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Cholesky Factorizations of Matrices Associated with r-Order Recurrent Sequences

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Abstract

In this paper we extend some results on the factorization of matrices associated to Lucas, Pascal, Stirling sequences by the Fibonacci matrix. We provide explicit factorizations of *any* matrix by the matrix associated with an r-order recurrent sequence U_n (having $U_0 = 0$). The Cholesky factorization for the symmetric matrix associated to U_n is also obtained.

1 Introduction and Motivation

Shapiro, et.al. [9] introduced the concept of Riordan matrix (element of the Riordan group $\mathcal{R}$) in an attempt to develop a tool to deal with large classes of combinatorial identities. An infinite lower triangular matrix $L = (l_{n,k})_{n,k\geq 0}$ is a *Riordan matrix* if there exist generating functions $g(z) = \sum g_n z^n$, $f(z) = \sum f_n z^n$, $f_0 = 0, f_1 \neq 0$, such that $l_{n,0} = g_n$ and $\sum_{n\geq k} l_{n,k} z^n = g(z)(f(z))^k$. We write $L = (g(z), f(z))$ (or (g, f)). Given two Riordan matrices $L_1 = (g_1, f_1)$ and $L_2 = (g_2, f_2)$ we define the multiplication by $L_1 L_2 = (g_1 g_2(f_1), f_2(f_1))$, and the inverse $L^{-1} = (\frac{1}{g(\bar{f})}, \bar{f})$, where $\bar{f}$ is the compositional inverse of f. The identity matrix $I = (1, z)$ is Riordan. Thus, $\mathcal{R}$ becomes a group under the previous operation. The underlying idea is to

*Associated with the Institute of Mathematics "Simion Stoilow" of The Romanian Academy, Bucharest, Romania find factorizations of some matrices by matrices for which one knows their Riordan components (g, f). Using the operation of $\mathcal{R}$, then one can find closed forms for some combinatorial sums. In [8] the authors proved that some matrices in the Riordan group can be factorized as $R = PCF$, where P is a Pascal-like matrix, C a Catalan sequence connected matrix and F a Fibonacci involving matrix.

In [1, 3, 4, 5, 6, 10] the authors concentrated on various factorizations of the Fibonacci matrix, Pascal, and Stirling matrices. They also studied the Cholesky factorization of the Fibonacci matrix. Besides the interest in its own right, Cholesky factorization can be applied in the resolution of linear equations. A linear equation $Ax = b$ can be solved twice as fast using its Cholesky factorization $A = LL^T$, than by LU-factorization.

The mentioned results were generalized in a recent paper [3] to a k-Fibonacci sequence. In [2] the author gave a few applications of the k-Lucas sequences in graph theory, and in [6] a few applications of the k-Fibonacci numbers in combinatorics and probabilities were provided.

It is our intent in this paper to point out that many of the results which were obtained in all these papers can be generalized to factorizations of *any* matrix by matrices constructed using any r-order recurrent sequence (having $U_0 = 0$ - see next). The Cholesky factorization for the associated symmetric matrix is also obtained here.

Since it is easier to write it down, we prove the results explicitly in the case of a second-order recurrent sequence pointing out in the general case only the differences (if they exist).

The present author was led to this investigation in an attempt to find divisors in Cayley graphs by looking at factorizations of the adjacency matrix of a graph.

2 Binary Recurrent Sequences

We consider the general nondegenerate second-order (binary) recurrent sequence $U_{n+1} = aU_n + bU_{n-1}$, a, b, U_0, U_1 integers, $\delta = \sqrt{a^2 + 4b} \neq 0$ real. We recall the Binet formula for the sequence U_n, namely $U_n = A\alpha^n - B\beta^n$, where $\alpha = \frac{1}{2}(a + \sqrt{a^2 + 4b})$, $\beta = \frac{1}{2}(a - \sqrt{a^2 + 4b})$ and $A = \frac{U_1 - U_0\beta}{\alpha - \beta}$, $B = \frac{U_1 - U_0\alpha}{\alpha - \beta}$. In this paper we assume that $U_0 = 0$, therefore $A = B$. We associate the sequence $V_n = \alpha^n + \beta^n$, which satisfies the same recurrence, with the initial conditions $V_0 = 2, V_1 = a$. Obviously, if $a = b = 1$ and $U_0 = 0, U_1 = 1$, then $U_n = F_n$, the Fibonacci sequence, and $V_n = L_n$, the Lucas sequence.

