..., A_m\}$ be a partion of $[0, \tau)$ with $|A_i|=\tau_i$ and M_i the number of jumps in A_i of the Y-process. Then*

$$E_\eta\{x_m^{M_1} \cdots x_m^{M_m} \mid Y_\tau=n\} = q_n\left(\tau^{-1} \sum_{i=1}^{m} \tau_i x_i\right) \Big/ q_n(1). \tag{2.11}$$

Proof. With (2.2), the independence of $\{M_t, t \geqq 0\}$ and $X_1, X_2, ...,$ (2.8), and (2.6) for $x=1$ we find, putting $k=k_1+\cdots+k_m$,

$$\begin{aligned}
&P_\eta(M_i = k_i,\ i = 1, ..., m \mid Y_\tau = n)\\
&\quad = P_\eta(M_i = k_i,\ i = 1, ..., m,\ S_k = n)/P_\eta(Y_\tau = n)\\
&\quad = \left\{\prod_{i=1}^{m} \tau_i^{k_i} \exp(-\tau_i)/k_i!\right\} P_\eta(S_k = n)/P_\eta(Y_\tau = n)\\
&\quad = k!\, q_{nk} \left\{\prod_{i=1}^{m} (\tau^{-1}\tau_i)^{k_i}/k_i!\right\}\Big/ q_n(1).
\end{aligned}$$

Multiplying with $x_1^{k_1} \cdots x_m^{k_m}$, summing first over $k_1, ..., k_m$ with $k_1 + \cdots + k_m = k$ using the multinomial theorem, and finally summing over k we find (2.11).

Umbral Composition, See [25, *Section* 7]

Let $p_n(x) = \sum p_{nk} x^k$, $n \in \mathbb{N}$, be the sequence of basic polynomials for $P = p(D)$ with $f = p^{-1}$. Then the sequence

$$r_n(x) = \sum_{k=0}^{n} p_{nk} q_k(x), \qquad n \in \mathbb{N}, \tag{2.12}$$

is basic for the delta operator $r(D) = p(q(D))$. Since $r^{-1} = q^{-1} \circ p^{-1} = g \circ f$ we have

$$\sum_{n=0}^{\infty} u^n r_n(x)/n! = \exp\{x g \circ f(u)\}. \tag{2.13}$$

We assume that f has nonnegative coefficients and a positive radius of convergence. We consider the set of $\theta > 0$ such that $g \circ f(\theta) < \infty$. Since $f(0) = 0$ this set is nonempty. We put

$$f(\theta) = \eta, \qquad g \circ f(\theta) = g(\eta) = \tau < \infty. \tag{2.14}$$

Let $U_1, U_2, ...$ be independent and also independent of the M_t and X_i. Let U_j have pgf

$$\tau^{-1} g \circ f(\theta u) = g_\eta \circ f_\theta(u), \tag{2.15}$$

where $g_\eta(u) = g(\eta u)/g(\eta)$ and $f_\theta(u) = f(\theta u)/f(\theta)$. Then in the same way as (2.6)

$$r_n(x)/n! = \theta^{-n} e^{\tau x} P_\theta(Z_{\tau x} = n), \qquad x \geqq 0, \tag{2.16}$$

where $Z_t = V(M_t)$, $V(0) = 0$, $V(n) = U_1 + \cdots + U_n$. Now let $\omega_1, \omega_2, ...$ be independent and also independent of the M_t and X_i and let ω_j have pgf f_θ. By conditioning with respect to S_n we see that $\Omega(S_n)$, with $\Omega(0) = 0$,

$\Omega(n)=\omega_1+\cdots+\omega_n$, $n\geqq 1$, has the same pgf as $V(n)$, $(g_\eta\circ f_\theta)^n$, cf. [15, Vol. I, Chap. XII.1]. So $Z_t=V(M_t)$ by (2.2) has the same distribution as $\Omega(S(M_t))=\Omega(Y_t)$. This also may be seen by conditioning with respect to Y_t in $\Omega(Y_t)$. So umbral composition corresponds to sampling the random walk $\Omega(n)$ at stochastic times $S(n)$, or Y_t, independent of the $\Omega(i)$. This type of random time substitution is known in probability theory as subordination, see [15, Vol. II].

As an example we take $g(u)=-\log(1-u)$. Then by (2.13)

$$\sum_{n=0}^{\infty} u^n r_n(x)/n! = \{1-f(u)\}^{-x}.$$

Sequences of this type are related to renewal sequences and will be discussed in another paper [27].

Sheffer Sets

We give two interpretations. First let η and Y_t be as in (2.2)–(2.4) and assume that $s_k(0)\geqq 0$, $k\in\mathbb{N}$, in (1.8) and

$$c(\eta)=\sum_{k=0}^{\infty} s_k(0)\,\eta^k/k! < \infty. \tag{2.17}$$

Let U be a random variable with pgf $c(\eta u)/c(\eta)$, independent of the Y_t-process. Writing (1.8) with (2.6) as

$$\eta^n s_n(x)/n! = c(\eta)\,e^{\tau x}\sum_{k=0}^{n} P_\eta(U=k)\,P_\eta(Y_{\tau x}=n-k),$$

we see that

$$s_n(x)/n! = \eta^{-n}c(\eta)\,e^{\tau x}P_\eta(U+Y_{\tau x}=n). \tag{2.18}$$

Here it is not necessary that $s_0(0)>0$. If $s_0(0)=0$, the interpretation (2.18) still holds and $s_n(x)$ is a polynomial of degree smaller than n. The process $U+Y_t$, $t\geqq 0$, has independent increments with the same distributions as those of Y_t, $t\geqq 0$. The difference is that the process does not start at 0 but at U. The relation (1.9), together with (2.6), reflects the fact that $U+Y(\tau x+\tau y)$ is the sum of the independent contributions $U+Y(\tau x)$ and $Y(\tau x+\tau y)-Y(\tau x)$.

In the same way as (2.8) and (2.10) we find

$$s_{nk}/n! = \eta^{-n}c(\eta)\,\tau^k\,P_\eta(U+S_k=n)/k!, \tag{2.19}$$

$$\begin{aligned} s_n(x)/s_n(1) &= E_\eta(x^{M_\tau}\mid U+Y_\tau=n)\\ &= \exp(\tau x-\tau)\,P_\eta(U+Y_{\tau x}=n)/P_\eta(U+Y_\tau=n), \qquad x\geqq 0, \end{aligned} \tag{2.20}$$

so that $s_n(x)/s_n(1)$ is the conditional pgf of the number of jumps in $[0, \tau)$ of $U + Y_t$, $t \geqq 0$, given $U + Y_\tau = n$. A result similar to Theorem 1 also holds for the s_n.

For the second interpretation we assume that $s_k(0) \geqq 0$, $k \geqq 0$, and that $\eta^k s_k(0)/k!$ increases with k to a limit $a \in (0, \infty)$, whereas $\tau = g(\eta) < \infty$. Let V be a nonnegative integer random variable, independent of the Y_t with $P(V \leqq k) = a^{-1}\eta^k s_k(0)/k!$. Then with (2.6) we may write (1.8) as

$$\begin{aligned} s_n(x)/n! &= a\eta^{-n} e^{\tau x} \sum_{k=0}^{n} P_\eta(V \leqq k)\, P_\eta(Y_{\tau x} = n-k) \\ &= a\eta^{-n} e^{\tau x} P_\eta(V + Y_{\tau x} \leqq n), \qquad x \geqq 0. \end{aligned} \tag{2.22}$$

3. Asymptotic Behavior

The local central limit theorem for sums of independent identically distributed random variables [17, Chap. 4.2] gives first-order asymptotics for the coefficients (2.8) and (2.19), holding for $|n - kE_\eta X_1| = O(n^{1/2})$ if X_1 has finite variance. This (n, k)-domain may be varied by different choices of η. An example is given in Section 4. The Edgeworth expansion [15, Vol. II, 23] even gives an asymptotic expansion.

A very difficult problem is determining all possible limit laws for $n \to \infty$ of the distribution (2.9) with centering and scaling, i.e., all limiting distributions of $(M_\tau - a_n)/b_n$ given $Y_\tau = n$, and finding necessary and sufficient conditions for convergence. In [1; 3] sufficient conditions for convergence to a normal law were given. One difficulty arises from the fact that the pgf (2.10) is the quotient of two probabilities in the tails of distributions. One could try to let $\tau \to \infty$ by increasing η, if possible, so that $E_\eta Y_\tau = n$ and apply a local central limit theorem to the numerator and denominator of (2.10). However, as η increases the aperiodicity of the distribution (2.4), needed for the local limit theorem, may decrease. There does not seem to be a substitute for the hard analysis used in [1; 3].