We define the Pascal matrix $\mathcal{P}_n$ with entries $\binom{i-1}{j-1}$ (with the standard convention that the binomial coefficients are equal to 0 if $i < j$).

Let $\mathcal{U}_n$ be the matrix with entries

$$u_{i,j} = \begin{cases} U_{i-j+1} & \text{if } i - j + 1 \geq 0 \\ 0 & \text{if } i - j + 1 < 0. \end{cases}$$

If $U_n = F_n$ the Fibonacci sequence, then $\mathcal{U}_n = \mathcal{F}_n$. In [5], the inverse of $\mathcal{F}_n$ was found to be the matrix $\mathcal{F}_n^{-1}$ with entries

$$f'_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i-2 \leq j \leq i-1 \\ 0 & \text{otherwise} \end{cases}$$

Define $X_n(\gamma) = (x_{i,j}(\gamma))_{i,j}$, where

$$x_{i,j}(\gamma) = \begin{cases} \gamma^{i-j+1} & \text{if } i-j+1 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For instance,

$$X_5(\gamma) = \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 \\ \gamma^2 & \gamma & 0 & 0 & 0 \\ \gamma^3 & \gamma^2 & \gamma & 0 & 0 \\ \gamma^4 & \gamma^3 & \gamma^2 & \gamma & 0 \\ \gamma^5 & \gamma^4 & \gamma^3 & \gamma^2 & \gamma \end{pmatrix}.$$

We note that $\mathcal{U}_n = AX_n(\alpha) - AX_n(\beta)$. It is a simple exercise to obtain the inverse of $X_n(\gamma)$.

Proposition 1. *The inverse of $X_n(\gamma)$ is the matrix with entries* $\begin{cases} 1/\gamma & \textit{if } i = j \\ -1 & \textit{if } j = i-1 \\ 0 & \textit{otherwise.} \end{cases}$

Proof. Straightforward. □

Proposition 2. *The inverse $\mathcal{U}_n^{-1}$ of $\mathcal{U}_n$ is the matrix with entries $u'_{i,j}$, where*

$$U_1\, u'_{i,j} = \begin{cases} 1 & \textit{if } i = j \\ -a & \textit{if } j = i-1 \\ -b & \textit{if } j = i-2 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. If $i = j$, then $\sum_{k=1}^{n} u_{i,k}u'_{k,i} = 1$. Assume $i \neq j$. Then,

$$\sum_{k=1}^{n} u_{i,k}u'_{k,j} = \frac{1}{U_1}(u_{i,j} - au_{i,j+1} - bu_{i,j+2}) = \frac{1}{U_1}(U_{i-j+1} - aU_{i-j} - bU_{i-j+1}) = 0.$$

The proposition is proved. □

Theorem 2.1 of [4] states that the Pascal matrix

$$\mathcal{P}_n = \mathcal{F}_n L_n,$$

where L_n is the matrix with entries $\binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}$. Also, Theorem 3.1 of [4] states that

$$\mathcal{S}_n(2) = \mathcal{F}_n M_n,$$

and Theorem 4.1 of [4] states that

$$\mathcal{S}_n(1) = Q_n \mathcal{F}_n,$$

where $\mathcal{S}_n(2), \mathcal{S}_n(1)$ are the matrices corresponding to the Stirling numbers of the first and second kind and M_n, Q_n are defined suitably using the Stirling matrices.