Here we give a partial solution to the problem of convergence without centering or scaling of the distribution with pgf (2.20) and its special case (2.9), (2.10). By the continuity theorem for generating functions [15, Vol. I, Chap. XI.6],

$$\lim_{n \to \infty} s_{nk}/s_n(1) = \alpha_k, \qquad k \in \mathbb{N}, \tag{3.1}$$

iff

$$\lim_{n \to \infty} s_n(x)/s_n(1) = \psi(x), \qquad 0 \leqq x < 1, \tag{3.2}$$

and then $\sum \alpha_k \leqq 1$, (3.2) holds for $|x|<1$ and

$$\psi(x)=\sum_{k=0}^{\infty} \alpha_k x^k, \qquad |x|<1. \tag{3.3}$$

Here something more is true.

THEOREM 2. *If* (3.1) *holds with* $\sum \alpha_k>0$, *we have* $\sum \alpha_k=1$ *and the series* (3.3) *converges and* (3.2) *holds for all* $x \in \mathbb{C}$.

Proof. If (3.3) converges and (3.2) holds for $|x|<b$, then the same is true for $|x|<3b/2$: Take $0<|x|<v<b$. From (3.2) we have $s_n(v)/s_n(\frac{1}{2}v) \leqq C(v)<\infty$, $n \in \mathbb{N}$. So with (1.9),

$$\begin{aligned} s_n(3v/2) &= \sum_{k=0}^{n} \binom{n}{k} s_k(v)\, q_{n-k}(\tfrac{1}{2}v) \\ &\leqq C(v) \sum_{k=0}^{n} \binom{n}{k} s_k\left(\tfrac{1}{2}v\right) q_{n-k}\left(\frac{1}{2}v\right)=C(v)\, s_n(v). \end{aligned}$$

So $s_n(3v/2)/s_n(1)$ is bounded, implying that

$$\lim_{N \to \infty} \sum_{k>N} s_{nk}|3x/2|^k/s_n(1)=0,$$

uniformly in n. Then by (3.1)

$$\lim_{n \to \infty} s_n(3x/2)/s_n(1)=\sum_k \alpha_k(3x/2)^k,$$

finite. Finally, $\sum \alpha_k=1$ follows by taking $x=1$.

The asymptotic behavior of the probability distribution (2.9) is fundamentally different in the following two case:

A. The convergence radius ρ of (1.5) is finite and $g(\rho)<\infty$.

B. Either $\rho=\infty$ or $\rho<\infty$ and $g(\rho)=\infty$.

To show this and identify the limiting distributions we need the inequality, holding if $a_n>0$, $b_n \geqq 0$, $n \geqq n_1$,

$$\begin{aligned} \liminf_{n \to \infty} b_n/a_n &\leqq \liminf_{n \to \infty} \sum_{k \geqq n} b_k \Big/ \sum_{k \geqq n} a_k \\ &\leqq \limsup_{n \to \infty} \sum_{k \geqq n} b_k \Big/ \sum_{k \geqq n} a_k \\ &\leqq \limsup_{n \to \infty} b_n/a_n, \end{aligned} \tag{3.4}$$

and the lemma, being the special case $\Phi(x)=x^m$ of [26, Theorems 1 and 4].

LEMMA 1. *If μ is a probability measure on $\mathbb{N}$ with $\mu[n,\infty)>0$, $n\geqq n_1$, we have for its m-fold convolution μ^{m*}*

$$\liminf_{n\to\infty} \mu^{m*}[n,\infty)/\mu[n,\infty)\geqq m, \qquad m\in\mathbb{N}, \tag{3.5}$$

If, moreover, $\mu_n=\mu(\{n\})>0$, $n\geqq n_1$, and $\sum_n \mu_n x^n=\infty$, $x>1$, then

$$\liminf_{n\to\infty} \mu_n^{m*}/\mu_n\leqq m, \qquad n\in\mathbb{N}. \tag{3.6}$$

THEOREM 3. *Under* A *and* B *we have, respectively,*

$$\limsup_{n\to\infty} q_n(x)/q_n(1)>0, \qquad x>0, \tag{3.7}$$

$$\liminf_{n\to\infty} q_n(x)/q_n(1)=0, \qquad 0\leqq x<1. \tag{3.8}$$

Proof. In (2.4) take $\eta=\rho$. Then by (2.5) the pgf of Y_t, $t>0$, is infinite for $u>1$. Also $P(Y_t=n)>0$, $n\in\mathbb{N}$, $t>0$, since $\pi_1>0$. So by (3.6), since the distribution of Y_{mt} is the m-fold convolution of the distribution of Y_t, we have for $x=m^{-1}$,

$$\limsup_{n\to\infty} P_\rho(Y_{\tau x}=n)/P_\rho(Y_\tau=n)\geqq m^{-1}, \tag{3.9}$$

and (3.7) follows from (2.10).

For $\eta<\rho$ and $0\leqq x<1$ we have from (2.10) and (3.4),

$$\begin{aligned}\liminf_{n\to\infty} & q_n(x)/q_n(1)\\ &\leqq \exp(\tau x-\tau)\liminf_{n\to\infty} P_\eta(Y_{\tau x}\geqq n)/P_\eta(Y_\tau\geqq n)\\ &\leqq \exp(\tau x-\tau),\end{aligned}$$

since $Y_{\tau x}\leqq Y_\tau$. Under B we have $\tau=g(\eta)\to\infty$ as $\eta\to\rho$ and (3.8) follows.

So A is necessary for $q_n(x)/q_n(1)$ to have a nonzero limit. We conjecture that $q_n(x)/q_n(1)\to 0$ under *B*.

We now show that the only nonzero limit for $n\to\infty$ of the distribution (2.9) is a Poisson distribution shifted 1 to the right.

THEOREM 4. *Under* A, *if*

$$\lim_{n\to\infty} q_n(x)/q_n(1)=\psi(x), \qquad 0\leqq x<1, \tag{3.10}$$

$$\psi(x)=x\exp\{g(\rho)(x-1)\}. \tag{3.11}$$

Proof. By (3.1)–(3.3) the relation (3.10) holds for $|x|<1$ and ψ is analytic on $\{|x|<1\}$. Take $\eta=\rho$ in (2.10) so that $\tau=g(\rho)$. We have

$$\lim_{n\to\infty} P_\rho(Y_{\tau x}=n)/P_\rho(Y_\tau=n)=\phi(x), \qquad |x|<1,$$

where $\phi(m^{-1})\geqq m^{-1}$ by (3.9). From (3.4),

$$\lim_{n\to\infty} P_\rho(Y_{\tau x}\geqq n)/P_\rho(Y_\tau\geqq n)=\phi(x), \qquad 0\leqslant x<1,$$

so $\phi(m^{-1})\leqq m^{-1}$ by (3.5). Since with ψ also ϕ is analytic on $\{|x|<1\}$ we have $\phi(x)=x$ and the theorem follows from (2.10).

Sufficient conditions for (3.10) are given in the theory of subexponential distributions, the latest surveys of which are [14, 13]. We need the following weaker version of the deep theorem 1 in [4], proved by real analysis methods in [12].

LEMMA 2. *In the notation of Lemma* 1, *if* $\mu_n>0$, $n\geqq n_1$, $\mu_{n+1}/\mu_n\to 1$, *and* $\mu_n^{2*}/\mu_n\to c\in(0,\infty)$, *then* $c=2$. *If, moreover,*

$$v_n=\sum_{k=0}^{\infty} c_k\mu_n^{k*},$$

and $\phi(z)=\sum c_k z^k$ *is analytic on* $\{|z|<1+\varepsilon\}$, *then* $v_n/\mu_n\to\Phi'(1)$.

If $g(\rho)<\infty$, $g_n>0$, $n\geqq n_1$, $g_{n+1}/g_n\to\rho^{-1}$, and $g_n^{2*}/g_n\to cg(\rho)$, then $P_\rho(X_1=n)$ by (2.4) satisfies the conditions for μ_n in Lemma 2 so that with $\Phi(z)=\exp(\tau xz)$ where $\tau=g(\rho)$, we have with (2.7)

$$P_\rho(Y_{\tau x}=n)/P_\rho(X_1=n)\to\tau x, \qquad x>0. \tag{3.12}$$

Then (3.10) and (3.11) follow with (2.10).