The previous results are not specific to the Fibonacci, Pascal, or Stirling matrices, rather they can be generalized to *any* matrix T_n. We do that in our first theorem. For an arbitrary matrix $T_n = (t_{i,j})_{1\le i,j\le n}$ and a binary sequence $U = (U_k)$ define the matrix $T_n(U)$ with entries $t'_{i,j} = t_{i,j} - at_{i-1,j} - bt_{i-2,j}$ if $i \ge 3$, and $t'_{1,j} = t_{1,j}, t'_{2,j} = t_{i,j} - at_{i-1,j}$. (We could give up the extra definition of $t'_{i,j}$ for $i \le 2$, if we assume that the matrix T_n is padded by zeros, so $t_{i,j} = 0$ if one of the indices is ≤ 0.)

Theorem 3. *Let $T_n(U)$ be the matrix associated to an arbitrary matrix T_n and the sequence $U = (U_k)_k$, defined as above. Then the matrix T_n can be factorized by $\mathcal{U}_n$ as*

$$T_n = \frac{1}{U_1}\mathcal{U}_n T_n(U).$$

Proof. We previously proved that the matrix $\mathcal{U}_n$ is invertible. Thus, it suffices to show now that $T_n(U) = U_1 \mathcal{U}_n^{-1} T_n$. Since the entries of $\mathcal{U}_n^{-1}$ are $1/U_1$ if $i = j$, $-a/U_1$ if $j = i-1$, $-b/U_1$ if $j = i-2$ and 0 otherwise, a straightforward computation proves our claim. □

As a simple corollary we have the following combinatorial identity.

Corollary 4. *If $i \ge j$, then*

$$t_{i,j} = \frac{1}{U_1}\sum_{k=1}^{i} U_{i-k+1}(t_{k,j} - at_{k-1,j} - bt_{k-2,j}).$$

One can obtain a plethora of seemingly complicated combinatorial identities by taking various suitably chosen matrices T_n and sequences $(U_k)_k$. In particular, one can obtain all combinatorial identities of [4]. As a simple example, by taking $t_{i,j} = x^{i-j}\binom{i}{j}$ $(i \ge j)$ we obtain

$$\sum_{k=j}^{i} U_{i-k+1}x^k\left(\binom{k}{j} - a\binom{k-1}{j}x^{-1} - b\binom{k-2}{j}x^{-2}\right) = U_1 x^i\binom{i}{j}.$$

Or, taking $t_{i,j} = U_{i-j+1}\binom{i}{j}$ $(i \geq j)$ we get

$$\sum_{k=j}^{i} U_{i-k+1}\binom{k}{j}\left(U_{k-j+1} - aU_{k-j}\frac{k-j}{k} - bU_{k-j-1}\frac{(k-j)(k-j-1)}{k(k-1)}\right) = U_1 U_{i-j+1}\binom{i}{j}.$$

The reader can amuse oneself, by taking arbitrary entries $t_{i,j}$ and derive other combinatorial identities.

Define the direct sum of two square matrices (not necessarily of the same dimension) $A \oplus B = \left[\begin{smallmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{smallmatrix}\right]$. Further, we define the matrices

$$\begin{aligned}
S_0 &= \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix}; \quad S_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} && (1) \\
S_k &= S_0 \oplus I_k; \quad \overline{\mathcal{U}_k} = [U_1] \oplus \mathcal{U}_{k-1}; && (2) \\
G_1 &= I_n;\ G_2 = I_{n-3} \oplus S_{-1};\ G_k = I_{n-k} \oplus S_{k-3},\ (n \text{ is fixed}); && (3) \\
C_k &= \begin{pmatrix} U_1 & 0 & 0 & \cdots & 0 \\ U_2 & 1 & 0 & \cdots & 0 \\ U_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_k & 0 & 0 & \cdots & 1 \end{pmatrix}. && (4)
\end{aligned}$$

We regard C_k as a *companion* matrix to the sequence $U = (U_n)_n$.

We prove now

Theorem 5. *The following identities are true*

$$\begin{aligned}
&\overline{\mathcal{U}}_k S_{k-3} = \mathcal{U}_k,\ k \geq 3, \\
&\mathcal{U}_n = U_1\, G_1 G_2 \cdots G_n, \\
&\mathcal{U}_n = C_n(I_1 \oplus C_{n-1})(I_2 \oplus C_{n-2}) \cdots (I_{n-1} \oplus C_1).
\end{aligned}$$

Proof. If $k = 3$, then we need to prove $([U_1] \oplus \mathcal{U}_2)S_0 = \mathcal{U}_3$. This follows from

$$\begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_1 & 0 \\ 0 & U_2 & U_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} U_1 & 0 & 0 \\ U_2 & U_1 & 0 \\ U_3 & U_2 & U_1 \end{pmatrix},$$

which is certainly true. The general case follows in the same manner observing that in $([U_1] \oplus \mathcal{U}_{k-1})(S_0 \oplus I_{k-3})$ multiplying the ith row of the first matrix by the 1st row of the second matrix renders U_{i+1} (using the recurrence on $(U_k)_k$) and multiplying the ith row by the 2nd, 3rd column etc., we get U_i, U_{i-1}, etc.

We prove now the factorization of $\mathcal{U}_n$. By the first part of our theorem, we have $\mathcal{U}_n = \overline{\mathcal{U}}_n S_{n-3} = \overline{\mathcal{U}}_n G_n$. Further,

$$\overline{\mathcal{U}}_n = [U_1] \oplus \mathcal{U}_{n-1} = [U_1] \oplus \overline{\mathcal{U}}_{n-1} S_{n-4} = ([U_1] \oplus \overline{\mathcal{U}}_{n-1})(I_1 \oplus S_{n-4}) = ([U_1] \oplus \overline{\mathcal{U}}_{n-1})G_{n-1}.$$

Continuing in this manner, and observing that $[U_1] \oplus [U_1] \oplus \cdots \oplus [U_1] = U_1 I_n = U_1 G_1$, we obtain the result. The last identity is done similarly. $\square$

The following is an example of the second identity

$$\begin{pmatrix} U_1 & 0 & 0 & 0 \\ U_2 & U_1 & 0 & 0 \\ U_3 & U_2 & U_1 & 0 \\ U_4 & U_3 & U_2 & U_1 \end{pmatrix} = U_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

One might think that Theorem 2.1 of [5] is incomplete, that is, that the product should end in $I_{n-1} \oplus C_1$. However, if $U_n = F_n$, the Fibonacci sequence, then $I_{n-1} \oplus C_1 = I_n$, which is not necessarily the case in this paper.

Corollary 6. *By Theorem* 5*, we have*

$$\begin{aligned} \mathcal{U}_n^{-1} &= (I_{n-1} \oplus C_1)^{-1}(I_{n-2} \oplus C_2)^{-1} \cdots (I_1 \oplus C_{n-1})^{-1} C_n^{-1} \\ &= (I_{n-1} \oplus C_1^{-1})(I_{n-2} \oplus C_2^{-1}) \cdots (I_1 \oplus C_{n-1}^{-1}) C_n^{-1}, \end{aligned}$$

where

$$C_k^{-1} = \begin{pmatrix} U_1/U_1 & 0 & 0 & \cdots & 0 \\ -U_2/U_1 & 1 & 0 & \cdots & 0 \\ -U_3/U_1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -U_k/U_1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

3 Cholesky Factorization

We define now the $n \times n$ *symmetric U-matrix* $\mathcal{S}_n(U)$ with the entries given by (for $i \leq j$)

$$q_{i,j} = q_{j,i} = \begin{cases} \sum_{k=1}^{i} U_k^2 & \text{if } i = j \\ a\, q_{i,j-1} + b\, q_{i,j-2} & \text{if } i+1 \leq j, \end{cases}$$

with the convention that $q_{i,j} = 0$ if $j \leq 0$.

Next we give explicit expressions for $q_{i,j}$'s.