Examples and simpler sufficient conditions are given in [12, 14, 21].

The pgf's (2.20) have a larger set of possible limits than (2.10). Roughly, the limiting behavior depends on whether $P(U=n)$ decreases slower with n than $P(X_1=n)$ or faster. We have the following partial results.

THEOREM 5. *If the sequence* $\{g_n\}$ *satisfies condition* B *and* $c(\eta)<\infty$, $\eta<\rho\leqq\infty$, *see* (2.17), *then*

$$\liminf_{n\to\infty} s_n(x)/s_n(1)=0, \qquad 0\leqq x<1.$$

Proof. The proof is the same as of (3.8) in Theorem 3, with (2.10) replaced by (2.20).

THEOREM 6. *If either* $(k+1)\,s_k(0)/s_{k+1}(0)\to\lambda\in(0,\rho)$ *as* $k\to\infty$, *or* $\lambda^k s_k(0)/k!\to a\in(0,\infty)$ *and* $g(\lambda)<\infty$,

$$\lim_{n\to\infty} s_n(x)/s_n(1)=\exp\{(x-1)\,g(\lambda)\}, \qquad x\in\mathbb{C}. \tag{3.13}$$

Proof. The first assertion follows by applying problem 178 in [24, p. 39] to the sequence $s_n(0)/n!$ and $q_n(x)/n!$ in (1.8); see also [12, Lemma 2.2], noting that the series (1.6) has convergence radius ρ.

If $g(\lambda)<\infty$ we have with (1.8) and (2.6) for $\eta=\lambda$,

$$s_n(x)/n!=\lambda^{-n}\{\exp xg(\lambda)\}\sum_{j=0}^{n} P_\lambda(Y_{\tau x}=j)\,\lambda^{n-j}\,s_{n-j}(0)/(n-j)!.$$

If $\lambda^k s_k(0)/k!\to a$, the sum converges to a by dominated convergence and (3.13) follows.

The Poisson pgf (3.13) is the limit of (2.20) if U is preponderant in $P(U+Y_{\tau x})$. Mixtures of the pgf's in (3.11) and (3.12) are also possible.

THEOREM 7. *If the sequence* $\{g_n\}$ *satisfies the conditions stated after Lemma* 2 *and, as* $k\to\infty$,

$$s_k(0)/(k!\,c(\rho))\sim\beta g_k/g(\rho) \tag{3.14}$$

with $0\leqq\beta<\infty$, *then for* $x\in\mathbb{C}$ *as* $n\to\infty$,

$$s_n(x)/s_n(1)\to(xg(\rho)+\beta)(g(\rho)+\beta)^{-1}\exp\{(x-1)\,g(\rho)\}.$$

Proof. By (2.4), (2.17), and (3.12) we may write (3.14) as

$$P_\rho(U=k)=\beta(\tau x)^{-1}\,P_\rho(Y_{\tau x}=k)+\varepsilon_k, \qquad x>0, \tag{3.15}$$

with $\varepsilon_k/P_\rho(Y_{\tau x}=k)\to 0$ as $k\to\infty$. Then

$$P_\rho(U+Y_{\tau x}=n)=\beta(\tau x)^{-1}\,P_\rho(Y(2\tau x)=n)+\sum_{k=0}^{n}\varepsilon_k P_\rho(Y_{\tau x}=n-k). \tag{3.16}$$

Since $\mu_n = P_\rho(Y_{\tau x} = n)$ by (3.12) satisfies the conditions of Lemma 2, we have by [4, Lemma 1, p. 260],

$$\sum_{k=0}^{n} \varepsilon_k P_\rho(Y_{\tau x} = n-k) \sim P_\rho(Y_{\tau x} = n) \sum_{k=0}^{\infty} \varepsilon_k, \tag{3.17}$$

where $\sum \varepsilon_k = 1 - \beta/(\tau x)$ by (3.15). From (3.16), (3.17), and (3.12) we have

$$P_\rho(U + Y_{\tau x} = n) \sim (\beta + \tau x)\, P_\rho(X_1 = n),$$

with $\tau = g(\rho)$ and the theorem follows with (2.20).