Lemma 7. *We have*

$$\begin{aligned} q_{1,j} &= q_{j,1} = U_1 U_j \text{ and} && (5) \\ q_{i,j} &= U_i U_j + U_{i-1} U_{j-1} + \cdots + U_1 U_{j-i+1}, \text{ if } j \geq i \geq 2 && (6) \end{aligned}$$

Proof. The first relation $q_{1,j} = q_{j,1} = U_1 U_j$ follows easily from the recurrence on $q_{i,j}$'s. We show that $q_{i,j} = U_i U_j + U_{i-1} U_{j-1} + \cdots + U_1 U_{j-i+1}$ by induction. The initial conditions certainly hold. Adding the relations

$$\begin{aligned} aq_{i,j-1} &= aU_i U_{j-1} + aU_{i-1} U_{j-2} + \cdots + aU_1 U_{j-i} \\ bq_{i,j-2} &= bU_i U_{j-2} + bU_{i-1} U_{j-3} + \cdots + bU_1 U_{j-i-1} \end{aligned}$$

and using the recurrence $U_{k+1} = aU_k + bU_{k-1}$ we obtain the expression of $q_{i,j}$. □

Let M^T be the transpose of M.

Theorem 8. *The Cholesky factorization of $\mathcal{S}_n(U)$ is given by $\mathcal{S}_n(U) = \mathcal{U}_n\mathcal{U}_n^T$.*

Proof. It suffices to prove that $U_1\mathcal{U}_n^{-1}\mathcal{S}_n(U) = U_1\mathcal{U}_n^T$. Proposition 2 implies that the matrix $U_1\mathcal{U}_n^{-1}\mathcal{S}_n(U)$ has the entries $x_{i,j}$, where

$$x_{i,j} = \begin{cases} q_{1,j}(= U_1U_j), & \text{if } i = 1, \\ q_{2,j} - aq_{1,j}, & \text{if } i = 2, \\ q_{i,j} - aq_{i-1,j} - bq_{i-2,j} & \text{if } i \geq 3. \end{cases}$$

Thus, we need to prove that $x_{i,j}$ are the entries of $U_1\mathcal{U}_n^T$, that is $x_{i,j} = u_{j,i} = U_1U_{j-i+1}$. For $i = 1, 2$, we note that the claim follows from $q_{1,j} = U_1U_j$ and $q_{2,j} - aq_{1,j} = U_2U_j + U_1U_{j-1} - aU_1U_j = U_1U_{j-1}$, since $U_2 = aU_1$. In general,

$$\begin{aligned} q_{i,j} - aq_{i-1,j} - bq_{i-2,j} &= \quad U_iU_j + U_{i-1}U_{j-1} + \cdots + U_3U_{j-i-1} + U_2U_{j-i} + U_1U_{j-i+1} \\ &\quad -aU_{i-1}U_j - aU_{i-2}U_{j-1} - \cdots - aU_2U_{j-i-1} - aU_2U_{j-i} \\ &\quad -bU_{i-2}U_j - bU_{i-3}U_{j-1} - \cdots - bU_1U_{j-i-1} \\ &= (U_i - aU_{i-1} - bU_{i-2})U_j + \cdots \\ &\quad +(U_3 - aU_2 - bU_1)U_{j-i-1} + (U_2 - aU_1)U_{j-i} + U_1U_{j-i+1} \\ &= U_1U_{j-i+1}, \end{aligned}$$

using the recurrence on the sequence $(U_k)_k$. □

4 r-Order Recurrent Sequences

In this section we generalize our previous results. We started with the particular case since the proofs are easier to write, but the general case is treated similarly. Assume r is a fixed positive integer. Let $U = (U_n)_n$ be an r-order recurrence, that is, U_n satisfies the recurrence

$$U_{n+r} = a_1U_{n+r-1} + a_2U_{n+r-2} + \cdots + a_rU_n, \tag{7}$$

and for simplicity we assume that $U_0 = U_{-1} = \ldots = U_{2-r} = 0$, U_1 arbitrary. Therefore, the Binet formula gives

$$U_n = A_1\alpha_1^n + A_2\alpha_2^n + \cdots + A_r\alpha_r^n,$$

for some suitable $A_1, \cdots, A_r$ (found by using Cramer's formulas from linear algebra applied to the system obtained by imposing the initial conditions), where α_i are the complex roots of the characteristic k-degree polynomial of (7). We associate a matrix $\mathcal{U}_n$ to this sequence, with entries

$$u_{i,j} = \begin{cases} U_{i-j+1} & \text{if } i - j + 1 \geq 0 \\ 0 & \text{if } i - j + 1 < 0. \end{cases}$$

Proposition 9. *The inverse of $\mathcal{U}_k$ is the matrix with entries $u'_{i,j}$, where*

$$U_1\, u'_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -a_k & \text{if } j = i-k, \text{ where } k = 1, 2, \ldots, r \\ 0 & \text{otherwise} \end{cases}$$

Proof. Straightforward, as in the proof of Proposition 2. □

Now we show that *any* matrix $T_n = (t_{i,j})_{i,j=1,2,\ldots,n}$ can be factorized by $\mathcal{U}_n$. We define $T_n(U)$ with entries $t'_{i,j}$ where

$$t'_{i,j} = t_{i,j} - \sum_{k=1}^{r} a_k t_{i-k,j},$$

where we regard T_n as being padded with zeros, so the previous relation makes sense for negative indices.

Theorem 10. *We have*

$$T_n = \frac{1}{U_1}\mathcal{U}_n T_n(U).$$

Proof. The proof follows closely the one of Theorem 3. □

Corollary 11. *If $i \geq j$, then*

$$t_{i,j} = \frac{1}{U_1}\sum_{k=1}^{i} U_{i-k+1}(t_{k,j} - a_1 t_{k-1,j} - a_2 t_{k-2,j} - \cdots - a_r t_{k-r,j}).$$

5 General Cholesky Factorization

As before, we define now the (generalized) $n \times n$ *symmetric matrix* $\mathcal{S}_n(U)$ with the entries given by $(i \leq j)$

$$q_{i,j} = q_{j,i} = \begin{cases} \sum_{k=1}^{i} U_k^2 & \text{if } i = j \\ \sum_{k=1}^{r} a_k\, q_{i,j-k} & \text{if } i+1 \leq j, \end{cases}$$

with the convention that $q_{i,j} = 0$ if $j \leq 0$. One can prove as in the case of $r = 2$ the following lemma.

Lemma 12. *We have for $i \leq j$,*

$$q_{i,j} = U_i U_j + U_{i-1}U_{j-1} + \cdots + U_1 U_{j-i+1}.$$

We define now the following matrices ($k \geq 1$ arbitrary integer; $i = 1, 2, \ldots, r$)

$$\begin{aligned}
C_k &= \begin{pmatrix} U_1 & 0 & 0 & \cdots & 0 \\ U_2 & 1 & 0 & \cdots & 0 \\ U_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_k & 0 & 0 & \cdots & 1 \end{pmatrix}; \quad A_i = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_i & 0 & 0 & \cdots & 1 \end{pmatrix}; \\
S_{i-r} &= I_{r-i} \oplus A_i \text{ (of order } (r+1) \times (r+1)); \\
S_{l-r-1} &= S_0 \oplus I_{l-r-1}, l \geq r+1 \text{ (of order } l \times l); \\
\overline{\mathcal{U}_n} &= [U_1] \oplus \mathcal{U}_{n-1}; \\
G_0 &= I_n; G_i = I_{n-i} \oplus A_{i-1},\ i = 1, \ldots, r \text{ and} \\
G_l &= I_{n-l} \oplus S_{l-r-1}, l \geq r+1,\ \text{where } n \text{ is fixed.}
\end{aligned}$$

The inverse of the companion matrix C_k was already computed by Corollary 6.