4. Examples

The generating function with $0 < \gamma < 1$

$$g(u) = 1 - (1-u)^\gamma = \sum_{k=1}^{\infty} (-1)^{k-1} \binom{\gamma}{k} u^k, \qquad |u| \leqq 1, \tag{4.1}$$

has positive coefficients and is a pgf, since $g(1) = 1$. If X_1 has pgf (4.1) we have by (2.5) with $\eta = 1$, $\tau = 1$,

$$E_1 u^{Y(x)} = \exp\{-x(1-u)^\gamma\}. \tag{4.2}$$

This is a discrete stable pgf as defined and studied in [28]. Expansion of the exponential and comparison with (2.6) gives

$$P_1(Y_x = n) = (-1)^n \sum_{k=0}^{\infty} \binom{k\gamma}{n} (-x)^k/k! > 0, \qquad x > 0,$$

$$q_n(x) = n!(-1)^n e^x \sum_{k=0}^{\infty} \binom{k\gamma}{n} (-x)^k/k!, \tag{4.3}$$

showing that (4.3) is a polynomial of binomial type with positive coefficients if $0 < \gamma < 1$. For $\gamma = 1$ the distribution of Y_x is Poisson, for $\gamma < 0$ or $\gamma > 1$ the q_n in (4.3) are still polynomials of binomial type.

From (4.1) we have, with $b_k = k^{1/\gamma}$,

$$\lim_{k\to\infty} g(\exp(-s/b_k)) = \exp(-s^\gamma), \qquad \operatorname{Re} s \geqq 0.$$

So if X_1 has pgf (4.1) the distribution of S_k/b_k converges to the stable distribution on $[0, \infty)$ with Laplace–Stieltjes transform $\exp(-s^\gamma)$, see [15, Vol. II, 17]. The density h of this distribution has no simple closed form, except if $\gamma = \frac{1}{2}$ when $h(x) = \frac{1}{2}\pi^{-1/2} x^{-3/2} \exp(-x^{-1}/4)$, see [15, Chap. XIII.3, 10, Sect. 3.4]. Much, however, is known about stable densities, see [17]. By the local limit theorem [17, Theorem 4.2.1] and (2.8) we have

$$k!\, q_{nk}/n! = P_1(S_k = n) = b_k^{-1} h(n/b_k) + \sigma(b_k^{-1}),$$

uniformly in n. Since $h(x) \to 0$ as $x \to \infty$ or $x \to 0$, see [15, Chap. XIII.6, 17, Chap. 2], this is only useful if $0 < c_1 < n/b_k < c_2 < \infty$.

If the pgf of X_1 is $g(\eta u)/g(\eta)$ with $0 < \eta < 1$ and g as in (4.1), all moments of X_1 are finite and the local central limit theorem [23, Chap. VII, Theorem 6] gives, with (2.8),

$$\eta^n \tau^{-k} k!\, q_{nk}/n! = (2\pi k\sigma^2)^{-1/2} \exp\{-(n-k\mu)^2/2k\sigma^2\} + O(k^{-1}),$$

where $\mu = E_\eta X_1$, $\sigma^2 = \mathrm{Var}_\eta X_1$, and which is useful if $|n - k\mu| = O(k^{1/2})$.

In [8] a method was developed that may give explicit expressions for the distributions (2.4) and (2.7) or their pgf's and therefore also for $\tau = g(\eta)$. This is useful since classical probability theory gives relatively few examples of it. We sketch the method in a form adapted to our notation. Let

$$A(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{4.4}$$

with $a_k \geqq 0$, $a_0 > 0$, and $a_h > 0$ for some $h \geqq 2$, converge for $z \in \mathbb{C}$ (the last assumption may be weakened). From the strict convexity of A on $[0, \infty)$ we see that there is $u_0 > 0$ such that the equation in z,

$$z = uA(z), \tag{4.5}$$

has two positive roots, $0 < t_1(u) < t_2(u)$, if $0 < u < u_0$, one positive root $t(u_0)$ if $u = u_0$, and no positive root if $u > u_0$. Also

$$uA'(t_1(u)) < 1, \qquad uA'(t_2(u)) > 1, \qquad 0 < u < u_0, \qquad u_0 A'(t(u_0)) = 1. \tag{4.6}$$