Theorem 13. *We have the following factorizations*

$$\begin{aligned}
&\overline{\mathcal{U}}_k S_{k-r-1} = \mathcal{U}_k,\ k \geq r+1, \\
&\overline{\mathcal{U}}_i A_{i-1} = \mathcal{U}_i,\ i \leq r, \\
&\mathcal{U}_n = U_1 G_1 G_2 \cdots G_n.
\end{aligned}$$

Proof. The first factorization follows by using the observation that the multiplication $([U_1] \oplus \mathcal{U}_{k-1})(S_0 \oplus U_{k-r-1})$ renders immediately the entries of $\mathcal{U}_k$ which are not on the first column. The entry on the first column in position $(j, 1)$ is $a_1 U_j + a_2 U_{j-1} + \cdots + a_r U_{j-r}$ which is certainly U_{j+1} by using the recurrence on the sequence U. The second factorization follows similarly.

By the first part of the theorem, if n is fixed, then $\mathcal{U}_n = \overline{\mathcal{U}}_n S_{n-r-1} = \overline{\mathcal{U}}_n G_n$. Now, $\overline{\mathcal{U}}_n = [U_1] \oplus \mathcal{U}_{n-1} = [U_1] \oplus \overline{\mathcal{U}}_{n-1} S_{n-r-2} = (U_1 I_2 \oplus \overline{\mathcal{U}}_{n-1})(I_2 \oplus S_{n-r-2}) = (U_1 I_2 \oplus \overline{\mathcal{U}}_{n-1}) G_{n-1}$. Continuing in this manner we obtain

$$\mathcal{U}_n = (U_1 I_{n-r-1} \oplus \overline{\mathcal{U}}_{r+1}) G_{r+1} \cdots G_n.$$

Next, $U_1 I_{n-r-1} \oplus \overline{\mathcal{U}}_{r+1} = U_1 I_{n-r} \oplus \mathcal{U}_r = U_1 I_{n-r} \oplus \overline{\mathcal{U}}_r A_{r-1} = (U_1 I_{n-r} \oplus \overline{\mathcal{U}}_r)(I_{n-r} \oplus A_{r-1}) = (U_1 I_{n-r} \oplus \overline{\mathcal{U}}_r) G_r$. Iterating the procedure and using $\overline{\mathcal{U}}_i A_{i-1} = \mathcal{U}_i$, for $i \leq r$ we get the factorization. $\square$

Finally, Theorem 8 can be further generalized into (the proof is omitted)

Theorem 14. *The Cholesky factorization of $\mathcal{S}_n(U)$ is*

$$\mathcal{S}_n(U) = U_n U_n^T.$$

Our proofs were more straightforward (we believe) because we disregarded some of the properties of the entries of various matrices considered in the previously mentioned research, since these properties are not relevant in this context.

6 Further Remarks

Let $a = [a_1, a_2, \ldots, a_n]$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ be two increasing sequences. The sequence a is said to be *majorized* by λ, denoted by $a \prec \lambda$, if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} \lambda_i, \ k = 1, \ldots, n-1$$
$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i.$$

The well known Schur-Horm Theorem [7] states that if H is a Hermitian matrix, and a and λ contain the diagonal entries and eigenvalues of H, respectively, then $a \prec \lambda$. Using this result, by taking $a = [U_1^2, U_1^2 + U_2^2, \ldots, \sum_{i=1}^{n} U_i^2]$ and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$, the eigenvalues of $\mathcal{U}_n$ in increasing order, we obtain the following result. (Certainly, over $\mathbf{R}$, a symmetric matrix is Hermitian.)

Theorem 15. *The eigenvalues of $\mathcal{U}_n$ satisfy*

$$\sum_{i=1}^{k} U_i^2 \leq \sum_{i=1}^{k} \lambda_i, \ k = 1, \ldots, n-1;$$
$$\sum_{i=1}^{k} U_i^2 = \sum_{i=1}^{k} \lambda_i.$$

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