For $0 < |u| < u_0$ and $t_1(|u|) < |z| < t_2(|u|)$, we have

$$|uA(z)| \leqq |u| \sum a_k |z|^k = |u|\; A(|z|) < |z|.$$

So by Lagrange's theorem [11, Section 38], Eq. (4.5) has a single root $z = z(u)$ in $\{|z| < t_2(|u|)\}$ given by the convergent series

$$z(u) = \sum_{k=1}^{\infty} u^k D^{k-1} A^k(0)/k!. \tag{4.7}$$

If $u > 0$ we have $z(u) = t_1(u)$. The series is of the form (1.5) with nonnegative coefficients and therefore may be taken for $g(u)$. By a continuity argument

$$z(u_0) \stackrel{\text{def}}{=} \lim_{u \uparrow u_0} z(0) = t(u_0), \tag{4.8}$$

and is still given by (4.7) with $u = u_0$. Again by Lagrange's theorem,

$$\begin{aligned}\exp\{xz(u)\} &= 1 + x \sum_{n=1}^{\infty} u^n D^{n-1} e^{xz} A^n(z)|_{z=0}/n! \\ &= 1 + \sum_{n=1}^{\infty} u^n \sum_{i=1}^{n-1} \binom{n-1}{i-1} x^i D^{n-1} A^n(0)/n!. \end{aligned} \tag{4.9}$$

So by (1.6), we see that $q_0(x) = 1$,

$$q_n(x) = \sum_{i=1}^{n=1} \binom{n-1}{i-1} x^i D^{n-1} A^n(0), \qquad n \geqq 1, \tag{4.10}$$

is a set of polynomials of binomial type with nonnegative coefficients. By (4.5) its delta operator is $D/A(D)$. Distributions with pgf (4.7) or $f(z(u))$ when A and f are pgf's, were called Lagrangian in [8]. Their properties and applications, e.g., in queueing, were studied in [9, 20], see also [5, Chap. II.3, App. 6].

By substituting positive values of z into $u = z/A(z)$ and consulting (4.6) we may find explicit values of $t_1(u)$ and u, or, in our notation (2.4), of $g(\eta)$ and $u = \eta$. As an example, we take $A(z) = c\exp(\alpha z)$, $c > 0$, $\alpha > 0$. By (4.5) and (4.6) we have $z = t_1(u)$ iff $z = uc\exp(\alpha z)$ and $uc\alpha\exp(\alpha z) < 1$ or $z = uc\exp(\alpha z)$ and $\alpha z < 1$, and $u = u_0$, $z = t(u_0)$ iff $z = uc\exp(\alpha z)$ and $\alpha z = 1$. We find $u_0 = (c\alpha e)^{-1}$, $t(u_0) = \alpha^{-1}$ and (4.7) and (4.9) give

$$z(u) = \sum_{k=1}^{\infty} u^k c^k (k\alpha)^{k-1}/k!, \qquad |u| < u_0,$$

$$\exp\{xz(u)\} = 1 + x \sum_{n=1}^{\infty} u^n c^n (x + n\alpha)^{n-1}/n!, \qquad |u| < u_0.$$

So that $c^{-n} q_n$ in (4.10) are the Abel polynomials, see [25, p. 744]. By the above remark we have $z(u) = \varepsilon\alpha^{-1}$ with $0 < \varepsilon < 1$ for $u = c^{-1}\varepsilon\alpha^{-1}\exp(-\varepsilon)$. Then (2.4) and (2.6) with $\eta = c^{-1}\varepsilon\alpha^{-1}\exp(-\varepsilon)$ and $\tau = g(\eta) = z(\eta) = \varepsilon\alpha^{-1}$ give us the pair

$$P_\eta(X_1 = k) = (\varepsilon k)^{k-1} \exp(-k\varepsilon)/k!, \tag{4.11}$$

$$P(Y(\varepsilon\alpha^{-1}x = n)) = \varepsilon\alpha^{-1}x(\varepsilon\alpha^{-1}x + n\varepsilon)^{n-1}\exp(-\varepsilon\alpha^{-1}x - n\varepsilon)/n!. \tag{4.12}$$

The two-parameter distribution (4.12) is the generalized Poisson distribution studied in [6, 7]. Note that the right-hand sides of (4.11) and (4.12) have sum 1 over k and n. For these relations also see [16].

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